Groups and topological dynamics

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Contents

Chapter 3

Groupoids

The notion of a topological groupoid is an interpolation of the notions of a group and of a topological space, and therefore fits well into the main subject of this book. They will be also important technical tools in the subsequent chapters. We will use groupoids in two different situations: as generalizations of dynamical systems and as "non-commutative spaces". The first approach will be also a source for construction of groups with interesting properties in Chapter 5. The non-commutative spaces appear naturally (as *orbispaces*, e.g., *Thurston orbifolds* for rational functions) in the study of sub-hyperbolic dynamical systems. They also naturally appear in the study of foliated spaces (e.g., in the case of Ruelle-Smale systems). Foliation theory is one of the main historical sources of the interest in groupoid theory, see.... The other important direction in theory of topological groupoids comes from the theory of C^*-algebras, see...

3.1. Basic definitions

3.1.1. General definition and terminology.

Definition 3.1.1. A *groupoid* is a set $\mathfrak{G}$ with a partially defined multiplication and everywhere defined operation of taking inverse satisfying the following conditions.

(1) If g_1g_2 and g_2g_3 are defined, then $(g_1g_2)g_3 = g_1(g_2g_3)$ and both products are defined.
(2) For every $g \in \mathfrak{G}$ the products gg^{-1} and gg^{-1} are defined.
(3) If g_1g_2 is defined, then $(g_1^{-1}g_1)g_2 = g_2$ and $g_1(g_2g_2^{-1}) = g_1$ and the corresponding products are defined.

Lemma 3.1.2. *For every $g \in \mathfrak{G}$ we have $(g^{-1})^{-1} = g^{-1}$. If $g_1 g_2$ is defined, then $(g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}$.*

Proof. We have

$$gg^{-1}(g^{-1})^{-1} = (gg^{-1})(g^{-1})^{-1} = (g^{-1})^{-1}$$

and

$$gg^{-1}(g^{-1})^{-1} = g(g^{-1}(g^{-1})^{-1}) = g,$$

hence $(g^{-1})^{-1} = g$.

We also have

$$g_1 g_2 g_2^{-1} g_1^{-1} g_1 = g_1(g_2 g_2^{-1}) g_1^{-1} g_1 = g_1 g_1^{-1} g_1 = g_1$$

Multiplying by $(g_1 g_2)^{-1}$ from the left side, we get

$$g_2^{-1} g_1^{-1} g_1 = (g_1 g_2)^{-1} g_1.$$

Multiplying by g_1^{-1} from the right side, we get

$$g_1^{-1} g_1^{-1} = (g_1 g_2)^{-1}.$$

□

Elements of the form gg^{-1} are called *units* of the groupoid. Define

$$\mathbf{s}(g) = g^{-1} g, \qquad \mathbf{r}(g) = gg^{-1}.$$

The units $\mathbf{s}(g)$ and $\mathbf{r}(g)$ are called the *source* and the *range* of g, respectively.

Lemma 3.1.3. *A product $g_1 g_2$ is defined if and only if $\mathbf{r}(g_2) = \mathbf{s}(g_1)$. If $g_1 g_2$ is defined, then $\mathbf{s}(g_1 g_2) = \mathbf{s}(g_2)$ and $\mathbf{r}(g_1 g_2) = \mathbf{r}(g_1)$.*

Proof. If a product $g_1 g_2$ is defined, then the product $g_1^{-1} g_1 g_2 g_2^{-1}$ is also defined, by the conditions of Definition 3.1.1. We have

$$\mathbf{r}(g_2) = g_2 g_2^{-1} = (g_1^{-1} g_1) g_2 g_2^{-1} = g_1^{-1} g_1 (g_2 g_2^{-1}) = g_1^{-1} g_1 = \mathbf{s}(g_1).$$

We also have

$$\mathbf{s}(g_1 g_2) = (g_1 g_2)^{-1} g_1 g_2 = g_2^{-1} g_1^{-1} g_1 g_2 = g_2^{-1} g_2 = \mathbf{s}(g_2)$$

and

$$\mathbf{r}(g_1 g_2) = (g_1 g_2)(g_1 g_2)^{-1} = g_1 g_2 g_2^{-1} g_1^{-1} = g_1 g_1^{-1} = \mathbf{r}(g_1).$$

□

We imagine, therefore, units of the groupoid as points, and elements g of the groupoid as arrows from $\mathbf{s}(g)$ to $\mathbf{r}(g)$. A composition $g_1 g_2$ of elements of the groupoid is defined if the arrows are aligned so that the end of the arrow g_2 is the beginning of the arrow g_1, see Figure 3.1. This leads to another formulation of Definition 3.1.1: a groupoid is a small category of

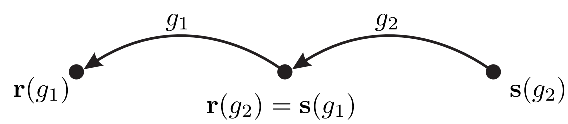

Figure 3.1. Product $g_1 g_2$

isomorphisms, i.e., a category in which all morphisms are isomorphisms, and the classes of objects and morphisms are sets.

We denote by $\mathfrak{G}^{(0)}$ the set of all units of Γ, and by $\mathfrak{G}^{(2)} = \{(g_1, g_2) \in \mathfrak{G} \times \mathfrak{G} \ : \ \mathbf{s}(g_1) = \mathbf{r}(g_2)\}$ the set of all *composable pairs*.

Definition 3.1.4. A *functor* (or a *homomorphism*) of groupoids is a map $\phi : \mathfrak{G}_1 \longrightarrow \mathfrak{G}_2$ such that for every $(g_1, g_2) \in \mathfrak{G}_1^{(2)}$ we have $(\phi(g_1), \phi(g_2)) \in \mathfrak{G}_2^{(2)}$ and $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$.

Two groupoids $\mathfrak{G}_1, \mathfrak{G}_2$ are said to be *isomorphic* if there exists an invertible map $\phi : \mathfrak{G}_1 \longrightarrow \mathfrak{G}_2$ such that ϕ and ϕ^{-1} are functors.

We will often consider groupoids as "atlases" of quotients of spaces by equivalence relations. From this point of view, Definition 3.1.4 is too restrictive, and a different more flexible notion of a morphism between groupoids will be used, see 3.2.

Example 3.1.5. A groupoid is said to be *trivial* if it consists of units only. Trivial groupoids are just sets.

Example 3.1.6. A group is a groupoid with one unit (and hence with everywhere defined multiplication).

Example 3.1.7. Let E be an equivalence relation on a set X, seen as a subset of $X \times X$. Then E has a natural groupoid structure with product defined by

$$(x_1, x_2)(x_2, x_3) = (x_1, x_3),$$

and $(x_1, x_2)^{-1} = (x_2, x_1)$. We have then $\mathbf{s}(x_1, x_2) = (x_2, x_2)$ and $\mathbf{r}(x_1, x_2) = (x_1, x_1)$. We usually identify a unit (x, x) with the point x.

As a mixture of the last two examples, we get the following general description of abstract groupoids.

Example 3.1.8. Let $\{G_i\}_{i \in I}$ be a collection of groups. Let X be a set with an equivalence relation E, and let $P : X \longrightarrow I$ be a map constant on the E-classes. Let $\mathfrak{G}$ be the set of all triples (g, x_1, x_2), where $(x_1, x_2) \in E$, and $g \in G_{P(x_1)}$. Define multiplication and taking inverse on $\mathfrak{G}$ by the rules

$$(g_1, x_1, x_2)(g_2, x_2, x_3) = (g_1 g_2, x_1, x_3), \qquad (g, x_1, x_2)^{-1} = (g^{-1}, x_2, x_1).$$

Then $\mathfrak{G}$ is a groupoid. Every groupoid is isomorphic to a groupoid of this class.

Example 3.1.9. As a partical case of Example 3.1.8, consider the following groupoid. Let G be a group, and let H be its subgroup. Consider the category whose objects are the left cosets of G modulo H, and whose morphisms are maps $x \mapsto gx$ between cosets $hH \longrightarrow ghH$ given by the left multiplication by elements of G. This is a small category of isomorphisms, i.e., a groupoid. We call it the *coset groupoid* of G modulo H.

Definition 3.1.10. Let $\mathfrak{G}$ be a groupoid. We say that two units $x, y \in \mathfrak{G}^{(0)}$ belong to the same *orbit* if there exists $g \in \mathfrak{G}$ such that $\mathbf{s}(g) = x$ and $\mathbf{r}(g) = y$.

It follows from Lemma 3.1.3 that belonging to one orbit is an equivalence relation. The corresponding equivalence classes are called *orbits* of the groupoid. For example, the orbits of the groupoid from Example 3.1.7 coincide with the equivalence classes of E.

Definition 3.1.11. A subset $A \subset \mathfrak{G}^{(0)}$ is a $\mathfrak{G}$-*transversal* if it intersects every $\mathfrak{G}$-orbit.

Definition 3.1.12. Let $x \in \mathfrak{G}^{(0)}$. The *isotropy group* of x is the group

$$\mathfrak{G}_x = \{g \in \mathfrak{G} \ : \ \mathbf{s}(g) = \mathbf{r}(g) = x\}.$$

An element $g \in \mathfrak{G}$ is called *isotropic* if $\mathbf{s}(g) = \mathbf{r}(g)$. A groupoid is called *principal* if all its isotropy groups are trivial.

A groupoid is principal if and only if it is isomorphic (as an abstract groupoid) to the groupoid associated with an equivalence relation (as in Example 3.1.7).

Definition 3.1.13. For $A \subset \mathfrak{G}^{(0)}$, the *restriction* $\mathfrak{G}|_A$ of $\mathfrak{G}$ to A is the groupoid $\{g \in \mathfrak{G} \ : \ \mathbf{s}(g), \mathbf{r}(g) \in A\}$.

3.1.2. Topological groupoids. Abstract groupoids, without any additional structure are not very interesting. Every one of them is isomorphic to a groupoid of the form described in Example 3.1.8, so it is a rather straightforward mixture of groups and equivalence relations. We will be interested in a much richer theory of *topological groupoids.*

Definition 3.1.14. A *topological groupoid* is a groupoid $\mathfrak{G}$ with a topology on it such that the operations of multiplication $(g_1, g_2) \mapsto g_1 g_2 : \mathfrak{G}^{(2)} \longrightarrow \mathfrak{G}$ and taking inverse $g \mapsto g^{-1} : \mathfrak{G} \longrightarrow \mathfrak{G}$ are continuous, and the maps $\mathbf{s}, \mathbf{r} : \mathfrak{G} \longrightarrow \mathfrak{G}^{(0)}$ are open. Here $\mathfrak{G}^{(2)}$ is taken with the relative topology of a subset of the direct product $\mathfrak{G} \times \mathfrak{G}$.

We assume (as a part of the definition) that a topological groupoid and its unit space are locally compact and locally Hausdorff, and that the maps $\mathbf{s}, \mathbf{r} : \mathfrak{G} \longrightarrow \mathfrak{G}^{(0)}$ are open.

(We will drop the condition of local compactness in one instance 3.4.3.)

We *do not* assume that $\mathfrak{G}$ is Hausdorff. Note also that we do not include Hausdorffness into the definition of a compact set.

Example 3.1.15. Let $f \circlearrowleft \mathcal{X}$ be an expansive homeomorphism, where $\mathcal{X}$ is a compact metric space. Recall, that it means that f is $\epsilon > 0$ such that $d(f^n(x), f^n(y)) \leqslant \epsilon$ for all $n \in \mathbb{Z}$ implies that $x = y$, see 1.4.5.

Consider the stable equivalence relation

$$x \sim y \Longleftrightarrow \lim_{n\to\infty} d(f^n(x), f^n(y))) = 0.$$

As any equivalence relation, it defines a groupoid with the set of units $\mathcal{X}$ consisting of pairs of equivalent points. A naïve definition of a topology on this groupoid would be the relative topology of a subset of $\mathcal{X} \times \mathcal{X}$. Note, however, that this topology is not locally compact. On the other hand, the set $\Delta \subset \mathcal{X} \times \mathcal{X}$ of pairs (x, y) such that $d(f^n(x), f^n(y)) \leqslant \epsilon$ for all $n \geqslant 0$ is compact, and the stable equivalence relation is equal to the increasing union of the sets $f^{-n}(\Delta)$, $n \geqslant 0$, see Lemma 1.4.21.

It is natural to consider the stable equivalence relation as the direct limit of the spaces $f^{-n}(\Delta)$ and $n \to \infty$, and to consider the inductive limit topology. We get hence two topological groupoids naturally associated with the stable and the unstable equivalence relations for an expansive homeomorphism.

Lemma 3.1.16. *Let $\mathfrak{G}$ be a topological groupoid. If $A, B \subset \mathfrak{G}$ are open, then AB is open. If $A, B \subset \mathfrak{G}$ are compact, then AB is compact.*

Proof. Suppose that A and B are open, and let $g \in AB$. Let $a_0 \in A$ and $b_0 \in B$ be such that $g = a_0 b_0$. Since $\mathbf{s}$ is an open map, $\mathbf{s}(B)$ is an open neighborhood of $\mathbf{s}(b_0)$ in $\Gamma^{(0)}$. Let U be a neighborhood of g such that $\mathbf{s}(U) \subset \mathbf{s}(B)$ (which exists, by continuity of $\mathbf{s}$). By continuity of multiplication, there exist neighborhoods $U' \subset U$ and $B' \subset B$ of g and b_0, respectively, such that $\mathbf{s}(U') \subset \mathbf{s}(B')$ and for every $h \in U'$ and $b \in B'$ such that $\mathbf{s}(b) = \mathbf{s}(h)$ we have $hb^{-1} \in A$. For every element $h \in U'$ there exists $b \in B'$ such that $\mathbf{s}(h) = \mathbf{s}(b)$, and by the choice of U' and B' we have then $a = hb^{-1} \in A$, hence $h = ab \in AB$. We prove that $U' \subset AB$, i.e., that a neighborhood of g is contained in AB.

Suppose that A and B are compact. We can represent A and B as a finite union of compact Hausdorff sets such that their images under $\mathbf{s}$ and $\mathbf{r}$ are Hausdorff. Therefore, we may assume that A, B, $\mathbf{s}(A), \mathbf{r}(B)$ are Hausdorff.

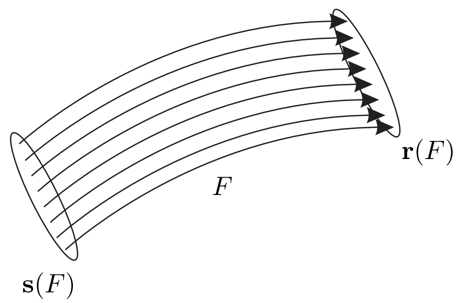

Figure 3.2. A bisection

Then the set $A \times B \subset \mathfrak{G} \times \mathfrak{G}$ is compact and Hausdorff, and $A \times B \cap \mathfrak{G}^{(2)}$ is its closed subset, hence it is also compact. Then AB is a continuous image of a compact set, hence it is also compact. □

History and literature for general topological groupoids....

Definition 3.1.17. A subset $F \subset \mathfrak{G}$ is a *bisection* (or a $\mathfrak{G}$-*bisection*) if the maps $\mathbf{s} : F \longrightarrow \mathbf{s}(F)$ and $\mathbf{r} : F \longrightarrow \mathbf{r}(F)$ are homeomorphisms.

A topological groupoid $\mathfrak{G}$ is said to be *étale* if there is a basis of topology on $\mathfrak{G}$ consisting of open bisections.

$\mathfrak{G}$-bisections are called sometimes $\mathfrak{G}$-sets, see...

In other words $\mathfrak{G}$ is étale if the maps $\mathbf{s}$ and $\mathbf{r}$ are local homeomorphisms.

Example 3.1.18. Let G be a discrete group acting by homeomorphisms on a space $\mathcal{X}$. Then $G \times \mathcal{X}$ has a natural groupoid structure defined by

$$\mathbf{s}(g, x) = x, \qquad \mathbf{r}(g, x) = g(x),$$

and

$$(g_1, g_2(x)) \cdot (g_2, x) = (g_1 g_2, x).$$

For every $g \in G$ and every open subset $U \subset \mathcal{X}$ the set $\{(g, x) : x \in U\}$ is an open bisection. We call the constructed groupoid *groupoid of the action*, and denote it $G \ltimes \mathcal{X}$.

3.1.3. Groupoids of germs.

Let G be a (discrete) group acting by homeomorphisms on a space $\mathcal{X}$. A *germ* is an equivalence class of a pair $(g, x) \in G \times \mathcal{X}$, where two pairs (g_1, x_1) and (g_2, x_2) are equivalent if $x_1 = x_2$ and there is a neighborhood U of x_1 such that the restrictions $g_1|_U$ and $g_2|_U$ are equal maps. The germ (g, x) "remembers" only the action of g on arbitrarily small neighborhood of x.

The set of germs has a natural groupoid structure with the same multiplication rule as for the groupoid of the action. It is easy to see that the equivalence relation in the definition of germs agrees with the groupoid operations, so that the groupoid of germs is a quotient of the groupoid of the action.

The natural topology on the groupoid of germs is given by the basis of open sets consisting of sets of the form $\mathcal{F}_{g,U} = \{(g, x) \ : \ x \in U\}$, where $g \in G$ and U is an open subset of $\mathcal{X}$. It is easy to see that the groupoid of germs is étale with respect to this topology.

Groupoids of germs can be defined not only for group actions, but for arbitrary pseudogroups.

Definition 3.1.19. A *pseudogroup of local homeomorphisms* $\mathcal{G}$ of a space $\mathcal{X}$ is a set of homeomorphisms between open subsets of $\mathcal{X}$ containing the identity homeomorphism $Id : \mathcal{X} \longrightarrow \mathcal{X}$ and closed under the following operations.

(1) Composition.
(2) Taking inverse.
(3) Restricting onto an open subset of the domain.
(4) Taking unions: if $F : U_1 \longrightarrow U_2$ is a homeomorphism between open subsets of $\mathcal{X}$ such that there exists a cover $\mathcal{U}$ of U_1 by open subsets such that $F|_U \in \mathcal{G}$ for all $U \in \mathcal{U}$, then $F \in \mathcal{G}$.

If $\mathcal{G}$ is a pseudogroup, then we can define its groupoid of germs in the same way as we defined the groupoid of germs of a group action. Note that the pseudogroup can be reconstructed from its groupoid of germs in the following way.

Let $\mathfrak{G}$ be an étale groupoid, and let $F \subset \mathfrak{G}$ be an open bisection. Then F naturally defines a homeomorphism $\mathbf{r} \circ \mathbf{s}^{-1} : \mathbf{s}(F) \longrightarrow \mathbf{r}(F)$ between the domain and the range of F. Note that if $\mathfrak{G}$ is a groupoid of germs of a pseudogroup $\mathcal{G}$, and F is an element of $\mathcal{G}$, then the set of germs of F is an open bisection defining F. The following is straightforward, and is left as an exercise.

Proposition 3.1.20. *Let $\mathfrak{G}$ be an étale groupoid. The set of all homeomorphisms defined by open bisections of $\mathfrak{G}$ is a pseudogroup. If $\mathfrak{G}$ is the groupoid of germs of a pseudogroup $\mathcal{G}$, then the set of homeomorphisms defined by open bisections of $\mathfrak{G}$ is equal to $\mathcal{G}$.*

We call the pseudogroup of open bisections of an étale groupoid $\mathfrak{G}$ the *associated pseudogroup* of $\mathfrak{G}$. If $\mathfrak{G}$ is an arbitrary étale groupoid, then the groupoid of germs of the associated pseudogroup of $\mathfrak{G}$ is a quotient of $\mathfrak{G}$. We call it the *effective quotient* of $\mathfrak{G}$. Groupoids of germs of pseudogroups are thus called effective groupoids. They can be characterized in the following way.

Proposition 3.1.21. *An étale groupoid is effective (i.e., is a groupoid of germs of a pseudogroup) if and only if for every $g \in \mathfrak{G} \setminus \mathfrak{G}^{(0)}$ and every neighborhood U of g there exists $h \in U$ such that $\mathbf{s}(h) \neq \mathbf{r}(h)$.*

We have thus two equivalent terminological approaches to the same object: pseudogroups of local homeomorphisms and effective groupoids. Different terminologies are convenient in different situations. But considering general (non-effective) groupoids is in some cases necessary even in the study of effective groupoids. For example, a restriction $\mathfrak{G}|_A$ of an effective groupoid is not always effective (though it is always étale).

Note that groupoids of germs are not always Hausdorff even if the space of units is Hausdorff. The following proposition gives some criteria of Hausdorfness that will be useful later.

Proposition 3.1.22. *A pseudogroup $\mathcal{G}$ acting on a Hausdorff space $\mathcal{X}$ has a Hausdorff groupoid of germs if and only if for every $F \in \mathcal{G}$ the interior of the set of fixed points of F is relatively closed in the domain of F.*

Proof. Let $\mathfrak{G}$ be the groupoid of germs of $\mathcal{G}$. If $g, h \in \mathfrak{G}$ are such that $\mathbf{s}(g) \neq \mathbf{s}(h)$, then there exist neighborhoods $U_g \ni g$ and $U_h \ni h$ such that $\mathbf{s}(U_g)$ and $\mathbf{s}(U_h)$ are disjoint neighborhoods of $\mathbf{s}(g)$ and $\mathbf{s}(h)$, respectively. Similarly, if $\mathbf{r}(g) \neq \mathbf{r}(h)$, then g and h can be separated by disjoint neighborhoods. If g and h do not have disjoint neighborhoods, then $h^{-1}g$ and $g^{-1}g = \mathbf{s}(g)$ do not have disjoint neighborhoods. Consequently, if $\mathfrak{G}$ is not Hausdorff, then there exists $F \in \mathcal{G}$ and $x \in \mathbf{s}(F)$ such that every neighborhood of the germ (F, x) and every neighborhood of (Id, x) have a non-empty intersection while $(F, x) \neq (Id, x)$. This is equivalent to the condition that for every neighborhood U of x the interior of the set of fixed points of $F|_U$ is non-empty, which in turn is equivalent to the condition that x belongs to the closure of the interior of the set of fixed points of F. On the other hand $(F, x) \neq (Id, x)$ is equivalent to the condition that x does not belong to the interior of the set of fixed points of F. We have proved that the groupoid of germs $\mathfrak{G}$ is non-Hausdorff if and only if there exists $x \in \mathcal{X}$ and $F \in \mathcal{G}$ such that x is the boundary point of the set of fixed points of F. □

It is easy to construct therefore examples of pseudogroups and group actions with non-Hausdorff groupoid of germs. It follows from Proposition 3.1.22 that the groupoid of germs of a group action is Hausdorff if and only if all points have Hausdorff groups of germs in the sense of Definition 2.1.13. In particular, the action described in Example 2.1.15 has a non-Hausdorff groupoid of germs.

3.1.4. Examples of étale groupoids.

3.1.4.1. *Groupoids generated by local homeomorphisms.* Let $f \circlearrowleft \mathcal{X}$ be a local homeomorphisms (e.g., a covering map). Consider the set $\mathcal{F}$ of all homeomorphisms of the form $f : U \longrightarrow f(U)$, where U is an open subset of $\mathcal{X}$. By the definition, the domains of the elements of $\mathcal{F}$ cover $\mathcal{X}$.

Definition 3.1.23. The groupoid of germs *generated by* $f \circlearrowleft \mathcal{X}$ is the groupoid of germs of the pseudogroup generated by $\mathcal{F}$.

Informally, the groupoid of germs $\mathfrak{F}_f$ generated by f is the groupoid generated by the germs of f.

Every element of the groupoid $\mathfrak{F}_f$ is equal to the product $(f^n, y)^{-1}(f^m, x)$, where (f^n, y) and (f^m, x) are the germs of the maps f^n and f^m at the points y and x, respectively, and $x, y \in \mathcal{X}$ are such $f^m(x) = f^n(y)$.

The orbits of $\mathfrak{F}_f$ are called the *grand orbits* of the map $f \circlearrowleft \mathcal{X}$: they are the classes of the equivalence relation generated by $x \sim f(x)$. A point $x \in \mathcal{X}$ has a non-trivial isotropy group in $\mathfrak{F}_f$ if and only if it is *eventually periodic*, i.e., if there exist $m > n \geqslant 0$ such that $f^m(x) = f^n(x)$. The isotropy groups are always cyclic.

A related étale groupoid was defined by ....

3.1.4.2. *Holonomy groupoids of local product structures and foliations.* Let $\mathcal{X}$ be a topological space, and let $\mathcal{R} = \{(R_i, [\cdot, \cdot]_i) : i \in I\}$ be an atlas of a local product structure on $\mathcal{X}$, see Definition 1.4.27. Let $R_i = A_i \times B_i$ be the canonical decomposition of R_i into the direct product. We assume that the spaces A_i are connected. Then the space $\mathcal{X}$ is partitioned into the *leaves*, where two points $x, y \in \mathcal{X}$ belong to one leaf if there exists a sequence $\mathsf{P}_1(R_{i_1}, x_1), \mathsf{P}_1(R_{i_2}, x_2), \ldots, \mathsf{P}_1(R_{i_n}, x_n)$, where $x \in \mathsf{P}_1(R_{i_1}, x_1)$, $y \in \mathsf{P}_1(R_{i_n}, x_n)$, and $\mathsf{P}_1(R_{i_k}, x_k) \cap \mathsf{P}_1(R_{i_{k+1}}, x_{k+1}) \neq \varnothing$ for all k. The partition into the leaves depends only on the local product structure, and does not depend on the choice of the atlas.

Typically, the quotient of $\mathcal{X}$ obtained by identifying all points belonging to the same leaf is a non-Hausdorff space, and the topology of the quotient space does not carry much useful information about the local product structure. Accordingly, the right thing to consider is not the quotient space, but the associated groupoid.

Let $x \in R_i \cap R_j$. Then there exists a rectangular open neighborhood U of x such that the restrictions of $[\cdot, \cdot]_i$ and $[\cdot, \cdot]_j$ to $U \cap R_i \cap R_j$ coincide. It follows that the sets $\mathsf{P}_2(R_i, x) \cap U$ and $\mathsf{P}_2(R_j, x) \cap U$ coincide. They are identified with the subsets $U_i \times \{x_1\} \subset B_i$ and $U_j \times \{x_2\} \subset B_j$, and we get a natural homeomorphism $H : U_i \longrightarrow U_j$ between the corresponding subsets of B_i and B_j, see Figure 3.3. The homeomorphism may depend on the choice of U, but its germ $\gamma_{x,i,j}$ depends only on x and R_i, R_j. All germs of the homeomorphism H are of the form $\gamma_{y,i,j}$ for some $y \in \mathsf{P}_1(R_i, x) \cap U$.

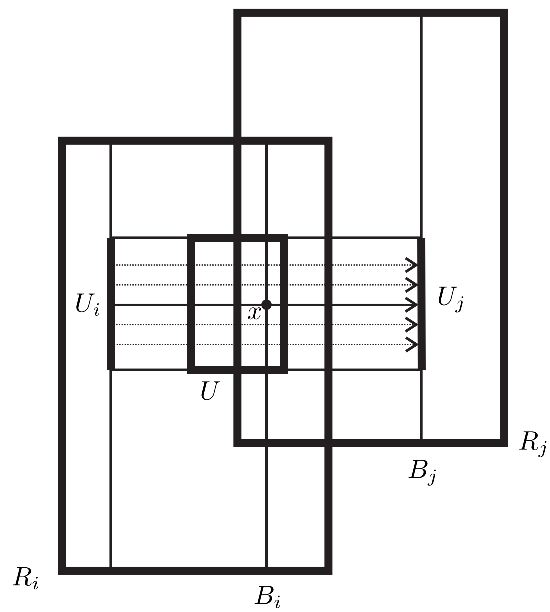

Figure 3.3. Generators of the holonomy groupoid

The *holonomy groupoid* of the *first direction* of the local product structure is the groupoid of germs of the pseudogroup of local homeomorphisms of the disjoint union $\bigsqcup_{i\in I} B_i$ generated by the homeomorphisms of the form $H : U_i \longrightarrow U_j$, as defined above.

If γ is an element of the holonomy groupoid, and $\mathbf{s}(\gamma) \in R_i, \mathbf{r}(\gamma) \in R_j$, then γ describes how the fiber $\mathsf{P}_2(R_i, \mathbf{s}(\gamma))$ is locally mapped to the fiber $\mathsf{P}_2(R_j, \mathbf{r}(\gamma))$ as a point travels from $\mathbf{s}(\gamma)$ to $\mathbf{r}(\gamma)$ along the leaf containing these points. Namely, there exists a rectangle $R = A \times B$ and a local homeomorphism $f : R \longrightarrow \mathcal{X}$ preserving the local product structures (see Definition 1.4.29) such that there exist $a_1, a_2 \in A$, and $b \in B$ such that $\mathbf{s}(\gamma) = f(a_1, b)$, $\mathbf{r}(\gamma) = f(a_2, b)$, and γ is the germ of the local homeomorphism the neighborhood $U_1 = f(\{a_1\} \times B)$ of $\mathbf{s}(\gamma)$ in $\mathsf{P}_2(\mathbf{s}(\gamma), R_i)$ to the neighborhood $U_2 = f(\{a_2\} \times B)$ of $\mathbf{s}(\gamma)$ in $\mathsf{P}_2(\mathbf{r}(\gamma), R_j)$ mapping $f(a_1, y)$ to $f(a_2, y)$ for $y \in B$. The rectangle R is a "thin" neighborhood in $\mathcal{X}$ of the leaf containing $\mathbf{s}(\gamma)$ and $\mathbf{r}(\gamma)$, see Figure 3.4.

The definition of the holonomy groupoid does not use the full strength of Definition 1.4.27. We only use the fact that every plaque $\mathsf{P}_1(y, R_i)$ intersects at most one plaque $\mathsf{P}_1(z, R_j)$. More precisely, the map from a subset of the direct product $A_i \times B_i$ to $A_j \times B_j$ identifying $U \cap R_i$ with $U \cap R_j$ does not have to be of the from $f(a, b) = (f_1(a), f_2(b))$. It is enough to require that the map is of the form $f(a, b) = (f_1(a, b), f_2(b))$, so that we still have a well defined map from a subset of B_i to a subset of B_j.

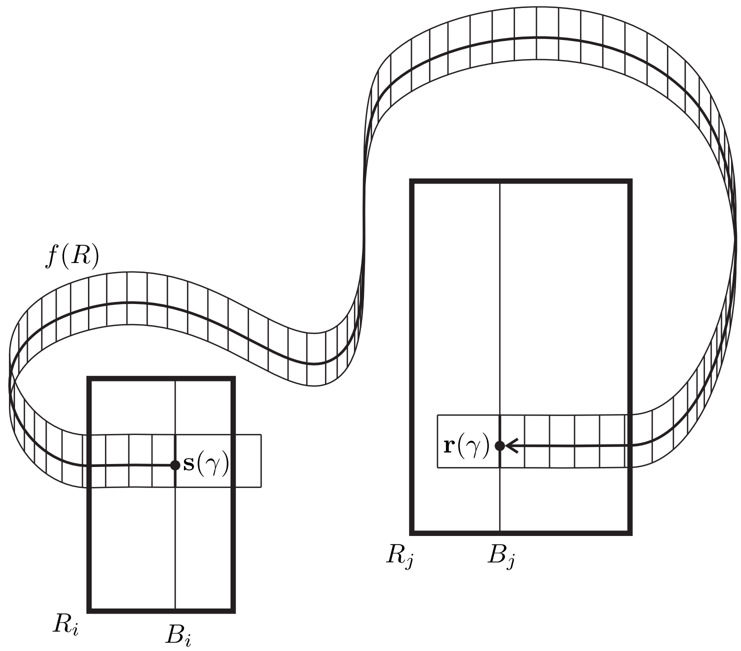

Figure 3.4. Holonomy groupoid

This weaker condition holds in the case of *foliations*, see... Holonomy groupoids of foliations is historically one of the main sources of interest in groupoid theory...

3.1.4.3. *Groupoids associated with Ruelle-Smale systems.* Let $f \circlearrowright \mathcal{X}$ be a Ruelle-Smale dynamical system. Suppose that $x, y \in \mathcal{X}$ are stably equivalent. Then there exists $n \geqslant 0$ such that $f^n(x)$ and $f^n(y)$ belong to one small rectangle. The germ at $f^n(x)$ of the holonomy from the unstable leaf of $f^n(x)$ to the unstable leaf of $f^n(y)$ does not depend on the rectangle for big n, since the direct product structure is locally unique. Applying f^{-n} to it, we get a well defined germ $S_{x,y}$ at x of the holonomy from the unstable leaf of x to the unstable leaf of y. If the stable leaves are path connected, then this germ of the holonomy is uniquely determined by the local product structure on $\mathcal{X}$ and coincides with the germ define in the previous example 3.1.4.2.

It follows from the uniqueness of the germ $S_{x,y}$ that if x, y, z are stably equivalent to each other, then $S_{x,z} = S_{y,z}S_{x,z}$ and that $S_{x,y}^{-1} = S_{y,x}$. It follows that the set of all germs of the form $S_{x,y}$ is a groupoid, which we will denote $\mathfrak{U}$.

For open every rectangle R of $\mathcal{X}$ and any two unstable plaques $\mathsf{W}_-(R, x)$ and $\mathsf{W}_-(R, y)$ we have the corresponding set of germs $S_{t,[t,y]_R}$ of the holonomy from $\mathsf{W}_-(R, x)$ to $\mathsf{W}_-(R, y)$ inside R. We declare the collection of such sets a basis of topology on the groupoid $\mathfrak{U}$. Then the sets of germs of holonomies inside rectangles of $\mathcal{X}$ become open bisections of $\mathfrak{U}$. The space of units of $\mathfrak{U}$ is the disjoint union of the unstable leaves of $\mathcal{X}$ with inductive limit topology on the leaves.

The groupoid $\mathfrak{U}$ is an étale version of the groupoid of the stable equivalence relation defined in 3.1.15. They both represent the "non-commutative space" of the stable equivalence classes. In fact, we will see later that these two groupoids are equivalent in a rigorous sense.

It is natural to replace $\mathfrak{U}$ by its restriction onto a transversal (for instance in order to make it second countable). For example, we can cover $\mathcal{X}$ by a finite number of open rectangles, choose an unstable plaque of each rectangle, and restrict $\mathfrak{U}$ to their union.

The groupoid $\mathfrak{S}$ of germs of the holonomies between stable leaves of $\mathcal{X}$ is defined in the same way (by changing f to f^{-1} in the definition).

Example 3.1.24. Let $f \circlearrowleft \mathsf{X}^{\mathbb{Z}}$ be the full shift. Two sequences $(x_n)_{n\in\mathbb{Z}}, (y_n)_{n\in\mathbb{Z}}$ are stably equivalent if and only if $x_n = y_n$ for all $n \geqslant n_0$ for some n_0. The unstable leaf of $(x_n)_{n\in\mathbb{Z}}$ is the set of sequences $(z_n)_{n\in\mathbb{Z}}$ such that $x_n = z_n$ for all n smaller than some index n_0. The set of all such $(z_n)_{n\in\mathbb{Z}}$ for a given index n_1 is a neighborhood of $(x_n)_{n\in\mathbb{Z}}$ in the unstable leaf (the neighborhood becomes smaller as n_1 becomes bigger). The germ $S_{(x_n)_{n\in\mathbb{Z}},(y_n)_{n\in\mathbb{Z}}}$ is the germ of the transformation replacing every coordinate z_n for $n < n_0$ by the coordinate y_n.

The local product structure on $\mathsf{X}^{\mathbb{Z}}$ is generated by the direct product structure of one rectangle $\mathsf{X}^{-\omega}\times\mathsf{X}^{\omega}$ given by the identification $(\ldots x_{-2}x_{-1}, (x_0x_1\ldots) \mapsto (\ldots x_{-2}x_{-1} \,.\, x_0x_1\ldots)$. If we restrict $\mathfrak{U}$ onto the unstable plaque of this rectangle, and identify it with X^{ω}, then the restriction becomes identified with the groupoid of germs of the transformations of the form $S_{v_1,v_2}v_1w \mapsto v_2w : v_1\mathsf{X}^{\omega} \longrightarrow v_2\mathsf{X}^{\omega}$, where $v_1, v_2 \in \mathsf{X}^*$ are finite words of equal lengths.

One can also consider the groupoid of germs generated by $\mathfrak{U}$ and the action of f on the unstable leaves, since the leaves are the stable equivalence relation are f-invariant. We call the obtained groupoid the *unstable Ruelle groupoid* of the system $f \circlearrowleft \mathcal{X}$. The *stable Ruelle groupoid* is defined analogously.

Example 3.1.25. The unstable Ruelle groupoid of the full $\mathbb{Z}$-shift from the previous example, in its version restricted to X^{ω} is the groupoid of germs of the transformations of the form $v_1w \mapsto v_2w : v_1\mathsf{X}^{\omega} \longrightarrow v_2\mathsf{X}^{\omega}$, where $v_1, v_2 \in \mathsf{X}^*$ are arbitrary finite words (of possibly different lengths).

3.1.5. Proper groupoids. The space of orbits of a groupoid with the quotient topology is usually not very well behaved (e.g., has anti-discrete topology if every orbit is dense). Here we describe a class of groupoids for which the space of orbits is Hausdorff.

Recall that a map $f : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ is said to be proper if for every compact subset $C \subset \mathcal{X}_2$ the set $f^{-1}(C)$ is compact.

Definition 3.1.26. A topological (not necessarily étale) groupoid $\mathfrak{G}$ is said to be *proper* if the map $(\mathbf{s}, \mathbf{r}) : \mathfrak{G} \longrightarrow \mathfrak{G}^{(0)} \times \mathfrak{G}^{(0)}$ is proper.

A subset $C \subset \mathfrak{G}^{(0)} \times \mathfrak{G}^{(0)}$ is compact if and only if it is closed and is contained in a set of the form $C_1 \times C_2$ for some compact sets $C_1, C_2 \subset \mathfrak{G}^{(0)}$. It follows that the map $(\mathbf{s}, \mathbf{r})$ is proper if and only if for every two compact subsets C_1, C_2 of $\mathfrak{G}^{(0)}$ the set of elements $g \in \mathfrak{G}$ such that $\mathbf{s}(g) \in C_1$ and $\mathbf{r}(g) \in C_2$ is compact. The next lemma then easily follows.

Lemma 3.1.27. *A groupoid $\mathfrak{G}$ is proper if and only if for every compact set $C \subset \mathfrak{G}^{(0)}$ the set of elements $g \in \mathfrak{G}$ such that $\{\mathbf{s}(g), \mathbf{r}(g)\} \subset C$ is compact.*

Example 3.1.28. An action $(G, \mathcal{X})$ of a discrete group on a topological space is called proper if for every compact set $C \subset \mathcal{X}$ the set of elements $g \in G$ such that $g(C) \cap C \neq \varnothing$ is finite. It is easy to see that the action is proper if and only if the groupoid of the action is proper. The properness of the action is also equivalent to the properness of the groupoid of germs.

The following property of proper groupoids is a generalization of a well known fact about group actions.

Proposition 3.1.29. *Suppose that $\mathfrak{G}$ is proper and $\mathfrak{G}^{(0)}$ is Hausdorff. Then the space of orbits of $\mathfrak{G}$ is Hausdorff with respect to the quotient topology.*

Proof. The quotient topology on the space of orbits is the smallest topology such that the quotient map from $\mathfrak{G}^{(0)}$ to the set of orbits is continuous. In other words, a subset of the set of orbits is open if and only if its preimage in $\mathfrak{G}^{(0)}$ is open.

Let $x, y \in \mathfrak{G}^{(0)}$ be two units belonging to different orbits. We have to show that there exist disjoint open neighborhoods $U_x, U_y \subset \mathfrak{G}^{(0)}$ equal to unions of $\mathfrak{G}$-orbits and such that $x \in U_x, y \in U_y$. Let V_x and V_y be disjoint compact neighborhoods of x and y, respetively. They exist by local compactness and Hausodrffness of $\mathfrak{G}^{(0)}$. Let $B_x = \{g \in \mathfrak{G} \ : \ \mathbf{s}(g) = x, \mathbf{r}(g) \in V_y\}$ and $B_y = \{g \in \mathfrak{G} \ : \ \mathbf{s}(g) = y, \mathbf{r}(g) \in V_x\}$. These sets are compact, by properness of the groupoid. It follows that the sets $\mathbf{r}(B_x)$ and $\mathbf{r}(B_y)$ are compact (as continuous images of compact sets). We have $x \notin \mathbf{r}(B_y)$ and $y \notin \mathbf{r}(B_x)$, since x and y belong to different $\mathfrak{G}$-orbits.

Since compact Hausdorff spaces are regular, there exist compact neighborhoods $V_x' \subset V_x$ and $V_y' \subset V_y$ such that $V_x' \cap \mathbf{r}(B_y) = \varnothing$ and $V_y' \cap \mathbf{r}(B_x) = \varnothing$. Consider the set $A = \{g \in \mathfrak{G} \ : \ \mathbf{s}(g) \in V_x', \mathbf{r}(g) \in V_y'\}$. It is compact, and $x \notin \mathbf{s}(A)$, since otherwise there exists $g \in \mathfrak{G}$ such that $\mathbf{s}(g) = x$ and $\mathbf{r}(g) \in V'_x \subset V_x$, hence $g \in B_x$ and $V'_x \cap \mathbf{r}(B_x) \neq \emptyset$, which is a contradiction. Similarly, $y \notin \mathbf{r}(A)$, since otherwise there exists g such that $\mathbf{s}(g^{-1}) = y$, $\mathbf{r}(g^{-1}) \in V'_y \subset V_y$. Let W_x and W_y be interiors of the sets $V'_x \setminus \mathbf{s}(A)$ and $V'_y \setminus \mathbf{r}(A)$, respectively. They are disjoint and, by the definition of A, there does not exist an element $h \in \mathfrak{G}$ such that $\mathbf{s}(h) \in W_x$ and $\mathbf{r}(h) \in W_y$.

Let U_x be the set of all points that can be represented as $\mathbf{r}(g)$ for $g \in \mathfrak{G}$ such that $\mathbf{s}(g) \in W_x$. Define U_y in the same way. Then U_x and U_y are unions of $\mathfrak{G}$-orbits. They are disjoint, since otherwise there exist elements $g_1, g_2 \in \mathfrak{G}$ such that $\mathbf{r}(g_1) = \mathbf{r}(g_2)$ and $\mathbf{s}(g_1) \in W_x, \mathbf{s}(g_2) \in W_y$, which implies $\mathbf{s}(h) \in W_x$ and $\mathbf{r}(h) \in W_y$ for $g_2 g_1^{-1}$. It remains to show that U_x and U_y are open. But we have

$$U_x = \mathbf{r}(\mathbf{s}^{-1}(W_x)), \quad U_y = \mathbf{r}(\mathbf{s}^{-1}(W_y)),$$

and since $\mathbf{s}, \mathbf{r}$ are continuous and open, the sets U_x and U_y are open. □

3.2. Actions and correspondences

3.2.1. Actions. The notion of an action of a groupoid on a topological space (see [**MRW87**] and [**BH99**, III.$\mathcal{G}$ Definition 3.11]) is a generalization of the notion of a group action. It is naturally modified to take into account the fact that groupoids have many units.

Definition 3.2.1. A *(right) action* $\mathcal{X} \curvearrowleft \mathfrak{G}$ of a groupoid $\mathfrak{G}$ on a topological space $\mathcal{X}$ *over a continuous map* $P : \mathcal{X} \longrightarrow \mathfrak{G}^{(0)}$ (called the *anchor* of the action) is a continuous map $\mathcal{X} \times_P \mathfrak{G} \longrightarrow \mathcal{X} : (x, g) \mapsto x \cdot g$, where

$$\mathcal{X} \times_P \mathfrak{G} = \{(x, g) \ : \ P(x) = \mathbf{r}(g)\},$$

such that $P(x \cdot g) = \mathbf{s}(g)$, and $(x \cdot g_1) \cdot g_2 = x \cdot g_1 g_2$ for all $x \in \mathcal{X}$ and $g_1, g_2 \in \mathfrak{G}$ such that $P(x) = \mathbf{r}(g_1)$ and $\mathbf{r}(g_2) = \mathbf{s}(g_1)$, see Figure 3.5.

In the same way as for groupoids, we always assume that the space $\mathcal{X}$ is locally compact and locally Hausdorff.

The *left action* $\mathfrak{G} \curvearrowright \mathcal{X}$ is defined in a similar way. It is a map $(g, x) \mapsto g \cdot x$ from $\mathfrak{G} \times_P \mathcal{X} = \{(g, x) \ : \ P(x) = \mathbf{s}(g)\}$ to $\mathcal{X}$ satisfying $P(g \cdot x) = \mathbf{r}(g)$ and $g_1 \cdot (g_2 \cdot x) = g_1 g_2 \cdot x$.

Note that it follows from the definition of a right action that $P(\mathcal{X})$ is a $\mathfrak{G}$-invariant subset of $\mathfrak{G}^{(0)}$.

Example 3.2.2. The natural *right action of* $\mathfrak{G}$ *on itself* is defined for $\mathcal{X} = \mathfrak{G}$ over the map $P(g) = \mathbf{s}(g)$, and is given by multiplication $(x, g) \mapsto x \cdot g = xg$.

Example 3.2.3. Every groupoid $\mathfrak{G}$ acts naturally on its space of units. Both actions are defined over the identical embedding $\mathfrak{G}^{(0)} \longrightarrow \mathfrak{G}$ and are

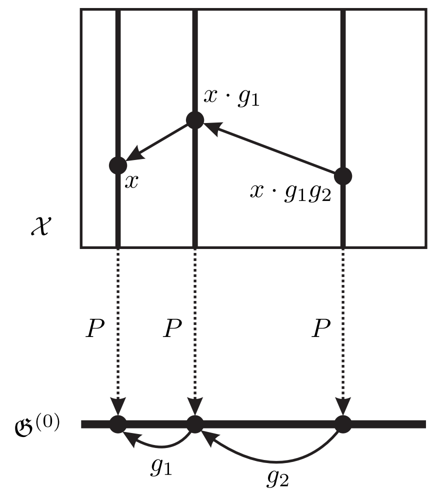

Figure 3.5. A right action

given by $g \cdot x = \mathbf{s}(g)$ for the left action and by $x \cdot g = \mathbf{r}(g)$ for the right action.

Example 3.2.4. Let $\mathcal{G}$ be a pseudogroup of local diffeomorphisms of a manifold $\mathcal{X}$, and let $\mathfrak{G}$ be its groupoid of germs. Let $P : T\mathcal{X} \longrightarrow \mathcal{X}$ be the tangent bundle. Then the map $(\vec{v}, g) \mapsto Dg(\vec{v})$ for $\vec{v} \in T_{\mathbf{r}(g)}\mathcal{X}$ is a well defined action of $\mathfrak{G}$ on the tangent bundle.

The notion of a groupoid of a group action (see...) has a natural generalization to actions of groupoids.

Definition 3.2.5. Suppose that we have a right action of a groupoid $\mathfrak{G}$ with anchor $P : \mathcal{X} \longrightarrow \mathfrak{G}^{(0)}$. The corresponding *groupoid of the action*, denoted $\mathcal{X} \rtimes \mathfrak{G}$, is the space $\mathcal{X} \times_P \mathfrak{G} = \{(x, g) \ : \ P(x) = \mathbf{r}(g)\}$ with multiplication

$$(x_1, g_1) \cdot (x_2, g_2) = (x_1, g_1 g_2),$$

where the product is defined if and only if $x_2 = x_1 \cdot g_1$.

Similarly, the groupoid $\mathfrak{G} \ltimes \mathcal{X}$ of a left action $\mathfrak{G} \curvearrowright \mathcal{X}$ with the anchor $P : \mathcal{X} \longrightarrow \mathfrak{G}^{(0)}$ is the space $\mathfrak{G} \times_P \mathcal{X} = \{(g, x) \ : \ P(x) = \mathbf{s}(g)\}$ with multiplication

$$(g_1, x_1) \cdot (g_2, x_2) = (g_1 g_2, x_2),$$

where the product is defined if and only if $x_1 = g_2 \cdot x_2$.

The source and range maps are given in $\mathcal{X} \rtimes \mathfrak{G}$ by

$$\mathbf{s}(x, g) = (x \cdot g, \mathbf{s}(g)), \quad \mathbf{r}(x, g) = (x, \mathbf{r}(g))$$

and in $\mathfrak{G} \ltimes \mathcal{X}$ by

$$\mathbf{s}(g, x) = (\mathbf{s}(g), x), \qquad \mathbf{r}(g, x) = (\mathbf{r}(g), g \cdot x).$$

The units of $\mathcal{X} \rtimes \mathfrak{G}$ (resp. $\mathfrak{G} \ltimes \mathcal{X}$) are of the form $(x, P(x))$ (resp. $(P(x), x))$), hence the space of units of $\mathcal{X} \rtimes \mathfrak{G}$ is naturally identified with $\mathcal{X}$.

Note that every right action $\mathcal{X} \curvearrowleft \mathfrak{G}$ can be transformed into a left action by the rule

$$g \cdot x = x \cdot g^{-1}.$$

Then the corresponding groupoid of the action $\mathfrak{G} \ltimes \mathcal{X}$ is isomorphic to the groupoid of the original action $\mathcal{X} \rtimes \mathfrak{G}$ under the isomorphism

$$(g, x) \mapsto (x \cdot g^{-1}, g).$$

We will call the maps $(x, g) \mapsto g : \mathcal{X} \rtimes \mathfrak{G} \longrightarrow \mathfrak{G}$ and $(g, x) \mapsto g : \mathfrak{G} \ltimes \mathcal{X} \longrightarrow \mathfrak{G}$ the *natural projections*. It is easy to see that they are functors of groupoids.

We say that $x_1, x_2 \in \mathcal{X}$ belong to one *orbit* of an action $\mathcal{X} \curvearrowleft \mathfrak{G}$ if there exists $g \in \mathfrak{G}$ such that $x_2 = x_1 \cdot g$. It is easy to see that this is an equivalence relation on $\mathcal{X}$. In fact, the orbits of the action coincide with the orbits of the groupoid of the action. We denote the set of orbits of the action by $\mathcal{X}/\mathfrak{G}$ for right actions and by $\mathfrak{G}\backslash\mathcal{X}$ for left actions.

Definition 3.2.6. A right action of $\mathfrak{G}$ over $P : \mathcal{X} \longrightarrow \mathfrak{G}^{(0)}$ is *free* if $x \cdot g = x$ implies that g is a unit (i.e., that $g = P(x)$).

The action is said to be *proper* if the groupoid of the action is proper, i.e., if the map

$$(x, g) \mapsto (x \cdot g, x) : \mathcal{X} \rtimes \mathfrak{G} \longrightarrow \mathcal{X} \times \mathcal{X}$$

is proper.

The action is free if and only if the groupoid of the action is principal. If the action is proper and $\mathcal{X}$ is Hausdorff then, by Proposition 3.1.29, the space of orbits $\mathcal{X}/\mathfrak{G}$ is Hausdorff.

Example 3.2.7. The (right or left) action of a groupoid on itself is free. It is also proper, by Lemma 3.1.16.

Example 3.2.8. The (right or left) action of a groupoid on its space of units is free if and only if the groupoid is principal. It is proper if and only if the groupoid is proper.

Example 3.2.9. Let $\mathfrak{G}$ be a groupoid, and let $\mathcal{F}$ be a topological space. Suppose that $G \curvearrowright \mathcal{F}$ is a topological group acting (from the left) by homeomorphisms on $\mathcal{F}$, and we have a *cocycle* $\sigma : \mathfrak{G} \longrightarrow G$, i.e., a continuous functor. The cocycle σ defines then a natural left action of $\mathfrak{G}$ on $\mathcal{F} \times \mathfrak{G}^{(0)}$ (over the projection $P : \mathcal{F} \times \mathfrak{G}^{(0)} \longrightarrow \mathfrak{G}^{(0)}$) by the rule

$$g \cdot (y, \mathbf{s}(g)) = (\sigma(g)(y), \mathbf{r}(g)).$$

Similarly, for a right action $\mathcal{F} \curvearrowleft G$ and a cocycle $\sigma : \mathfrak{G} \longrightarrow G$ the natural right action of $\mathfrak{G}$ on $\mathfrak{G}^{(0)} \times \mathcal{F}$ is defined by

$$(\mathbf{r}(g), y) \cdot g = (\mathbf{s}(g), (y) \cdot \sigma(g)).$$

We denote the groupoid of the left action by $\sigma \rtimes \mathfrak{G}$, and the groupoid of the left action by $\mathfrak{G} \ltimes \sigma$.

The natural projection $\sigma \rtimes \mathfrak{G} \longrightarrow \mathfrak{G}$ (respectively, $\mathfrak{G} \ltimes \sigma \longrightarrow \mathfrak{G}$) is called the *fiber bundle associated with the cocylce σ*.

Example 3.2.10. Let $\mathfrak{G}$ be a groupoid of germs of a pseudogroup of local diffeomorphisms of $\mathbb{R}^n$. Then for every germ (g, x) the differential Dg evaluated at x is a well defined cocycle from $\mathfrak{G}$ to $\mathrm{GL}_n(\mathbb{R})$ (acting on $\mathbb{R}^n$). The associated fiber bundle is, by definition, the *tangent bundle* of $\mathfrak{G}$.

3.2.2. Biactions. The most straightforward notion of a morphism between groupoids is the notion of a functor, i.e., a map $F : \mathfrak{G}_1 \longrightarrow \mathfrak{G}_2$ which is continuous and preserves the groupoid operations, see Definition 3.1.4. This approach is satisfactory in many situations. On the other hand, if we consider groupoids as non-commutative quotient spaces, then the same quotient space can be described by different *equivalent* groupoids. It becomes natural from this perspective to relax the definition of a morphism. A convenient definition is via the notion of a biaction (analogous to the notion of a bimodule over a C^*-algebra, see...).

Definition 3.2.11. A *biaction* $\mathfrak{G} \curvearrowright \mathcal{M} \curvearrowleft \mathfrak{H}$ consists of two actions: right action $\mathcal{M} \curvearrowleft \mathfrak{H}$ over $P_{\mathfrak{H}} : \mathcal{M} \longrightarrow \mathfrak{H}^{(0)}$ and a left action $\mathfrak{G} \curvearrowright \mathcal{M}$ over $P_{\mathfrak{G}} : \mathcal{M} \longrightarrow \mathfrak{G}^{(0)}$ such that the actions commute, i.e.,

$$(g \cdot x) \cdot h = g \cdot (x \cdot h)$$

for all $g \in \mathfrak{G}$, $h \in \mathfrak{H}$, $x \in \mathcal{M}$ such that $P_{\mathfrak{H}}(x) = \mathbf{r}(h)$ and $P_{\mathfrak{G}}(x) = \mathbf{s}(g)$.

Definition 3.2.12. We say that biactions $\mathfrak{G} \curvearrowright \mathcal{M} \curvearrowleft \mathfrak{H}$ and $\mathfrak{G} \curvearrowright \mathcal{M}' \curvearrowleft \mathfrak{H}$ are *isomorphic* if there exists a homeomorphism $F : \mathcal{M} \longrightarrow \mathcal{M}'$ such that

$$F(g \cdot x \cdot h) = g \cdot F(x) \cdot h$$

for all $g \in \mathfrak{G}$, $h \in \mathfrak{H}$, $x \in \mathcal{M}$ such that the left-hand expression is defined (and the expression on the left-hand side is defined if and only if the expression of the right-hand side is defined).

Every biaction $\mathfrak{G} \curvearrowright \mathcal{M} \curvearrowleft \mathfrak{H}$ defines a relation between $\mathfrak{G}^{(0)}$ and $\mathfrak{H}^{(0)}$ equal to the image of $\mathcal{M}$ in $\mathfrak{G}^{(0)} \times \mathfrak{H}^{(0)}$ by the map $(P_{\mathfrak{G}}, P_{\mathfrak{H}})$. We say

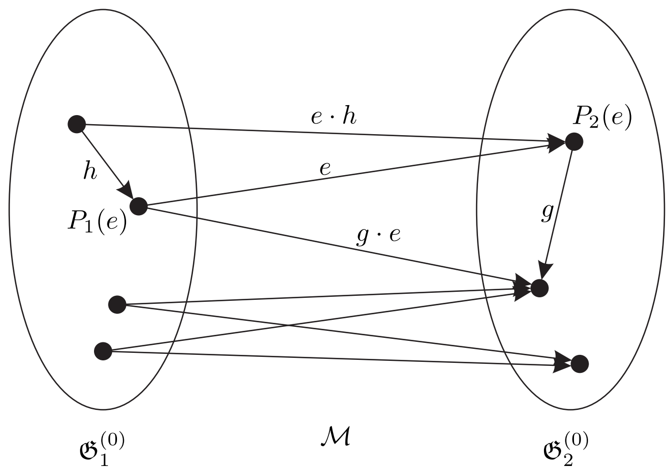

Figure 3.6. Biaction

that $x \in \mathfrak{G}^{(0)}$ and $y \in \mathfrak{H}^{(0)}$ are $\mathcal{M}$-*related* if there exists $e \in \mathcal{M}$ such that $x = P_{\mathfrak{G}}(e)$ and $y = P_{\mathfrak{H}}(e)$. It is useful to imagine $\mathcal{M}$ as a set of "connections" between units of $\mathfrak{G}$ and $\mathfrak{H}$, and to interpret the left and the right actions of the groupoids on $\mathcal{M}$ as post- and pre-compositions of these connections with the elements of the groupoid, see Figure 3.6.

Note that the relation between the unit spaces defined by a biaction is $\mathfrak{G}$- and $\mathfrak{H}$-invariant: if $\mathbf{s}(g)$ and $\mathbf{s}(h)$ are $\mathcal{M}$-related for some $g \in \mathfrak{G}$ and $h \in \mathfrak{H}$, then $\mathbf{r}(g)$ and $\mathbf{r}(h)$ are also $\mathcal{M}$-related. In other words, the biaction induces a relation (a correspondence) between the set of $\mathfrak{G}$-orbits and the set of $\mathfrak{H}$-orbits. In the same way as groupoids uniformize the quotient spaces, the biactions uniformize correspondences between the quotient spaces. Therefore, we consider biactions as correspondences between groupoids.

Correspondences can be naturally inverted in the following way. If $\mathfrak{G} \curvearrowright \mathcal{M} \curvearrowleft \mathfrak{H}$ is a biaction, then we denote by $\mathcal{M}^{-1}$ the biaction consisting of a set $\mathcal{M}^{-1}$ which is in a homeomorphic bijection $a \mapsto a^{-1} : \mathcal{M} \longrightarrow \mathcal{M}^{-1}$ with $\mathcal{M}$, and actions $\mathfrak{H} \curvearrowright \mathcal{M}^{-1} \curvearrowleft \mathfrak{G}$ given by

$$h \cdot a^{-1} \cdot g = (g^{-1} \cdot a \cdot h^{-1})^{-1}.$$

Let $\mathfrak{G} \curvearrowright \mathcal{M} \curvearrowleft \mathfrak{H}$ be a biaction. For every open set $U \subset \mathfrak{G}^{(0)}$ the set $P_{\mathfrak{G}}^{-1}(U) \subset \mathcal{M}$ is open and $\mathfrak{H}$-invariant. Consequently, the map $P_{\mathfrak{G}}$ induces a continuous map from the quotient space $\mathcal{M}/\mathfrak{H}$ to $\mathfrak{G}^{(0)}$. By the same argument, the map $P_{\mathfrak{H}}$ induces a continuous map from $\mathfrak{G}\backslash\mathcal{M}$ to $\mathfrak{H}^{(0)}$.

Let us show how to compose correspondences. Suppose $\mathfrak{G}_2 \curvearrowright \mathcal{M}_1 \curvearrowleft \mathfrak{G}_1$ and $\mathfrak{G}_3 \curvearrowright \mathcal{M}_2 \curvearrowleft \mathfrak{G}_2$ are biactions. Let they be defined over the maps $P_1 : \mathcal{M}_1 \longrightarrow \mathfrak{G}_1^{(0)}$, $P_2' : \mathcal{M}_1 \longrightarrow \mathfrak{G}_2^{(0)}$, $P_2'' : \mathcal{M}_2 \longrightarrow \mathfrak{G}_2^{(0)}$, $P_3 : \mathcal{M}_2 \longrightarrow \mathfrak{G}_3^{(0)}$.

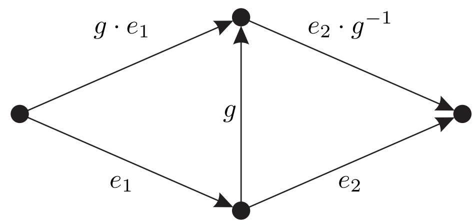

Figure 3.7. Composing biactions

Since we consider a point $e_1 \in \mathcal{M}_1$ as an arrow from $P_1(e_1)$ to $P_2'(e_1)$, and a point $e_2 \in \mathcal{M}_2$ as an arrow from $P_2''(e_2)$ to $P_3(e_2)$, the set of composable pairs of arrows is

$$\mathcal{M}_2 \,{}_{P_2''}\!\times_{P_2'} \mathcal{M}_1 = \{(e_2, e_1) \,:\, P_2''(e_2) = P_2'(e_1)\} \subset \mathcal{M}_2 \times \mathcal{M}_1.$$

We have to identify the composable pairs (e_2, e_1) producing the same correspondence from $P_1(e_1)$ to $P_3(e_2)$. These identifications are produced by the actions of $\mathfrak{G}_2$: the pair (e_2, e_1) is equivalent to $(e_2 \cdot g^{-1}, g \cdot e_1)$, see Figure 3.7.

We will denote the quotient $\mathfrak{G}_2 \backslash \left(\mathcal{M}_2 \,{}_{P_2''}\!\times_{P_2'} \mathcal{M}_1\right)$ by $\mathcal{M}_2 \otimes_{\mathfrak{G}_2} \mathcal{M}_1$ or just $\mathcal{M}_2 \otimes \mathcal{M}_1$. The groupoids $\mathfrak{G}_1$ and $\mathfrak{G}_3$ act naturally on $\mathcal{M}_2 \otimes \mathcal{M}_1$ since their actions commute with $\mathfrak{G}_2$. We get a biaction $\mathfrak{G}_3 \curvearrowright \mathcal{M}_2 \otimes \mathcal{M}_1 \curvearrowleft \mathfrak{G}_1$.

The role of the identical correspondence $\mathfrak{G} \longrightarrow \mathfrak{G}$ is played by the groupoid $\mathfrak{G}$ itself with the natural left and right actions, i.e., the natural biaction $\mathfrak{G} \curvearrowright \mathfrak{G} \curvearrowleft \mathfrak{G}$. Note that both actions of $\mathfrak{G}$ on itself are proper and free.

The next statement follows directly from the definitions.

Proposition 3.2.13. *Let $\mathfrak{G}_1 \curvearrowright \mathcal{M} \curvearrowleft \mathfrak{G}_2$ be a biaction. Then the map $x \otimes g \mapsto x \cdot g$ induces an isomorphism of the biaction $\mathcal{M} \otimes \mathfrak{G}_2$ with $\mathcal{M}$. Similarly, the biaction $\mathfrak{G}_1 \otimes \mathcal{M}$ is naturally isomorphic to $\mathcal{M}$.*

The process of taking a quotient used in the definition of $\mathcal{M}_1 \otimes_{\mathfrak{G}_2} \mathcal{M}_2$ maybe not well behaved topologically. However, if one of the actions of $\mathfrak{G}_2$ on $\mathcal{M}_1$ or $\mathcal{M}_2$ is free and proper, it is not problematic. Namely, we have the following.

Proposition 3.2.14. *Suppose that the action of $\mathfrak{G}_2$ on $\mathcal{M}_1$ is free and proper. Then the action of $\mathfrak{G}_2$ on $\mathcal{M}_2 \,{}_{P_2''}\!\times_{P_2'} \mathcal{M}_1$ over the map $(e_2, e_1) \mapsto P_2'(e_1)$ given by*

$$g \cdot (e_2, e_1) = (e_2 \cdot g^{-1}, g \cdot e_1),$$

is free and proper.

Proof. The action of $\mathfrak{G}_2$ on $\mathcal{M}_2 \,{}_{P_2''}\!\times_{P_2'} \mathcal{M}_1$ is free, since the action of $\mathfrak{G}_2$ on $\mathcal{M}_1$ is free. Let us show that it is proper. For every compact set

$K \subset \mathcal{M}_2\ {}_{P_2''}\times_{P_2'}\ \mathcal{M}_1$ the projection K_1 of K to $\mathcal{M}_1$ is compact as a continuous image of a compact set. The element g is uniquely determined by a pair $(e_1, g \cdot e_1)$, i.e., by its action on the projection K_1, since the action of $\mathfrak{G}_2$ on $\mathcal{M}_1$ is free. It follows that the set of elements $g \in \mathfrak{G}_1$ such that $g \cdot K \cap K \neq \varnothing$ is compact, i.e., that the action of $\mathfrak{G}_1$ on $\mathcal{M}_{2\,P_2''} \times_{P_2'} \mathcal{M}_1$ is proper. □

We see that biactions can be naturally composed if one of the middle actions is proper and free.

Maps are particular cases of correspondences, namely they are such that every point of one set is in a correspondence with exactly one point of the other set. This condition (that the correspondence is a map from the space of $\mathfrak{G}$-orbits to the space of $\mathfrak{H}$-orbits) can be naturally formulated in terms of biactions in the following way.

Definition 3.2.15. We say that a biaction $\mathfrak{G} \curvearrowright \mathcal{M} \curvearrowleft \mathfrak{H}$ is a *univalent correspondence from* $\mathfrak{G}$ *to* $\mathfrak{H}$ (or a *morphism*), which we will denote $\mathfrak{G} \curvearrowright \underrightarrow{\mathcal{M}} \curvearrowleft \mathfrak{H}$, if the action of $\mathfrak{H}$ on $\mathcal{M}$ is free and proper, and the map $P_{\mathfrak{G}}/\mathfrak{H} : \mathcal{M}/\mathfrak{H} \longrightarrow \mathfrak{G}^{(0)}$ induced by $P_{\mathfrak{G}}$ is a homeomorphism.

The choice of the sides in the conditions of Definition 3.2.15 is arbitrary. If $\mathfrak{G} \curvearrowright \mathcal{M}$ is free and proper, and the map $\mathfrak{G}\backslash\mathcal{M} \longrightarrow \mathfrak{H}^{(0)}$ induced by $P_{\mathfrak{H}}$ is a homeomorphism, then we write $\mathfrak{G} \curvearrowright \underleftarrow{\mathcal{M}} \curvearrowleft \mathfrak{H}$. In particular, we identify the morphism $\mathfrak{G} \curvearrowright \underrightarrow{\mathcal{M}} \curvearrowleft \mathfrak{H}$ with the morphism $\mathfrak{H} \curvearrowright \underleftarrow{\mathcal{M}^{-1}} \curvearrowleft \mathfrak{G}$.

It is not always convenient to check that $P_{\mathfrak{G}}/\mathfrak{H}$ is a homeomorphism. Instead, one can use the following reformulation of the definition of a morphism.

Proposition 3.2.16. *A biaction* $\mathfrak{G} \curvearrowright \mathcal{M} \curvearrowleft \mathfrak{H}$ *is a univalent correspondence* $\mathfrak{G} \curvearrowright \underrightarrow{\mathcal{M}} \curvearrowleft \mathfrak{H}$ *if and only if the following conditions are satisfied:*

(1) *The action of* $\mathfrak{H}$ *on* $\mathcal{M}$ *is free and proper.*

(2) *The action of* $\mathfrak{H}$ *is transitive on the set* $P_{\mathfrak{G}}^{-1}(x)$ *for every* $x \in \mathfrak{G}^{(0)}$.

(3) *The map* $P_{\mathfrak{G}} : \mathcal{M} \longrightarrow \mathfrak{G}^{(0)}$ *is onto and open.*

Proof. The conditions that $P_{\mathfrak{G}}/\mathfrak{H} : \mathcal{M}/\mathfrak{H} \longrightarrow \mathfrak{G}^{(0)}$ is one-to-one and onto are equivalent to the conditions that action of $\mathfrak{H}$ is transitive on the sets $P_{\mathfrak{G}}^{-1}(x)$ and that $P_{\mathfrak{G}}$ is onto, respectively. Since $P_{\mathfrak{G}}$ is continuous, the map $P_{\mathfrak{G}}/\mathfrak{H}$ is continuous, i.e., preimages by $P_{\mathfrak{G}}/\mathfrak{H}$ of open subsets of $\mathcal{M}/\mathfrak{H}$ are open. The map $P_{\mathfrak{G}}/\mathfrak{H}$ is a homeomorphism if and only if it is a bijection and open. It is open if and only if $P_{\mathfrak{G}}$ is open, since preimages in $\mathcal{M}$ of open subsets of $\mathcal{M}/\mathfrak{H}$ are precisely open $\mathfrak{H}$-invariant subsets of $\mathcal{M}$. □

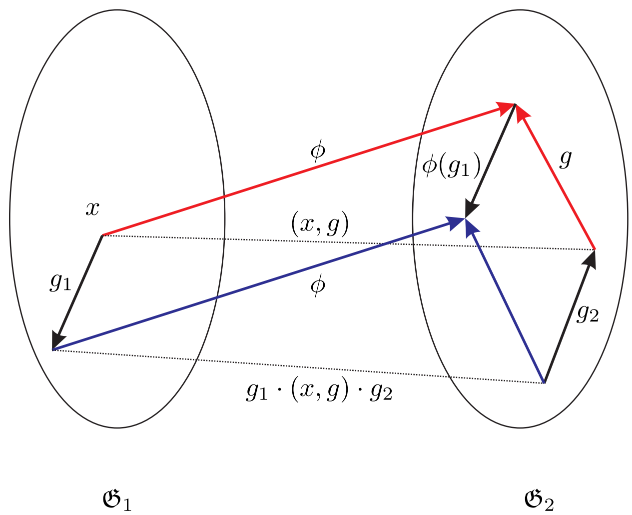

Figure 3.8. Functor as a biaction

Proposition 3.2.17. *Suppose that $\mathfrak{G}_1 \curvearrowright \underrightarrow{\mathcal{M}_1} \curvearrowleft \mathfrak{G}_2$ and $\mathfrak{G}_2 \curvearrowright \underrightarrow{\mathcal{M}_2} \curvearrowleft \mathfrak{G}_3$ are morphisms. Then the biaction $\mathfrak{G}_1 \curvearrowright \underrightarrow{\mathcal{M}_1 \otimes \mathcal{M}_2} \curvearrowleft \mathfrak{G}_3$ is a morphism.*

Proof. .... □

Suppose that $\mathfrak{G} \curvearrowright \mathcal{M} \curvearrowleft \mathfrak{H}$ is a biaction, and let $A \subset \mathfrak{G}^{(0)}$. Denote by $\mathcal{M}|_A$ the subspace $P_{\mathfrak{G}}^{-1}(A)$. Then the action $\mathfrak{G} \curvearrowright \mathcal{M}$ naturally restricts to an action $\mathfrak{G}|_A \curvearrowright \mathcal{M}|_A$. The set $\mathcal{M}|_A$ is $\mathfrak{H}$-invariant, hence we get a biaction $\mathfrak{G}|_A \curvearrowright \mathcal{M}|_A \curvearrowleft \mathfrak{H}$. If the action $\mathcal{M} \curvearrowleft \mathfrak{H}$ is free and proper, then its restriction to any $\mathfrak{H}$-invariant subset is also free and proper. Consequently, if $\mathfrak{G} \curvearrowright \underrightarrow{\mathcal{M}} \curvearrowleft \mathfrak{H}$ is a morphism, then for any subset $\mathfrak{G}|_A$ we get a morphism $\mathfrak{G}|_A \curvearrowright \underrightarrow{\mathcal{M}|_A} \curvearrowleft \mathfrak{H}$. We call it the *restriction* of the morphism $\mathfrak{G} \curvearrowright \underrightarrow{\mathcal{M}} \curvearrowleft \mathfrak{H}$ to A.

3.2.3. Functors as biactions. Let $\phi : \mathfrak{G}_1 \longrightarrow \mathfrak{G}_2$ be a functor. Set $\mathcal{M} = \{(x, g) \in \mathfrak{G}_1^{(0)} \times \mathfrak{G}_2 \ : \ \phi(x) = \mathbf{r}(g)\}$ with the biaction with the anchors $P_{\mathfrak{G}_1}(x, g) = x$, $P_{\mathfrak{G}_2}(x, g) = \mathbf{s}(g)$, and given by

$$g_1 \cdot (x, g) \cdot g_2 = (\mathbf{r}(g_1), \phi(g_1) g g_2),$$

see Figure 3.8.

The action $\mathcal{M} \curvearrowleft \mathfrak{G}_2$ is free and proper, since the right action of $\mathfrak{G}_2$ on itself is free and proper. If $P_{\mathfrak{G}_1}(x_1, g_1) = P_{\mathfrak{G}_1}(x_2, g_2)$, then $x_1 = x_2$, hence $\mathbf{r}(g_1) = \phi(x_1) = \phi(x_2) = \mathbf{r}(g_2)$. Then we have $(x_1, g_1) \cdot g_1^{-1} g_2 = (x_2, g_2)$, which shows that the action of $\mathfrak{G}_2$ is transitive on the fibers of $P_{\mathfrak{G}_1}$. The map $P_{\mathfrak{G}_1}$ is obviously onto. For every open neighborhood U of (x, g) there exists open neighborhoods U_g and U_x of g and x in $\mathfrak{G}_2$ and $\mathfrak{G}_1^{(0)}$, respectively, such that $\{(x', g') : x' \in U_x, g' \in U_g, \mathbf{r}(g') = \phi(x')\} \subset U$. The set $U'_x = U_x \cap \phi^{-1}(\mathbf{r}(U_g))$ is an open neighborhood of x, since ϕ is continuous and $\mathbf{r}$ is open. Then $P_{\mathfrak{G}_1}(\{(x', g') : x' \in U'_x, g' \in U_g, \mathbf{r}(g') = \phi(x')\})$ contains U'_x. We have shown that every point of $P_{\mathfrak{G}_1}(U)$ is internal, i.e., that $P_{\mathfrak{G}_1}$ is an open map.

We see that the defined biaction is a morphism. We say that $\mathfrak{G}_1 \curvearrowright \underrightarrow{\mathcal{M}} \curvearrowleft \mathfrak{G}_2$ is the morphism *defined* by the functor ϕ.

If $\mathfrak{G}_1 \curvearrowright \underrightarrow{\mathcal{M}} \curvearrowleft \mathfrak{G}_2$ is a morphism defined by a functor $\phi : \mathfrak{G}_1 \longrightarrow \mathfrak{G}_2$, as above, then the map $x \mapsto (\phi(x), x)$ is a section (i.e., a right inverse) of the map $P_{\mathfrak{G}_1} : \mathcal{M} \longrightarrow \mathfrak{G}_1^{(0)}$, since $P_{\mathfrak{G}_1}(\phi(x), x) = x$. Conversely, existence of such a section is equivalent to the condition that the biaction is defined by a functor.

Proposition 3.2.18. *Suppose that $\mathfrak{G}_1 \curvearrowright \underrightarrow{\mathcal{M}} \curvearrowleft \mathfrak{G}_2$ is a morphism, and suppose that there exists a section $\psi : \mathfrak{G}_1^{(0)} \longrightarrow \mathcal{M}$ of the map $P_{\mathfrak{G}_1} : \mathcal{M} \longrightarrow \mathfrak{G}_1^{(0)}$. Then for every $g \in \mathfrak{G}_1$ the point $g \cdot \psi(\mathbf{s}(g))$ can be written in a unique way as $\psi(\mathbf{r}(g)) \cdot h$ for some $h \in \mathfrak{G}_2$.*

The map $\phi : g \mapsto h$ is a continuous functor and the morphism $\mathfrak{G}_1 \curvearrowright \underrightarrow{\mathcal{M}} \curvearrowleft \mathfrak{G}_2$ is isomorphic to the morphism defined by the functor ϕ.

Proof. ....

□

We leave the following as an exercise.

Proposition 3.2.19. *Let $\phi_1 : \mathfrak{G}_1 \longrightarrow \mathfrak{G}_2$ and $\phi_2 : \mathfrak{G}_1 \longrightarrow \mathfrak{G}_2$ be functors, and suppose that there exists a continuous map $\delta : \mathfrak{G}_2^{(0)} \longrightarrow \mathfrak{G}_2$ such that*

$$\phi_2(g) = \delta(\mathbf{r}(\phi_1(g))) \cdot \phi_1(g) \cdot \delta(\mathbf{s}(\phi_1(g)))^{-1}.$$

Then the functors ϕ_1 and ϕ_2 define isomorphic morphisms.

Note that every continuous map $\delta : \mathfrak{G}^{(0)} \longrightarrow \mathfrak{G}$ defines an *inner automorphism* of $\mathfrak{G}$ equal to the map

$$g \mapsto \delta(\mathbf{r}(g)) \cdot g \cdot \delta(\mathbf{s}(g))^{-1}.$$

The above proposition tells us that the isomorphism class of the biaction defined by a functor depends only on the functor modulo inner automorphisms of $\mathfrak{G}_2$.

Example 3.2.20. Suppose that groupoids $\mathfrak{G}_1 = G_1$ and $\mathfrak{G}_2 = G_2$ are discrete groups, and let $G_1 \curvearrowright \underrightarrow{\mathcal{M}} \curvearrowleft G_2$ be a morphism. The maps $P_{G_1} : \mathcal{M} \longrightarrow G_1^{(0)}$ and $P_{G_2} : \mathcal{M} \longrightarrow G_2^{(0)}$ are constant, since $G_1^{(0)}$ and $G_2^{(0)}$ are singletons (units of the groups). It follows that any map $\psi : G_1^{(0)} \longrightarrow \mathcal{M} : 1_{G_1} \mapsto e$ is a section of P_{G_1}.

After we choose the point $e = \psi(1_{G_1})$, we transform $G_1 \curvearrowright \underrightarrow{\mathcal{M}} \curvearrowleft G_2$ into a homomorphism of groups $\phi : G_1 \longrightarrow G_2$ uniquely defined by the condition

$$g \cdot e = e \cdot \phi(g),$$

since the action of G_2 on $\mathcal{M}$ is free and transitive (properness follows from freeness for discrete groups).

We see that the notion of a groupoid morphisms between groups is equivalent to the notion of a group homomorphism (except for the fact that different choices of $e \in \mathcal{M}$ produce homomorphism that differ from each other by an inner automorphism of $\mathfrak{G}_2$).

3.2.4. Examples of morphisms.

3.2.4.1. *Morphism from a groupoid to a space.* Suppose that $\mathfrak{H}$ is a trivial groupoid, i.e., a topological space $\mathcal{X}$. Then every $\mathfrak{H}$-action $\mathfrak{H} \curvearrowright \mathcal{M}$ is trivial, i.e., $h \cdot x = x$ for all $h \in \mathfrak{G} = \mathcal{X}, x \in \mathcal{M}$ such that $P_{\mathfrak{G}}(x) = \mathbf{s}(h) = h$.

Every $\mathfrak{H}$-action is free, since $\mathfrak{H}$ contains only units. It is also proper, since the map $P_{\mathfrak{H}}$ is continuous, so maps compacts sets to compact sets.

Proposition 3.2.21. *Let $\mathfrak{G} \curvearrowright \underrightarrow{\mathcal{M}} \curvearrowleft \mathcal{X}$ be a morphism to a trivial groupoid. Then there exists a continuous map $f : \mathfrak{G}^{(0)} \longrightarrow \mathcal{X}$ constant on $\mathfrak{G}$-orbits such that $\mathcal{M}$ is isomorphic to the biaction $\mathfrak{G} \curvearrowright \mathfrak{G}^{(0)} \curvearrowleft \mathcal{X}$, where $\mathfrak{G} \curvearrowright \mathfrak{G}^{(0)}$ is the natural action and $\mathfrak{G}^{(0)} \curvearrowleft \mathcal{X}$ is the trivial action over the map $f : \mathfrak{G}^{(0)} \longrightarrow \mathcal{X}$.*

In particular, a morphism between two trivial groupoids is just a continuous map.

Proof. As the action of $\mathcal{X}$ is trivial and has to be transitive on the fibers of $P_{\mathfrak{G}}$, the fibers of $P_{\mathfrak{G}}$ are singletons. The map $P_{\mathfrak{G}} : \mathcal{M} \longrightarrow \mathfrak{G}^{(0)}$ is therefore bijective and open, hence homeomorphism. Moreover the groupoid $(\mathfrak{G} \ltimes \mathcal{M})/\mathcal{X} = \mathfrak{G} \ltimes \mathcal{M}$ is naturally isomorphic to $\mathfrak{G}$ (see Exercise 7). It follows that $\mathfrak{G} \curvearrowright \mathcal{M}$ is the natural action of $\mathfrak{G}$ on its unit space, and the morphism $\mathfrak{G} \curvearrowright \underrightarrow{\mathcal{M}} \curvearrowleft \mathcal{X}$ is given by a continuous map $P_{\mathfrak{H}} : \mathfrak{G}^{(0)} \longrightarrow \mathcal{X}$. $\square$

3.2.4.2. *Morphisms from a space to a groupoid.* Suppose that $\mathfrak{H}$ is a principal proper groupoid, and denote by $\mathcal{X}$ the associated space $\mathfrak{H}^{(0)}/\mathfrak{H}$ of $\mathfrak{H}$-orbits. We have then a natural morphism $\mathcal{X} \curvearrowright \underrightarrow{\mathfrak{H}^{(0)}} \curvearrowleft \mathfrak{H}$, where $\mathcal{X} \curvearrowright \mathfrak{H}$ is defined by the anchor mapping a unit to its orbit, and the action $\mathfrak{H}^{(0)} \curvearrowleft \mathfrak{H}$ is natural. Note that the inverse correspondence is also a morphism. (It is an *equivalence of groupoids*, see 3.2.5.)

This gives us a method of constructing morphism from topological spaces to groupoids. If $\mathcal{X}$ is a topological space, then we can take a principal proper groupoid $\mathfrak{H}$ with the space of orbits homeomorphic to $\mathcal{X}$, and then compose the morphism $\mathcal{X} \curvearrowright \underrightarrow{\mathfrak{H}^{(0)}} \curvearrowleft \mathfrak{H}$ with a morphism $\mathfrak{H}\underrightarrow{\mathcal{M}} \curvearrowleft \mathfrak{G}$ (for example, defined by a functor $\mathfrak{H} \longrightarrow \mathfrak{G}$).

In fact, this described method is general. Suppose that $\mathcal{X} \curvearrowright \underrightarrow{\mathcal{M}} \curvearrowleft \mathfrak{G}$ is a morphism from a trivial groupoid. Then $\mathfrak{G}$ acts on the fibers of $P_{\mathcal{X}}$, and its action on $\mathcal{M}$ is free and proper. The map $P_{\mathcal{X}}$ induces a homeomorphism from the space of orbits $\mathcal{M}/\mathfrak{G}$ to $\mathcal{X}$, by definition of a morphism. The natural projection $(x, g) \mapsto g$ is a functor from the action groupoid $\mathcal{M} \rtimes \mathfrak{G}$ to $\mathfrak{G}$. The space of orbits of the action $\mathcal{M} \curvearrowleft \mathfrak{G}$ is naturally identified with the space of orbits of the principal proper groupoid $\mathcal{M} \curvearrowleft \mathfrak{G}$. It is not hard to check that the biaction $\mathcal{X} \curvearrowright \underrightarrow{\mathcal{M}} \curvearrowleft \mathfrak{G}$ is isomorphic to the composition of the natural morphism from $\mathcal{X}$ to the space of orbits $\mathcal{M}/\mathfrak{G}$ of $\mathcal{M} \rtimes \mathfrak{G}$ with the morphism defined by the projection functor $\mathcal{M} \rtimes \mathfrak{G} \longrightarrow \mathfrak{G}$.

A particular case $\mathcal{X} = [0, 1]$, i.e., of *paths*, will be studied in more detail in 3.3.

3.2.4.3. *The natural morphism from the unit space to the groupoid.* Let $\mathfrak{G}$ be a groupoid. Then we have a natural morphism $\mathfrak{G}^{(0)} \curvearrowright \underrightarrow{\mathfrak{G}} \curvearrowleft \mathfrak{G}$, where the action $\mathfrak{G}^{(0)} \curvearrowright \underrightarrow{\mathfrak{G}}$ is defined by the anchor $\mathbf{s} : \mathfrak{G} \longrightarrow \mathfrak{G}^{(0)}$, and $\mathfrak{G} \curvearrowleft \mathfrak{G}$ is the natural right action of $\mathfrak{G}$ on itself. The latter is free and proper, see Example 3.2.7. The constructed morphism from $\mathfrak{G}^{(0)}$ to $\mathfrak{G}$ can be seen as the natural "quotient map" from $\mathfrak{G}^{(0)}$ to the non-commutative space of $\mathfrak{G}$-orbits.

3.2.4.4. *Fundamental group of a space.* Let $\mathcal{X}$ be a path connected and semilocally simply connected space. Let $\widetilde{\mathcal{X}}$ be its universal covering, and let $\pi_1(\mathcal{X})$ be the fundamental group. Let $\mathfrak{G}_1 = \mathcal{X}$ be the trivial groupoid, and let $\mathfrak{G}_2 = \pi_1(\mathcal{X})$ be the group seen as a groupoid (with one unit). Let $P_1 : \widetilde{\mathcal{X}} \longrightarrow \mathcal{X}$ be the universal covering map, let $P_2 : \widetilde{\mathcal{X}} \longrightarrow \mathfrak{G}_2^{(0)}$ be the only possible map: the constant identity element of the fundamental group. Take the trivial action of $\mathfrak{G}_1$ and the natural action of the fundamental group on the universal covering for $\mathfrak{G}_2$. Both actions are free and proper. We get a biaction $\mathcal{X} \curvearrowright \widetilde{\mathcal{X}} \curvearrowleft \pi_1(\mathcal{X})$.

The action of $\pi_1(\mathcal{X})$ on $\widetilde{\mathcal{X}}$ is transitive on the fibers of P_1, and the map $P_1 : \widetilde{\mathcal{X}} \longrightarrow \mathcal{X}$ is onto and open. We see that there is a natural groupoid morphism $\mathcal{X} \curvearrowright \underrightarrow{\widetilde{\mathcal{X}}} \curvearrowleft \pi_1(\mathcal{X})$ from a space to its fundamental group.

3.2.5. Equivalence of groupoids.

Definition 3.2.22. An *equivalence* between groupoids $\mathfrak{G}$ and $\mathfrak{H}$ is a biaction $\mathfrak{G} \curvearrowright \mathcal{E} \curvearrowleft \mathfrak{H}$ such that both $\mathcal{E}$ and $\mathcal{E}^{-1}$ are morphisms.

Let us spell out Definition 3.2.22, using Proposition 3.2.16. An equivalence is a biaction $\mathfrak{G} \curvearrowright \mathcal{E} \curvearrowleft \mathfrak{H}$ satisfying the following properties.

(1) The actions $\mathfrak{G} \curvearrowright \mathcal{E}$ and $\mathcal{E} \curvearrowleft \mathfrak{H}$ are free and proper.

(2) The action of $\mathfrak{G}$ is transitive on the fibers of $P_{\mathfrak{H}}$, and the action of $\mathfrak{H}$ is transitive on the fibers of $P_{\mathfrak{G}}$.

(3) The maps $P_{\mathfrak{G}}$ and $P_{\mathfrak{H}}$ are onto and open. Moreover, they define homeomorphisms $\mathcal{M}/\mathfrak{H} \longrightarrow \mathfrak{G}^{(0)}$ and $\mathfrak{G}\backslash\mathcal{M} \longrightarrow \mathfrak{H}^{(0)}$.

The above definition of equivalence coincides with the one given in [**MRW87**].

Proposition 3.2.23. *Composition of equivalences is an equivalence. If $\mathcal{E}$ is an equivalence, then the compositions $\mathcal{E} \otimes \mathcal{E}^{-1}$ and $\mathcal{E}^{-1} \otimes \mathcal{E}$ are isomorphic to the identical morphisms.*

Proof. The first statement follows directly from Proposition 3.2.17.

Let $\mathfrak{G} \curvearrowright \mathcal{E} \curvearrowleft \mathfrak{H}$ be an equivalence. Consider an element of $\mathcal{E} \otimes \mathcal{E}^{-1}$ represented by the pair (e_1, e_2^{-1}) of $\mathcal{E}_{P_{\mathfrak{H}}} \times_{P_{\mathfrak{H}}} \mathcal{E}^{-1}$. Then $P_{\mathfrak{H}}(e_1) = P_{\mathfrak{H}}(e_2)$, hence there exists a unique $g \in \mathfrak{G}$ such that $g \cdot e_1 = e_2$. Note that g depends only on the corresponding element of $\mathcal{E} \otimes \mathcal{E}^{-1}$, since the action of $\mathfrak{G}$ on $\mathcal{E}_{P_{\mathfrak{H}}} \times_{P_{\mathfrak{H}}} \mathcal{E}^{-1}$ is by the transformations $(e_1, e_2^{-1}) \mapsto (e_1 \cdot h, (e_2 \cdot h)^{-1})$, so that $g \cdot e_1 = e_2$ is equivalent to $g \cdot (e_1 \cdot h) = (e_2 \cdot h)$. Let us denote the defined element g by $\phi(e_1 \otimes e_2^{-1})$. We want to prove that the map $\phi : \mathcal{E} \otimes \mathcal{E}^{-1}$ is an isomorphism. It is easy to check that ϕ agrees with the left and right actions of $\mathfrak{G}$ on $\mathcal{E} \otimes \mathcal{E}^{-1}$, so it is enough to show that ϕ is a homeomorphism. Let $g \in \mathfrak{G}$. Since $P_{\mathfrak{G}}$ is onto, there exists $e \in \mathcal{E}$ such that $P_{\mathfrak{G}}(e) = \mathbf{s}(g)$. We have $P_{\mathfrak{H}}(e) = P_{\mathfrak{H}}(g \cdot e)$, so that $(g \cdot e) \otimes e^{-1}$ is an element of $\mathcal{E} \otimes \mathcal{E}^{-1}$. Suppose that $e' \in \mathcal{E}$ is another element such that $P_{\mathfrak{G}}(e') = \mathbf{s}(g)$. Then there exists $h \in \mathfrak{H}$ such that $e' = e \cdot h$, and we have $(g \cdot e') \otimes (e')^{-1} = g \cdot e \cdot h \otimes (e \cdot h)^{-1} = g \cdot e \otimes e^{-1}$. We have shown that $g \cdot e \otimes e^{-1}$ does not depend on e. The map $g \mapsto g \cdot e \otimes e^{-1}$ is inverse to the map ϕ. Let us show that both maps are continuous.... $\square$

It is sometimes more convenient to define equivalence of groupoids using functors, so we need to understand when a functor defines an equivalence of groupoids.

Proposition 3.2.24. *Let $\phi : \mathfrak{G} \longrightarrow \mathfrak{H}$ be a functor. It defines an equivalence if and only if the following conditions are satisfied.*

(1) *If $x, y \in \mathfrak{G}^{(0)}$ and $h \in \mathfrak{H}$ are such that $\phi(x) = \mathbf{s}(h)$ and $\phi(y) = \mathbf{r}(h)$ then there exists a unique $g \in \mathfrak{G}$ such that $\phi(g) = h$.*

(2) *The map $\phi : \mathfrak{G}^{(0)} \longrightarrow \mathfrak{H}^{(0)}$ is open and onto.*

Proof. Recall that the morphism $\mathfrak{G} \curvearrowright \underset{\rightarrow}{\mathcal{F}} \curvearrowleft \mathfrak{H}$ defined by ϕ is the space $\mathcal{F} = \{(x, h) \in \mathfrak{G}^{(0)} \times \mathfrak{H} \ : \ \mathbf{r}(g) = \phi(x)\}$ with the biaction

$$g_1 \cdot (x, h) \cdot h_1 = (\mathbf{r}(g_1), \phi(g_1)hh_1).$$

We have to understand when $\mathcal{F}^{-1}$ is a morphism, i.e., when the action of $\mathfrak{G}$ on $\mathcal{F}$ is free, proper, and transitive on the $P_{\mathfrak{H}}$-fibers, and when the map $P_{\mathfrak{G}}$ is onto and open.

Freeness of the action means that $g \cdot (x, h) = (x, h)$ is equivalent to $g \in \mathfrak{G}^{(0)}$. The equality $g \cdot (x, h) = (x, h)$ is equivalent to $\mathbf{s}(g) = x$, $\mathbf{r}(g) = x$, $\phi(g)h = h$, i.e., that g belongs to the isotropy group of x and that $\phi(g)$ is a unit. It follows that the freeness of $\mathfrak{G} \curvearrowright \mathcal{F}$ is equivalent to the condition that ϕ is injective on isotropy groups.

Transitivity on the $P_{\mathfrak{H}}$-fibers means that whenever (x_1, h_1) and $(x_2, h_2) \in \mathcal{F}$ are such that $\mathbf{s}(h_1) = \mathbf{s}(h_2)$, then there exists $g \in \mathfrak{G}$ such that $g \cdot (x_1, h_1) = (x_2, h_2)$. Recall that we have $\mathbf{r}(h_i) = \phi(x_i)$, $\mathbf{s}(g) = x_1$, and $g \cdot (x_1, h_1) = (\mathbf{r}(g), \phi(g)h_1)$. We need $\mathbf{r}(g) = x_2$ and $h_2 = \phi(g)h_1$, i.e., $\phi(g) = h_2h_1^{-1}$. It follows that $\mathfrak{G}$ is transitive on the $P_{\mathfrak{H}}$-fibers if and only if for any $h \in \mathfrak{H}$ and $x_1, x_2 \in \mathfrak{G}^{(0)}$ such that $\phi(x_1) = \mathbf{s}(h)$ and $\phi(x_2) = \mathbf{r}(h)$, there exists $g \in \mathfrak{G}$ such that $\mathbf{s}(g) = x_1$, $\mathbf{r}(g) = x_2$ and $\phi(g) = h$. Uniqueness of the element g is equivalent to injectivity of ϕ on the isotropy groups.

We see that freeness and transitivity of of the $\mathfrak{G}$-action is equivalent to the condition that the map $\Phi : g \mapsto (\mathbf{s}(g), \mathbf{r}(g), \phi(g))$ from $\mathfrak{G}$ to the space $\{(x, y, h) \in \mathfrak{G}^{(0)} \times \mathfrak{G}^{(0)} \times \mathfrak{H} \ : \ \phi(x) = \mathbf{s}(h), \phi(y) = \mathbf{r}(h)\}$ is bijective. Since the spaces are locally compact and locally Hausdorff, the inverse is also continuous if it exists. So, freeness and transitivity is equivalent to the condition that this map is a homeomorphism.....

The action of $\mathfrak{G}$ on $\mathcal{F}$ is proper if and only if the map

$$(g, x, h) \mapsto (\mathbf{r}(g), \phi(g)h, x, h)$$

from $\{(g, x, h) \ : \ \mathbf{s}(g) = x, \mathbf{r}(h) = \phi(x)\}$ to $\mathcal{F} \times \mathcal{F}$ is proper. Let $K \subset \mathcal{F}$ be compact. The preimage of $K \times K$ under the map is the set of triples (g, x, h) such that $(x, h) \in K$ and $(\mathbf{r}(g), \phi(g)h) \in K$. We have $\phi(g)h, h \in K$ if and only if $h \in K$ and $\phi(g) \in KK^{-1}$. Since the ... Using the fact that Φ is a homeomorphism... □

3.2.6. Equivalence as an ambient groupoid. Let $\mathfrak{G} \curvearrowright \mathcal{E} \curvearrowleft \mathfrak{H}$ be an equivalence.

Suppose that $e_1, e_2 \in \mathcal{E}$ are such that $P_{\mathfrak{G}}(e_1) = P_{\mathfrak{G}}(e_2)$. Then there exists a unique element $h \in \mathfrak{H}$ such that $e_1 \cdot h = e_2$. Similarly, if $P_{\mathfrak{H}}(e_1) = P_{\mathfrak{H}}(e_2)$, then there exists a unique element $g \in \mathfrak{G}$ such that $g \cdot e_1 = e_2$.

Let us rename $P_{\mathfrak{G}} : \mathcal{E} \longrightarrow \mathfrak{G}, P_{\mathfrak{H}} : \mathcal{E} \longrightarrow \mathfrak{H}$ by $\mathbf{r}, \mathbf{s}$, respectively, and define $\mathbf{s}(e^{-1}) = \mathbf{r}(e)$ and $\mathbf{r}(e^{-1}) = \mathbf{s}(e)$ for $e^{-1} \in \mathcal{E}^{-1}$. We also define $(e^{-1})^{-1} = e$. Let us denote the disjoint union $\mathfrak{G} \sqcup \mathcal{E} \sqcup \mathcal{E}^{-1} \sqcup \mathfrak{H}$ by $\mathfrak{G} \vee_{\mathcal{E}} \mathfrak{H}$. We have defined maps $\mathbf{s}, \mathbf{r} : \mathfrak{G} \vee_{\mathcal{E}} \mathfrak{H} \longrightarrow \mathfrak{G}^{(0)} \sqcup \mathfrak{H}^{(0)}$. Suppose that $h_1, h_2 \in \mathfrak{G} \vee_{\mathcal{E}} \mathfrak{H}$ are such that $\mathbf{s}(h_1) = \mathbf{r}(h_2)$. Define then the product $h_1 h_2$ in the following way:

(1) if $h_1, h_2 \in \mathfrak{G}$, or $h_1, h_2 \in \mathfrak{H}$, then $h_1 h_2$ is the usual product in $\mathfrak{G}$ or $\mathfrak{H}$;

(2) if $h_1 \in \mathfrak{G}$ and $h_2 \in \mathcal{E}$, or $h_1 \in \mathcal{E}$ and $h_2 \in \mathfrak{H}$, then $h_1 h_2$ is the result of the action of $\mathfrak{G}$ or $\mathfrak{H}$ on $\mathcal{E}$;

(3) if $h_1 \in \mathcal{E}^{-1}$ and $h_2 \in \mathfrak{G}$, or $h_1 \in \mathfrak{H}$ and $h_2 \in \mathcal{E}^{-1}$, then $h_1 h_2$ is the result of the action of $\mathfrak{G}$ or $\mathfrak{H}$ on $\mathcal{E}^{-1}$;

(4) if $h_1 \in \mathcal{E}$ and $h_2 \in \mathcal{E}^{-1}$, then $h_1 h_2$ is the unique element of $\mathfrak{G}$ such that $h_1 h_2 \cdot h_2^{-1} = h_1$;

(5) if $h_1 \in \mathcal{E}^{-1}$ and $h_2 \in \mathcal{E}$, then $h_1 h_2$ is the unique element $\mathfrak{H}$ such that $h_1^{-1} \cdot h_1 h_2 = h_2$.

Proposition 3.2.25. *The set $\mathfrak{G} \vee_{\mathcal{E}} \mathfrak{H}$ with the above defined multiplication is a topological groupoid with respect to the topology of the disjoint union of the topological spaces $\mathfrak{G} \sqcup \mathcal{E} \sqcup \mathcal{E}^{-1} \sqcup \mathfrak{H}$.*

Proof. We leave it to the reader to check that the above multiplication introduces a structure of a groupoid on $\mathfrak{G} \vee_{\mathcal{E}} \mathfrak{H}$. We only prove here that it is topological, i.e., that multiplication is continuous. Continuity of the operation of taking inverse is obvious. Continuity of multiplication at pairs $(h_1, h_2) \in (\mathfrak{G} \vee_{\mathcal{E}} \mathfrak{H})^{(2)}$ from the cases (1)–(2) follow from the continuity of multiplication in $\mathfrak{G}$ and $\mathfrak{H}$, continuity for the cases (3)–(6) follow from the continuity of the actions.

Let us prove the continuity at a pair $(h_1, h_2) \in \mathcal{E} \times \mathcal{E}^{-1}$ such that $\mathbf{s}(h_1) = \mathbf{r}(h_2)$, i.e., $P_2(h_1) = P_2(h_2^{-1})$. Consider the map $\mu : (g, h) \mapsto (g \cdot h, h^{-1})$ from $\mathfrak{G} \times_{P_2} \mathcal{E}$ to $\{(h_1, h_2) \in \mathcal{E} \times \mathcal{E}^{-1} : P_2(h_1) = P_2(h_2^{-1})\}$. It is continuous, by continuity of the $\mathfrak{G}$-action. It is invertible, by freeness of the action. The inverse map is given in terms of the groupoid $\mathfrak{G} \vee_{\mathcal{E}} \mathfrak{H}$ by $(h_1, h_2) \mapsto (h_1 h_2, h_2^{-1})$. By the definition of proper actions, the map μ is proper. It is known that a proper bijective continuous map between locally compact locally Hausdorff spaces is a homeomorphism. Consequently, the inverse map μ^{-1}, which is the multiplication in $\mathfrak{G} \vee_{\mathcal{E}} \mathfrak{H}$, restricted to $\mathcal{E} \times \mathcal{E}^{-1}$, is continuous. Continuity of the multiplication restricted to $\mathcal{E}^{-1} \times \mathcal{E}$ is proved in the same way, using properness of the $\mathfrak{H}$-action. □

Conversely, the notion of equivalence of groupoids can be defined in the following way. Recall that a subset A of a topological space $\mathcal{X}$ is said to be *locally closed* if it is equal to the intersection of an open and a closed subsets of $\mathcal{X}$.

Proposition 3.2.26. *Topological groupoids $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are equivalent if and only if there exists a topological groupoid $\mathfrak{H}$ and homomorphisms $\phi_1 : \mathfrak{G}_1 \longrightarrow \mathfrak{H}$ and $\phi_2 : \mathfrak{G}_2 \longrightarrow \mathfrak{H}$ such that the following conditions hold.*

(1) *The maps $\phi_i : \mathfrak{G}_i \longrightarrow \phi_i(\mathfrak{G}_i)$ are isomorphisms of topological groupoids.*

(2) *The groupoids $\phi_i(\mathfrak{G}_i)$ are equal to the restrictions of $\mathfrak{H}$ to $\phi_i(\mathfrak{G}_i^{(0)})$.*

(3) *The sets $\phi_i(\mathfrak{G}_i^{(0)})$ are locally closed $\mathfrak{H}$-transversals.*

In other words, two topological groupoids are equivalent if and only if they can be realized as restrictions of one groupoid to locally closed transversals.

Proof. We have already proved the "only if" part in by constructing the groupoid $\mathfrak{G} \vee_{\mathcal{E}} \mathfrak{H}$, which satisfies all the conditions of the proposition, see....

Conversely, suppose that $\mathfrak{H}$ is a topological groupoid, and let $\mathfrak{G}_1, \mathfrak{G}_2$ be restrictions of $\mathfrak{H}$ to locally closed $\mathfrak{H}$-transversals $\mathfrak{G}_1^{(0)}$ and $\mathfrak{G}_2^{(0)}$. Let $\mathcal{E} = \{h \in \mathfrak{H} : \mathbf{s}(h) \in \mathfrak{G}_2^{(0)}, \mathbf{r}(h) \in \mathfrak{G}_1^{(0)}\}$. Then $\mathfrak{G}_1$ and $\mathfrak{G}_2$ act on $\mathcal{E}$ from the left and the right, respectively, by multiplication. The actions commute, are obviously free, and satisfy condition (2) of Definition 3.2.22. Properness of the actions follows from Lemma 3.1.16. The unit spaces $\mathfrak{G}_1^{(0)}$ and $\mathfrak{G}_2^{(0)}$ are locally compact, since they are locally closed subsets of a locally compact locally Hausdorff space. Local compactness of $\mathfrak{G}_1$ and $\mathfrak{G}_2$ follows from the local compactness of $\mathfrak{H}$ and the fact that $\mathfrak{G}_1^{(0)}$ and $\mathfrak{G}_2^{(0)}$ are locally closed. □

Example 3.2.27. Let $f \circlearrowleft \mathcal{X}$ be a homeomorphism, and consider the corresponding action $(\mathbb{Z}, \mathcal{X})$. Let $\mathfrak{G} = \mathcal{X} \rtimes_f \mathbb{Z}$ be the groupoid of the action. Let $\mathcal{Y} \subset \mathcal{X}$ be an open set such that for every point $x \in \mathcal{X}$ there exist positive integers n_1 and n_2 such that $f^{n_1}(x) \in \mathcal{Y}$ and $f^{-n_2}(x) \in \mathcal{Y}$. For example, if $(\mathbb{Z}, \mathcal{X})$ is minimal, then $\mathcal{Y}$ can be any non-empty open subset of $\mathcal{X}$.

Example 3.2.28. IF $f \circlearrowleft \mathcal{X}$ is a minimal homeomorphism of a Cantor set. Then for every non-empty clopen subset $\mathcal{Y} \subset \mathcal{X}$ the groupoid of the $\mathbb{Z}$-action generated by f is equivalent to the groupoid of the $\mathbb{Z}$-action generated by the first return map $f_{\mathcal{Y}} \circlearrowleft \mathcal{Y}$ induced by f on $\mathcal{Y}$. It follows that if two minimal homeomorphisms are Kakutani equivalent, then the associated groupoids of actions are equivalent. See 1.3.6 for a discussion of Kakutani equivalence of minimal homeomorphisms and its relation to Vershik-Bratteli diagrams.

Example 3.2.29. The groupoids associated with the stable and unstable equivalence relations for a Ruelle-Smale system $f \circlearrowleft \mathcal{X}$ defined in Example 3.1.15 are equivalent to the groupoids $\mathfrak{S}$ and $\mathfrak{U}$ defined in Example 3.1.4.3

(both to the groupoid with the space of units equal to the disjoint union of all leaves and to its restriction to the union of plaques of a cover by rectangles). Namely, the groupoid from Example 3.1.4.3 with the space of units equal to the union of plaques of a finite cover by open rectangles is a restriction to a locally closed transversal both of the groupoid from Example 3.1.15 and of the groupoid with the space of units equal to the disjoint union of the leaves.

3.2.7. Equivalence for étale groupoids.

Proposition 3.2.30. *Let $\mathcal{M} \curvearrowleft \mathfrak{G}$ be an action of an étale groupoid. Then the groupoid $\mathcal{M} \rtimes \mathfrak{G}$ is étale.*

Proof. The source and range maps of the action groupoid $\mathcal{M} \rtimes \mathfrak{G}$ are

$$\mathbf{s}(x, g) = (x \cdot g, \mathbf{s}(g)), \qquad \mathbf{r}(x, g) = (x, \mathbf{r}(g)).$$

Suppose that $U \ni g$ is an open $\mathfrak{G}$-bisection. Consider the set $U' = \mathcal{E} \rtimes \mathfrak{G} \cap \mathcal{E} \times U$. It is an open neighborhood of (x, g) for every $x \in P^{-1}(\mathbf{r}(g))$, and the restrictions of the source and range maps to U' have continuous inverses:

$$(x, P(x)) \mapsto (x \cdot (\mathbf{s}^{-1}(P(x)))^{-1}, \mathbf{s}^{-1}(P(x))),$$

and

$$(x, P(x)) \mapsto (x, \mathbf{r}^{-1}(P(x))),$$

respectively, where $\mathbf{s}^{-1}$ and $\mathbf{r}^{-1}$ are the inverses of $\mathbf{s} : U \longrightarrow \mathbf{s}(U)$ and $\mathbf{r} : U \longrightarrow \mathbf{r}(U)$. ☐

Proposition 3.2.31. *Suppose that $\mathfrak{G}$ is étale, and the action $\mathcal{M} \curvearrowleft \mathfrak{G}$ is free and proper. Then the quotient map $\mathcal{M} \longrightarrow \mathcal{M}/\mathfrak{G}$ is étale, i.e., is a local homeomorphism.*

Proof. Take a point $x \in \mathcal{M}$, and let N be a compact Hausdorff neighborhood of x. Since the action $\mathcal{M} \curvearrowleft \mathfrak{G}$ is proper, the set A of elements $(y, g) \in \mathcal{M} \rtimes \mathfrak{G}$ such that $y, y \cdot g \in N$ is compact. For every non-unit element (y, g) of A we have $y \neq y \cdot g$, as the action $\mathcal{M} \curvearrowleft \mathfrak{G}$ is free. The groupoid $\mathcal{M} \curvearrowleft \mathfrak{G}$ is étale, so there exists a bisection $U_{(y,g)} \subset \mathcal{M} \rtimes \mathfrak{G}$ such that $\mathbf{s}(U_{(y,g)})$ and $\mathbf{r}(U_{(y,g)})$ have disjoint closures. If $(y, g) = (y, P_{\mathfrak{G}}(y))$ is a unit, we set $U_{(y,g)} = N$ (here we, as usual, identify a unit $(y, P_{\mathfrak{G}}(y))$ with the point y). Then there exists a finite cover of A by the sets of the form $U_{(y,g)}$. It follows that for a sufficiently small compact neighborhood $N' \subset N$ of x the set of elements $(y, g) \in \mathcal{M} \curvearrowleft \mathfrak{G}$ such that $y, y \cdot g \in N'$ consists of units only. It follows that the quotient map $\mathcal{M} \longrightarrow \mathcal{M}/\mathfrak{G}$ is injective on N', hence is a homeomorphism from N' onto its image. ☐

Proposition 3.2.32. *Let $\mathfrak{G}$ and $\mathfrak{H}$ be étale groupoids. Suppose that they are equivalent, and let $\mathfrak{G} \vee \mathfrak{H}$ be the corresponding groupoid described in Proposition 3.2.25. Then $\mathfrak{G} \vee \mathfrak{H}$ is étale.*

Proof. It is enough to prove that if $\mathfrak{G} \curvearrowright \underrightarrow{\mathcal{E}} \curvearrowleft \mathfrak{H}$ is an equivalence between étale groupoids $\mathfrak{G}$ and $\mathfrak{H}$, then the anchors $P_{\mathfrak{G}} : \mathcal{E} \longrightarrow \mathfrak{G}^{(0)}$ and $P_{\mathfrak{H}} : \mathcal{E} \longrightarrow \mathfrak{H}^{(0)}$ are local homeomorphisms.

Let U be a compact Hausdorff neighborhood of a point $x \in \mathcal{E}$. Since the $\mathfrak{H}$-action is proper, the set $C = \{(y, h) \in \mathcal{E} \rtimes \mathfrak{H} \;:\; y \in U, y \cdot h \in U\}$ is compact. The set of units of $\mathcal{E} \rtimes \mathfrak{H}$ is open by Proposition 3.2.30, hence the set $C' = C \setminus (\mathcal{E} \rtimes \mathfrak{H})^{(0)}$ is compact. Let $(y, h) \in C'$ be an arbitrary point. If $x \neq y$ and $x \neq y \cdot h$, then there exists an open relatively compact neighborhood $V_{(y,h)}$ of (y, h) such that $x \notin \mathbf{s}(\overline{V}_{(y,h)}) \cup \mathbf{r}(\overline{V}_{(y,h)})$. If $x = y$, then there exists an open relatively compact neighborhood $V_{(y,h)}$ of (y, h) such that $x \in \mathbf{s}(V_{(y,h)})$ and $x \notin \mathbf{r}(\overline{V}_{(y,h)})$. If $x = y \cdot h$, then we can find an open relatively compact neighborhood $V_{(y,h)}$ of (y, h) such that $x \notin \mathbf{s}(\overline{V}_{(y,h)})$ and $x \in \mathbf{r}(V_{(y,h)})$. Note that we can not have $y = y \cdot h$, since the action groupoid $\mathcal{E} \rtimes \mathfrak{H}$ is principal, and (y, h) is not a unit.

There exists a finite set $\mathcal{A}$ of sets of the form $V_{(y,h)}$, $(y, h) \in C'$ covering C'. Then the set

$$U' = U \setminus \left(\bigcup_{V \in \mathcal{A}, x \notin \mathbf{s}(V)} \mathbf{s}(\overline{V}) \cup \bigcup_{V \in \mathcal{A}, x \notin \mathbf{r}(V)} \mathbf{r}(\overline{V}) \right)$$

is a neighborhood of x such that there does not exist $(y, h) \in \mathcal{E} \rtimes \mathfrak{H} \setminus (\mathcal{E} \rtimes \mathfrak{H})^{(0)}$ such that $\{y, y \cdot h\} \in U'$. Consider the restriction of the map $P_{\mathfrak{G}}$ to U'. If $P_{\mathfrak{G}}(y_1) = P_{\mathfrak{G}}(y_2)$ for $y_1, y_2 \in U'$ such that $y_1 \neq y_2$, then there exists $h \in \mathfrak{H}$ such that $y_2 = y_1 \cdot h$. Then $(y_1, h) \in U$, hence there exists $V \in \mathcal{A}$ such that $(y_1, h) \in V$. Since $y_1, y_2 \in U'$, we have $x \in \mathbf{s}(V)$ and $x \in \mathbf{r}(V)$, which is not allowed. We get a contradiction showing that $P_{\mathfrak{G}} : U' \longrightarrow \mathfrak{G}^{(0)}$ is injective. It follows that $P_{\mathfrak{G}}$ is a local homeomorphism. The same arguments show that $P_{\mathfrak{H}}$ is a local homeomorphism. □

A convenient method of replacing an étale groupoid by an equivalent one is *pull-back* and *localization* defined in the following way.

Definition 3.2.33. Let $\mathfrak{G}$ be an étale groupoid, and let $F : \mathcal{X} \longrightarrow \mathfrak{G}^{(0)}$ be a surjective local homeomorphism. Denote by $F^*(\mathfrak{G})$ groupoid equal as a topological space to $\{(x_1, g, x_2) \in \mathcal{X} \times \mathfrak{G} \times \mathcal{X} \;:\; F(x_1) = \mathbf{s}(g), F(x_2) = \mathbf{r}(g)\}$ with operations

$$\mathbf{s}(x_1, g, x_2) = x_1, \quad \mathbf{r}(x_1, g, x_2) = x_2$$

and

$$(x_2, g_2, x_3)(x_1, g_1, x_2) = (x_1, g_2 g_1, x_3).$$

We call $F^*(\mathfrak{G})$ the *pull-back* of $\mathfrak{G}$ by the map F.

We leave it to the reader as an exercise to show that $F^*(\mathfrak{G})$ is an étale groupoid.

Proposition 3.2.34. *Let $F^*(\mathfrak{G})$ be as in Definition 3.2.33. Then $F^*(\mathfrak{G})$ is equivalent to $\mathfrak{G}$.*

Proof. ..... □

Example 3.2.35. Let $\mathcal{X}$ be a path connected and semilocally simply connected space, and let $F : \widetilde{\mathcal{X}} \longrightarrow \mathcal{X}$ be the universal covering. Then the pull-back of the trivial groupoid $\mathcal{X}$ by F is the groupoid $\widetilde{\mathcal{X}} \rtimes \pi_1(\mathcal{X})$ of the action of the fundamental group on the universal covering.

As an important particular case of the pull-back is the *localization* on an open cover, which we will use almost every time when dealing with equivalences of étale groupoids.

Definition 3.2.36. Let $\mathfrak{G}$ be an étale groupoid, and let $\mathcal{U}$ be an open cover of $\mathfrak{G}^{(0)}$. Let $\mathcal{X}$ be the disjoint union of the elements of $\mathcal{U}$, and let $F : \mathcal{X} \longrightarrow \mathfrak{G}^{(0)}$ be the natural map equal to the identity on each element of $\mathcal{U}$. Then F is a local homeomorphism, by definition. The pull-back $F^*(\mathfrak{G})$ is called the *localization* of $\mathfrak{G}$ to $\mathcal{U}$. We will denote it by $\mathfrak{G}|_{\mathcal{U}}$.

If $\mathcal{U} = \{U_i\}_{i \in I}$ is an open cover of $\mathfrak{G}^{(0)}$, then we will represent the disjoint union $\mathcal{X}$ of the elements of $\mathcal{U}$ as the space $\bigsqcup_{i \in I} U_i \times \{i\}$, so that a point of $\mathcal{X}$ is a pair (x, i) for $x \in U_i$. The elements of the localization are represented by triples (i_1, g, i_2), where $\mathbf{s}(g) \in U_{i_1}$, $\mathbf{r}(g) \in U_{i_2}$, so that the groupoid operations in the localization are

$$\mathbf{s}(i_1, g, i_2) = (\mathbf{s}(g), i_1), \quad \mathbf{r}(i_1, g, i_2) = (\mathbf{r}(g), i_2),$$

and

$$(i_2, g_2, i_3)(i_1, g_1, i_2) = (i_1, g_2 g_1, i_3).$$

Unlike in many other books on groupoids, we did not include the condition that the space of units is Hausdorff into the definition of a topological groupoid. The main reason for this was to include actions on non-Hausdorff spaces, which are essential for the general definition of morphisms between groupoids. On the other hand, since we can always consider the localization of an étale groupoid to a cover by open Hausdorff sets, every étale groupoid is equivalent to an étale groupoid with a Hausdorff space of units, so adding the condition of Hausdorffness to the definition of étale groupoids does not make it more general from the point of view of equivalence classes of groupoids.

Proposition 3.2.37. *Two étale groupoids $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are equivalent if and only if there exist open covers $\mathcal{U}_i$ of $\mathfrak{G}_i^{(0)}$ such that $\mathfrak{G}_1|_{\mathcal{U}_1}$ and $\mathfrak{G}_2|_{\mathcal{U}_2}$ are isomorphic.*

Proof. ..... □

Localizations are also convenient ways of defining morphisms from étale groupoids.

Definition 3.2.38. We say that two biactions $\mathfrak{G}_1 \curvearrowright \mathcal{M}_1 \curvearrowleft \mathfrak{H}_1$ and $\mathfrak{G}_2 \curvearrowright \mathcal{M}_2 \curvearrowleft \mathfrak{H}_2$ are *equivalent* if there exist equivalences $\mathfrak{G}_1 \curvearrowright \mathcal{E} \curvearrowleft \mathfrak{G}_2$ and $\mathfrak{H}_1 \curvearrowright \mathcal{F} \curvearrowleft \mathfrak{H}_2$ such that $\mathcal{M}_1$ is isomorphic to $\mathcal{E} \otimes \mathcal{M}_2 \otimes \mathcal{F}^{-1}$.

Proposition 3.2.39. *Every morphism $\mathfrak{G} \curvearrowright \underrightarrow{\mathcal{M}} \curvearrowleft \mathfrak{H}$, where $\mathfrak{G}$ is étale is equivalent to the morphism defined by a functor $f : \mathfrak{G}|_{\mathcal{U}} \longrightarrow \mathfrak{H}$, where $\mathcal{U}$ is an open cover of $\mathfrak{G}^{(0)}$.*

Proof. .... □

Proposition 3.2.40. *Let $G_1 \curvearrowright \mathcal{X}_1$ and $G_2 \curvearrowright \mathcal{X}_2$ be continuous actions of locally compact topological groups. Then the groupoids $G_1 \ltimes \mathcal{X}_1$ and $G_2 \ltimes \mathcal{X}_2$ are equivalent if and only if there exists a space $\mathcal{E}$ and a free proper action of $G_1 \times G_2 \curvearrowright \mathcal{E}$ such that the actions $G_1 \curvearrowright \mathcal{E}/G_2$ and $G_2 \curvearrowright \mathcal{E}/G_1$ are topologically conjugate to the actions $G_1 \curvearrowright \mathcal{X}_1$ and $G_2 \curvearrowright \mathcal{X}_2$, respectively.*

Proof. .... □

3.2.8. Flow equivalence for $\mathbb{Z}$-actions. Let $f \circlearrowleft \mathcal{X}$ be a homeomorphism of a locally compact Hausdorff space. Consider its *mapping torus* $\mathcal{T}_f$ defined as the quotient of $[0,1] \times \mathcal{X}$ by the equivalence relation $(1,x) \sim (0, f(x))$. The associated *flow* $\mathbb{R} \curvearrowright \mathcal{T}_f$ is given by $T_a(t,x) = \big(t + a - \lfloor a \rfloor, f^{\lfloor a \rfloor}\big)$, i.e., it is just the natural flow along the line $\ldots [0,1] \times \{f^{-1}(x)\} \cup [0,1] \times \{x\} \cup [0,1] \times \{f(x)\} \ldots$.

Since $\{0\} \times \mathcal{X}$ is a closed transversal of the flow $\mathbb{R} \curvearrowright \mathcal{T}_f$, the groupoids $\mathbb{Z} \ltimes_f \mathcal{X}$ and $\mathbb{R} \ltimes \mathcal{T}_f$ are equivalent.

Proposition 3.2.41. *Two actions $\mathbb{Z} \curvearrowright_{f_i} \mathcal{X}_i$ are groupoid equivalent if and only if the associated mapping torus flows $\mathbb{R} \curvearrowright_{f_i} \mathcal{T}_{f_i}$ are topologically conjugate.*

Proof. Since every $\mathbb{Z}$-action is groupoid equivalent to the associated $\mathbb{R}$-flow, it is enough to show that equivalent $\mathbb{Z}$-actions have topologically conjugate $\mathbb{R}$-flows on the mapping tori.

Let $\mathbb{R} \curvearrowright \mathcal{S}$ be an arbitrary flow, and let $x_1, x_2, \ldots, x_n \in \mathcal{S}$ be points belonging to one orbit. Then there exists $x \in \mathcal{S}$ and real numbers $t_1, t_2, \ldots, t_n$ such that $x_i = T_{t_i}(x)$. Let $p_1, p_2, \ldots, p_n$ be real numbers such that $p_1 + p_2 + \cdots + p_n = 1$. Consider the point $T_{p_1 t_1 + p_2 t_2 + \cdots + p_n t_n}(x)$. If we change x to another point $T_t(x)$ in the orbit, then we will replace $t_1, t_2, \ldots, t_n$ by $t_1 - t, t_2 - t, \ldots, t_n - t$, and get $T_{p_1(t_1-t)+p_2(t_2-t)+\cdots+p_n(t_n-t)}(T_t(x)) = T_{p_1 t_1 + p_2 t_2 + \cdots + p_n t_n - t}(T_t(x)) = T_{p_1 t_1 + p_2 t_2 + \cdots + p_n t_n}(x)$. It follows that $T_{p_1 t_1 + p_2 t_2 + \cdots + p_n t_n}(x)$ does not depend on the choice of x. We will denote it $p_1 x_1 + p_2 x_2 + \cdots + p_n x_n$.

Let $\mathbb{Z} \ltimes_{f_1} \mathcal{X}_1 \curvearrowright \mathcal{M} \curvearrowleft \mathbb{Z} \ltimes_{f_2} \mathcal{X}_2$ be an equivalence of groupoids. The projections $P_i = P_{\mathbb{Z} \ltimes_{f_i} \mathcal{X}_i}$ of $\mathcal{M}$ onto the unit spaces $\mathcal{X}_i$ are local homeomorphisms, by Proposition 3.2.32. It follows that there is a collection $\mathcal{U}$ of open subsets of $\mathcal{M}$ such that for every $U \in \mathcal{U}$ the maps $P_{i,U} : U \longrightarrow P_i(U)$ are homeomorphisms and the sets $\{P_i(U) \ : \ U \in \mathcal{U}\}$ are open covers of $\mathcal{X}_i$. Let $\phi_U : U \longrightarrow [0,1]$ be a partition of unity subordiate to $\{P_1(U) \ : \ U \in \mathcal{U}\}$. Let $x \in \mathcal{X}_i$, and consider all the sets $U \in \mathcal{U}$ such that $x \in P_1(U)$. Let $x_U = P_{2,U} \circ P_{1,U}^{-1}(x)$. All points x_U belong to one orbit of the flow $\mathbb{R} \curvearrowright \mathcal{T}_{f_2}$, i.e., are of the form $T_{t_U}(x_0)$ for some $x_0 \in \mathcal{T}_{f_2}$ and $t_U \in \mathbb{R}$. Consider the average $t_x = \sum_{U \in \mathcal{U}} \phi_U(x) t_U$, and let $\Phi(x) = T_{t_x}(x_0)$. In other words, we take the average of the points x_U along the $\mathbb{R}$-orbit using the weights $\phi_U(x)$. The point $\Phi(x)$ does not depend on the choice of x_0 and the map $\Phi : \mathcal{X}_1 \longrightarrow \mathcal{T}_{f_2}$ is continuous. It remains to show that the map $\mathcal{T}_{f_1} \longrightarrow \mathcal{T}_{f_2}$ given by $(t, x) \mapsto T_t(\Phi(x))$ is a homeomorphism, where T_t is the action $\mathbb{R} \curvearrowright_{f_2} \mathcal{T}_{f_2}$... $\square$

See an application of Proposition 3.2.41 for Kakutani equivalence of minimal homeomorphisms in Exercise 14.

3.3. Fundamental groups

3.3.1. $\mathfrak{G}$-paths and the fundamental group.

Definition 3.3.1. Let $\mathfrak{G}$ be a groupoid. A $\mathfrak{G}$-*path* is a morphism $[0,1] \curvearrowright \underset{\rightarrow}{\mathcal{F}} \curvearrowleft \mathfrak{G}$ together with a choice of points $x_0, x_1 \in \mathfrak{G}^{(0)}$ that are $\mathcal{M}$-related with $0, 1 \in [0,1]$, respectively. We say that x_0 and x_1 are the *beginning* and the *end* of the path. Two $\mathfrak{G}$-paths are *isomorphic* if their endpoints coincide and the corresponding morphisms are isomorphic.

We will assume, unless explicitly stated otherwise, that $\mathfrak{G}$ is étale. Every morphism $[0,1] \curvearrowright \underset{\rightarrow}{\mathcal{F}} \curvearrowleft \mathfrak{G}$ is equivalent to a morphism defined by a functor $F : [0,1]|_{\mathcal{U}} \longrightarrow \mathfrak{G}$, where $[0,1]|_{\mathcal{U}}$ is the localization onto a finite open cover (see Proposition 3.2.39). It follows that every $\mathfrak{G}$-path can be described by following data.

(1) A partition $t_0 = 0 < t_1 < t_2 < \ldots < t_n = 1$ of the interval $[0,1]$.

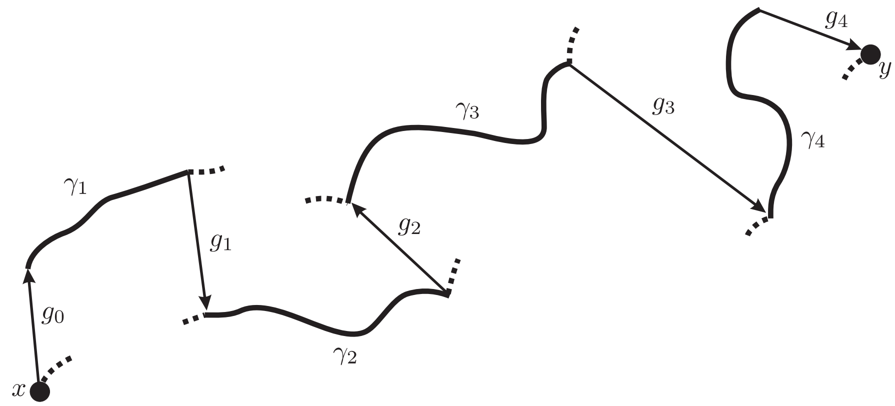

Figure 3.9. A $\mathfrak{G}$-path

(2) Continuous maps $\gamma_i : [t_i, t_{i+1}] \longrightarrow \mathfrak{G}^{(0)}$ for every $i = 0, 1, \ldots, n-1$.

(3) Elements $g_i \in \mathfrak{G}_i$ such that $\mathbf{s}(g_i)$ is the end of γ_i and the beginning of γ_{i+1}.

The beginning of this path is $\mathbf{s}(g_0)$ and its end is $\mathbf{r}(g_n)$. The path is defined by the functor from the localization onto the cover $\{[0, t_1 + \epsilon), (t_1 - \epsilon, t_2 + \epsilon), \ldots, (t_{n-1} - \epsilon, t_n]\}$, where ϵ is a small positive number. We will encode the above data by the sequence $(g_0, \gamma_1, g_1, \gamma_2, \ldots, \gamma_n, g_n)$. See Figure 3.9 for a schematic description of a path $(g_0, \gamma_1, g_1, \ldots, \gamma_4, g_4)$ connecting a point x to a point y. We get the same notion of a $\mathfrak{G}$-path as in [**BH99**]...

The conditions for two functors to define isomorphic morphisms of groupoids are given in Proposition 3.2.19. It implies that two sequences represent isomorphic $\mathfrak{G}$-paths if and only if they can be obtained from each other using a sequence of the following operations and their inverses applied to a path $(g_0, \gamma_1, \ldots, \gamma_n, g_n)$.

(1) *Subdivision:* Add a new point $t \in (t_{i-1}, t_i)$ and replace γ_i by the sequence $\gamma_i|_{[t_{i-1},t]}, \gamma_i(t), \gamma_i|_{[t,t_i]}$, where $\gamma_i(t)$ is seen as a unit element of $\mathfrak{G}$.

(2) $\mathfrak{G}$*-action:* For each $i = 1, \ldots, n$, choose a continuous function $h_i : [t_{i-1}, t_i] \longrightarrow \mathfrak{G}$ such that $\mathbf{s}(h_i(t)) = \gamma_i(t)$ for all $t \in [t_{i-1}, t_i]$, and replace γ_i by $\mathbf{r} \circ h_i$, replace g_i by $h_{i+1}(t_i) g_i h_i(t_i)^{-1}$ for all $i = 1, \ldots, n-1$, replace g_0 by $h_1(t_0) g_0$, and g_n by $g_n h_n(t_n)^{-1}$. See Figure 3.10.

Definition 3.3.2. The $\mathfrak{G}$-paths $[0,1] \curvearrowright \underrightarrow{\mathcal{F}_1} \curvearrowleft \mathfrak{G}$ and $[0,1] \curvearrowright \underrightarrow{\mathcal{F}_2} \curvearrowleft \mathfrak{G}$ are *homotopic* if their beginnings and ends coincide and there exists a morphism $[0,1]^2 \curvearrowright \underrightarrow{\mathcal{M}} \curvearrowleft \mathfrak{G}$ such that restriction of $\mathcal{M}$ to $[0,1] \times \{0\}$ is isomorphic to $\mathcal{F}_1$, restriction of $\mathcal{M}$ to $[0,1] \times \{1\}$ is isomorphic to $\mathcal{F}_2$, and the restrictions

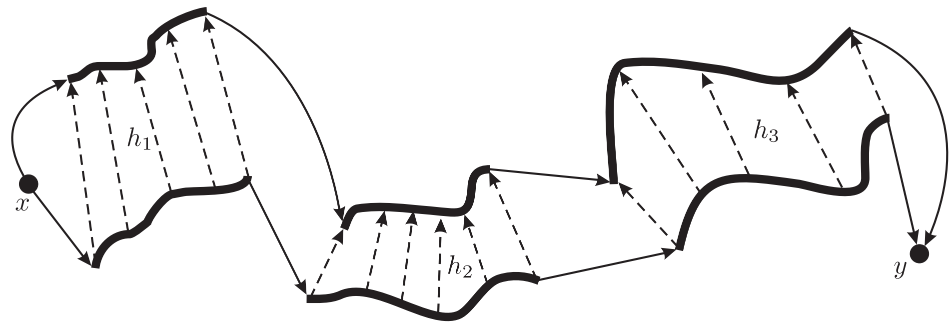

Figure 3.10. Isomorphism of $\mathfrak{G}$-paths

to $\{0\} \times [0,1]$ and $\{1\} \times [0,1]$ do not depend on the second coordinate (i.e., are isomorphic to compositions of the projection $\{x\} \times [0,1] \longrightarrow \{x\}$ with a morphism from $\{x\}$).

Again, as for $\mathfrak{G}$-paths, we can make the definition more concrete in the étale case by using functors from localizations. We get the following description.

Proposition 3.3.3. *Two paths $\alpha = (g_0, \alpha_1, \dots, \alpha_n, g_n)$ and $\beta = (h_0, \beta_1, \dots, \beta_m, h_m)$ are homotopic if and only if they can be obtained from each other by a sequence of the following operations.*

(1) *Subdivision and $\mathfrak{G}$-action, as in the description of isomorphism of paths.*
(2) Elementary homotopies*: a family of paths $\gamma^s = (g_0^s, \gamma_1^s, \dots, \gamma_n^s, g_n^s)$, where $s \in [0,1]$ is a real parameter; the path γ^s is defined over a subdivision $0 = t_0^s < t_1^s < \dots < t_n^s = 1$; the values t_i^s, g_i^s, $\gamma_i^s(t)$ depend continuously on s, and the elements g_0^s and g_n^s do not depend on s. The elementary homotopy replaces γ^0 by γ^1.*

If α is a path from x to y and β is a path from y to z, then we can *concatenate* the paths to get a path from x to z. We concatenate paths in the same order as we compose functions and groupoid elements, so that the path from x to z is denoted $\beta\alpha$. If $\alpha = (g_0, \alpha_1, \dots, \alpha_n, g_n)$ and $\beta = (h_0, \beta_1, \dots, \beta_m, h_m)$, then

$$\beta\alpha = (g_0, \alpha_1, \dots, \alpha_n, h_0 g_n, \beta_1, \dots, \beta_m, h_m).$$

It is natural therefore, to write a path $(g_0, \gamma_1, g_1, \dots, g_{n-1}, \gamma_n, g_n)$ as the concatenation

$$g_n \gamma_n g_{n-1} \cdots g_1 \gamma_1 g_0.$$

The set of the homotopy classes of $\mathfrak{G}$-paths is a groupoid with respect to this concatenation operation. We call it the *fundamental groupoid* and denote it $\pi_1(\mathfrak{G})$. It is not a topological groupoid yet. The set of units of $\pi_1(\mathfrak{G})$ is naturally identified with the set of units $\mathfrak{G}^{(0)}$ of $\mathfrak{G}$. The groupoid $\mathfrak{G}$ is also naturally identified with a subgroupoid of $\pi_1(\mathfrak{G})$, since any element of $\mathfrak{G}$ can be seen as a $\mathfrak{G}$-path.

Definition 3.3.4. The *fundamental group* $\pi_1(\mathfrak{G}, x)$, for $x \in \mathfrak{G}^{(0)}$, is the isotropy group of x in the fundamental groupoid $\pi_1(\mathfrak{G})$, i.e., the group of homotopy classes of $\mathfrak{G}$-paths starting and ending in x.

It follows directly from the definitions that $\pi_1(\mathfrak{G}, x)$ depends only on the equivalence class of $\mathfrak{G}$. More precisely, if $\mathfrak{G}_1 \curvearrowright \mathcal{E} \curvearrowleft \mathfrak{G}_2$ is an equivalence, and the units $x_i \in \mathfrak{G}_i^{(0)}$ are $\mathcal{E}$-related, then the groups $\pi_1(\mathfrak{G}_1, x_1)$ and $\pi_1(\mathfrak{G}_2, x_2)$ are isomorphic. The isomorphism maps a loop $[0,1] \curvearrowright \underrightarrow{\mathcal{M}} \curvearrowleft \mathfrak{G}_1$ to the loop $[0,1] \curvearrowright \underrightarrow{\mathcal{M} \otimes \mathcal{E}} \curvearrowleft \mathfrak{G}_2$.

Definition 3.3.5. We say that a groupoid $\mathfrak{G}$ is *connected* if it can not be represented as a disjoint union of two non-empty open sub-groupoids. It is *path connected* if any two points of $\mathfrak{G}^{(0)}$ can be connected by a $\mathfrak{G}$-path, i.e., if its fundamental groupoid is transitive. A groupoid $\mathfrak{G}$ is *locally connected*, resp. *locally simply connected*, if $\mathfrak{G}^{(0)}$ is a locally connected, resp. locally simply connected, topological space.

It is easy to see that a path-connected groupoid is connected.

Suppose now that $\mathfrak{G}$ is locally simply connected. We can introduce then a natural topology on $\pi_1(\mathfrak{G})$ making it a topological groupoid. Let $\gamma = g_n\gamma_n \cdots \gamma_1 g_0$ be a $\mathfrak{G}$-path. Let G_0, G_n be open simply connected (as usual topological spaces) $\mathfrak{G}$-bisections containing g_0 and g_n. For every $g_0' \in G_0$ and $g_n' \in G_n$ consider a path γ_1' inside $\mathbf{r}(G_0)$ from $\mathbf{r}(g_0')$ to $\mathbf{r}(g_0)$ and a path γ_n' inside $\mathbf{s}(G_n)$ from $\mathbf{s}(g_n)$ to $\mathbf{s}(g_n')$. Then the homotopy classes of the paths γ_1' and γ_n' depend only on g_0' and g_n'. Consider the set of all paths of the form

$$g_n'\gamma_n'\gamma_n g_{n-1} \cdots g_2\gamma_2 g_1\gamma_1\gamma_1' g_0'.$$

We set it to be a neighborhood of element γ of $\pi_1(\mathfrak{G})$. This will define a topology on $\pi_1(\mathfrak{G})$. If $\mathfrak{G}$ is path connected, then the fundamental groupoid $\pi_1(\mathfrak{G})$ is equivalent to the fundamental group $\pi_1(\mathfrak{G}, x)$, since $\pi_1(\mathfrak{G})$ has only one orbit.

We have a canonical morphism $\mathfrak{G} \curvearrowright \underrightarrow{\mathcal{M}} \curvearrowleft \pi_1(\mathfrak{G})$, where $\mathcal{M}$ is equal to $\pi_1(\mathfrak{G})$ with the natural right action of $\pi_1(\mathfrak{G})$ on itself and the natural action of $\mathfrak{G}$ on $\pi_1(\mathfrak{G})$ (since $\mathfrak{G}$ is a sub-groupoid of $\pi_1(\mathfrak{G})$). Since $\pi_1(\mathfrak{G})$ is equivalent to the fundamental group, we also get a canonical morphism from $\mathfrak{G}$ to its fundamental group.

We finish this subsection with some examples.

Example 3.3.6. Let $G \curvearrowright \mathcal{X}$ be an action of a discrete group a topological space, and consider the action groupoid $G \ltimes \mathcal{X}$. Then every $G \ltimes \mathcal{X}$-path is equivalent to a path of the form $(g, x) \cdot \gamma$, where $g \in G$, γ is a path in $\mathcal{X}$, and x is the end of γ. Namely, we can use the group action to "collect" all the curves in a $G \ltimes \mathcal{X}$-path to one curve.

If $\mathcal{X}$ is simply connected, then the homotopy class of such a path depends only on g and the endpoints of γ. It follows that every element of the fundamental group $\pi_1(G \ltimes \mathcal{X}, t)$ can be represented by $(g, g^{-1}(t)) \cdot \gamma$ and is uniquely determined by $g \in G$. It is easy to see now that we have $\pi_1(G \ltimes \mathcal{X}, t) \cong G$ for every action on a simply connected space $\mathcal{X}$.

Example 3.3.7. Suppose that $\mathcal{X}$ is a path connected and semi-locally simply connected topological space, and let $G \curvearrowright \mathcal{X}$ be an action of a discrete group. Then $\pi_1(G \ltimes \mathcal{X}, t)$ is isomorphic to the group $\tilde{G}$ of all lifts of the homeomorphisms $g \in G$ to the universal covering $\tilde{\mathcal{X}}$ of $\mathcal{X}$, since the groupoid $G \ltimes \mathcal{X}$ is equivalent to the groupoid $\tilde{G} \ltimes \tilde{\mathcal{X}}$, see Proposition 3.2.34.

Example 3.3.8. Let $\mathfrak{G}$ be the groupoid of the $\mathbb{Z}$-action generated by the irrational rotation $x \mapsto x + \theta$ of the circle $\mathbb{R}/\mathbb{Z}$, then $\pi_1(\mathfrak{G}, t)$ is isomorphic to $\mathbb{Z}^2$, since the group of lifts of this $\mathbb{Z}$-action to the universal covering $\mathbb{R}$ is the group of the transformations of the form $x \mapsto x + a\theta + b$ for $a, b \in \mathbb{Z}$.

Example 3.3.9. Holonomy groupoid of a local product structure Let $\mathcal{X}$ be a space with a local product structure, and consider the corresponding holonomy groupoid ... Explain why the fundamental group of this groupoid is $\pi_1(\mathcal{X})$ in the locally simply connected case...

3.3.2. Universal covering and developability. Let $\mathfrak{G}$ be a path connected and locally simply connected étale groupoid. Fix a basepoint $t \in \mathfrak{G}^{(0)}$, and consider the subset $\mathcal{X}_t = \mathbf{s}^{-1}(t)$ of the fundamental groupoid $\pi_1(\mathfrak{G})$, i.e., the space of homotopy classes of $\mathfrak{G}$-paths starting in t. The groupoid $\pi_1(\mathfrak{G})$ and its sub-groupoid $\mathfrak{G}$ naturally act on $\mathcal{X}_t$ from the left over the anchor mapping a path to its endpoint.

The fundamental group $\pi_1(\mathfrak{G}, t)$ acts on $\mathcal{X}_t$ it from the right, and the corresponding actions commute, i.e., we have a natural bi-action $\mathfrak{G} \curvearrowright \mathcal{X}_t \curvearrowleft \pi_1(\mathfrak{G}, t)$.

Definition 3.3.10. The groupoid $\mathfrak{G}$ is called *developable* if the action $\mathfrak{G} \curvearrowright \mathcal{X}_t$ is free and proper. Then the space $\mathfrak{G}\backslash\mathcal{X}_t$ is called the *universal covering* of $\mathfrak{G}$.

Theorem 3.3.11. *If $\mathfrak{G}$ is developable, then it is equivalent to the groupoid of the action of $\pi_1(\mathfrak{G}, t)$ on the universal covering of $\mathfrak{G}\backslash\mathcal{X}_t$, and the universal covering is a simply connected topological space.*

Conversely, for any action $G \curvearrowright \mathcal{X}$ of a discrete group on a simply connected topological space the groupoid $G \ltimes \mathcal{X}$ is developable, and $G \curvearrowright \mathcal{X}$ is topologically conjugate with the action of the fundamental group of $G \ltimes \mathcal{X}$ on its universal covering.

Proof. .... $\square$

Examples..

3.4. Orbispaces and complexes of groups

3.4.1. Orbispaces. The approach of Section 3.2 to groupoids was interpreting them as representations of some “quotient spaces” of orbits. The morphisms between such quotient spaces are given by biactions. Usual topological spaces are represented by trivial groupoids. Any principal proper étale groupoid $\mathfrak{G}$ is equivalent to the trivial groupoid on the quotient space of orbits of $\mathfrak{G}$. Therefore, usual topological spaces are represented in this approach by principal proper groupoids.

The natural next step, not far from usual topological spaces will be proper étale groupoids. Their spaces of orbits are still Hausdorff (see Proposition 3.1.29), but points come with the associated (necessarily finite) isotropy groups, which are preserved under equivalence of groupoids. Therefore, informally, orbispaces are sometimes defined as spaces locally described as quotients of topological spaces by actions of finite groups.

Definition 3.4.1. An *orbispace* is defined by a proper étale groupoid. Every equivalent groupoid is called an *atlas* of the orbispace. The associated pseudogroup is called the *pseudogroup of changes of charts* of the atlas. The space of orbits of the atlas is called the *underlying space* of the orbispace.

Morphisms (or *maps*) between two orbispaces are given by a morphism in the sense of Definition 3.2.15. Note that we do not get a category of orbispaces, since composition of morphisms is associative only up to isomorphism of biactions. We get rather a *weak bicategory*, see...

There are several version of Definition 3.4.1 in the literature, with different degrees of generality. For instance, it is customary to require the groupoid to be a Hausdorff groupoid of germs.

An *orbifold* is a orbispace given by a proper groupoid of germs of a pseudogroup of local diffeomorphisms of open subsets of $\mathbb{R}^n$. Similarly, one can define other structures on an orbispace by requiring the associated pseudogroup of the groupoid to preserve some structure (e.g., affine, conformal, measure).

Orbispaces as local quotients by finite group actions... Define good open covers, and restrict the atlas onto it, so that we get a disjoint union of finite group actions and changes of charts between them... Write it as a proposition .....

3.4.2. Covering maps between orbispaces. Morphisms between orbispaces...

Definition using cocycles into the symmetric group... Show how coverings of non-singular spaces can be realized this way... Hint (without a proof) that a more natural definition is equivalent to the given...

3.4.3. Groupoid simplicial complexes.

Definition 3.4.2. A *groupoid simplicial complex* is an abstract (discrete) groupoid $\mathfrak{G}$ identified with the set of all simplices of a simplicial complex satisfying the following conditions.

(1) If $(g, h) \in \mathfrak{G}^{(2)}$, then the simplices g and h have equal dimensions and we can order the sets of vertices of the simplices $g = \{v_0, v_1, \ldots, v_d\}$ and $h = \{u_0, u_1, \ldots, u_d\}$ so that $gh = \{v_0u_0, v_1u_1, \ldots, v_du_d\}$.
(2) If $g \in \mathfrak{G}$ is isotropic (i.e., $\mathbf{s}(g) = \mathbf{r}(g)$), then all vertices of the simplex g are isotropic.

Example 3.4.3. Let $G \curvearrowright \Gamma$ be an action of a group on a simplicial complex such that if $g \in G$ leaves a simplex of Γ invariant, then it fixes it pointwise. Then the groupoid of the action $G \ltimes \Gamma$ consisting of pairs (g, Δ), where $g \in G$ and Δ is a simplex of Γ, is a groupoid simplicial complex in a natural way.

Example 3.4.4. We can consider a quotient of the groupoid given in the last example by identifying two pairs (g_1, Δ) and (g_2, Δ) if the actions of g_1 and g_2 on all simplices intersecting Δ coincide. This will be also a groupoid simplicial complex.

A *geometric realization* of a groupoid simplicial complex is its geometric realization as a simplicial complex in the usual way, seen as a topological groupoid. Note that it may be not locally compact. Etale...?

Let $\mathfrak{G}$ be a groupoid simplicial complex. If $g \in \mathfrak{G}$, and α is a sub-simplex of $\mathbf{s}(g)$, it follows from Definition 3.4.2 that there exists a unique sub-simplex $h \in \mathfrak{G}$ of g such that $\mathbf{s}(h) = \alpha$. We will denote it $g|_\alpha$. If β is a subsimplex of α, then we obviously have

$$g|_\beta = g|_\alpha|_\beta \tag{3.1}$$

and

$$(g|_\alpha)^{-1} = (g^{-1})|_{\mathbf{r}(g|_\alpha)}. \tag{3.2}$$

Choose one simplex (i.e., and element of $\mathfrak{G}^{(0)}$) in each $\mathfrak{G}$-orbit. Let $T \subset \Gamma^{(0)}$ be the set of chosen simplices. If α is an element of $\mathfrak{G}^{(0)}$, then we denote by $\overline{\alpha}$ the element of T in the same orbit as α. For every $\alpha \in \mathfrak{G}^{(0)}$, choose an element $t_\alpha \in \mathfrak{G}$ such that $\mathbf{s}(t_\alpha) = \alpha$ and $\mathbf{r}(t_\alpha) = \overline{\alpha}$. Denote by G_α is isotropy group of $\alpha \in \mathfrak{G}^{(0)}$ in $\mathfrak{G}$.

Suppose that β is a subsimplex of α (we write then $\alpha \supset \beta$). Denote then by $\psi_{\alpha,\overline{\beta}} : G_\alpha \longrightarrow G_{\overline{\beta}}$ the homomorphism given by

$$\psi_{\alpha,\overline{\beta}}(g) = t_\beta g|_\beta t_\beta^{-1}.$$

In other words, it is the restriction homomorphism $g \mapsto g|_\beta$ from G_α to G_β conjugated by the identification of G_β with $G_{\overline{\beta}}$ defined by t_β.

Suppose that $\alpha \supset \beta \supset \gamma$. Let $\gamma_1 = \mathbf{r}(t_\beta|_\gamma)$, see Figure... We have then three morphisms between isotropy groups: $\psi_{\alpha,\overline{\beta}} : G_\alpha \longrightarrow G_{\overline{\beta}}$, $\psi_{\overline{\beta},\overline{\gamma}} : G_{\overline{\beta}} \longrightarrow G_{\overline{\gamma}}$, and $\psi_{\alpha,\overline{\gamma}} : G_\alpha \longrightarrow G_{\overline{\gamma}}$. It follows from (3.1)–(3.2) that $x|_{\gamma_1} = (t_\beta t_\beta^{-1} x t_\beta t_\beta^{-1})|_{\gamma_1} = t_\beta|_\gamma t_\beta^{-1} x|_\beta t_\beta t_\gamma t_\beta|_\gamma^{-1}$. Therefore,

$$\begin{aligned}\psi_{\overline{\beta},\overline{\gamma}} \circ \psi_{\alpha,\overline{\beta}}(g) = \\ t_{\gamma_1} t_\beta|_\gamma (t_\beta^{-1} t_\beta g|_\beta t_\beta^{-1} t_\beta)|_\gamma t_\beta|_\gamma^{-1} t_{\gamma_1}^{-1} = t_{\gamma_1} t_\beta|_\gamma g|_\beta|_\gamma t_\beta|_\gamma^{-1} t_{\gamma_1}^{-1} = \\ t_{\gamma_1} t_\beta|_\gamma g|_\gamma (t_\beta|_\gamma)^{-1} t_{\gamma_1}^{-1} = t_{\gamma_1} t_\beta|_\gamma t_\gamma^{-1} \psi_{\alpha,\overline{\gamma}}(g) t_\gamma t_\beta|_\gamma^{-1} t_{\gamma_1}^{-1}.\end{aligned}$$

We get for every triple $\alpha \supset \beta \supset \gamma$ a *twisting element*

$$t_{\alpha,\beta,\gamma} = t_{\gamma_1} t_\beta|_\gamma t_\gamma^{-1} \in G_{\overline{\gamma}}$$

satisfying

$$\psi_{\overline{\beta},\overline{\gamma}} \circ \psi_{\overline{\alpha},\overline{\beta}}(g) = t_{\alpha,\beta,\gamma} \psi_{\alpha,\overline{\gamma}}(g) t_{\alpha,\beta,\gamma}^{-1}. \tag{3.3}$$

Consider the category $\mathcal{G}$ whose objects are the elements of T, and whose morphisms are pairs $(\overline{\alpha}, \beta)$, where $\overline{\alpha} \in T$, and $\beta \subset \overline{\alpha}$ (note that $\beta \in \mathfrak{G}^{(0)}$ is not necessarily an element of T). The source of $(\overline{\alpha}, \beta)$ is $\overline{\alpha}$, its range is $\overline{\beta}$. Suppose that $(\overline{\alpha}, \beta)$, and $(\overline{\beta}, \gamma)$ is a pair of composable morphisms. Denote $\gamma_0 = \mathbf{r}(t_\beta^{-1}|_\gamma)$. Then $t_\beta|_{\gamma_0}$ satisfies $\mathbf{s}(t_\beta|_{\gamma_0}) = \gamma_0$ and $\mathbf{r}(t_\beta|_{\gamma_0}) = \gamma$. Note that γ_0 does not depend on the choice of t_β (due to the second condition in the definition of a simplicial groupoid).

We define then $(\overline{\beta}, \gamma) \circ (\overline{\alpha}, \beta) = (\overline{\alpha}, \gamma_0)$. It is easy to check that we get in this way a category. It is a *scwol*: small category without loops (the latter condition means that the only endomorphisms in this category are identical isomorphisms).

We have associated a group G_α with every object of the category $\mathcal{G}$, and homomorphisms $\psi_e = \psi_{\overline{\alpha},\overline{\beta}} : G_{\mathbf{s}(e)} \longrightarrow G_{\mathbf{r}(e)}$ with every morphism $e = (\overline{\alpha}, \beta)$. For each pair $e_1 = (\overline{\beta}, \gamma), e_2 = (\overline{\alpha}, \beta)$ of composable morphisms as in the previous paragraph, we have

$$\psi_{e_1} \circ \psi_{e_2}(x) = t_{e_1,e_2}\psi_{e_1e_2}(x)t_{e_1,e_2}^{-1}, \tag{3.4}$$

for $t_{e_1,e_2} = t_\gamma t_\beta|_{\gamma_0} t_{\gamma_0}^{-1}$, where, as before, γ_0 is the sub-simplex of β equal to $\mathbf{r}(t_\beta^{-1}|_\gamma)$.

Proposition 3.4.5. *For every triple e_1, e_2, e_3 of composable morphisms of the category $\mathcal{G}$, we have the following cocycle identity*

$$\psi_{e_1}(t_{e_2,e_3})t_{e_1,e_2e_3} = t_{e_1,e_2}t_{e_1e_2,e_3}. \tag{3.5}$$

Proof. If the product $e_1 \cdot e_2 \cdot e_3$ is defined, then there exists a sequence of simplices $\alpha \supset \beta \supset \gamma \supset \delta$, such that $e_1 = (\gamma_2, \delta_2)$, $e_2 = (\beta_1, \gamma_1)$, $e_3 = (\alpha, \beta)$, where $\beta_1 = \overline{\beta}$, $\gamma_1 = \mathbf{r}(t_\beta|_\gamma)$, $\gamma_2 = \overline{\gamma}$, and $\delta_2 = \mathbf{r}(t_{\gamma_1}|_{\delta_1})$, for $\delta_1 = \mathbf{r}(t_\beta|_\delta)$. See Figure... where all the simplices and maps between them are shown.

We have $e_1e_2 = (\gamma_2, \delta_2)(\beta_1, \gamma_1) = (\beta_1, \delta_1)$, $e_2e_3 = (\beta_1, \gamma_1)(\alpha, \beta) = (\alpha, \gamma)$, and $e_1e_2e_3 = (\alpha, \delta)$. It follows that

$$\begin{aligned}
\psi_{e_1}(g) &= t_{\delta_2} g|_{\delta_2} t_{\delta_2}^{-1}, \\
t_{e_2,e_3} &= t_{\gamma_1} t_\beta|_\gamma t_\gamma^{-1}, \\
t_{e_1,e_2e_3} &= t_{\delta_2} t_\gamma|_\delta t_\delta^{-1}, \\
t_{e_1,e_2} &= t_{\delta_2} t_{\gamma_1}|_{\delta_1} t_{\delta_1}^{-1}, \\
t_{e_1e_2,e_3} &= t_{\delta_1} t_\beta|_\delta t_\delta^{-1}.
\end{aligned}$$

We have

$$\psi_{e_1}(t_{e_2,e_3}) = t_{\delta_2}(t_{\gamma_1} t_\beta|_\gamma t_\gamma^{-1})|_{\delta_2} \cdot t_{\delta_2}^{-1} = t_{\delta_2} t_{\gamma_1}|_{\delta_1} t_\beta|_\delta t_\gamma|_\delta^{-1} t_{\delta_2}^{-1},$$

so

$$\psi_{e_1}(t_{e_2,e_3}) \cdot t_{e_1,e_2e_3} = t_{\delta_2} t_{\gamma_1}|_{\delta_1} t_\beta|_\delta t_\gamma|_\delta^{-1} t_{\delta_2}^{-1} \cdot t_{\delta_2} t_\gamma|_\delta t_\delta^{-1} = t_{\delta_2} t_{\gamma_1}|_{\delta_1} t_\beta|_\delta t_\delta^{-1}.$$

On the other hand,

$$t_{e_1,e_2} \cdot t_{e_1e_2,e_3} = t_{\delta_2} t_{\gamma_1}|_{\delta_1} t_{\delta_1}^{-1} \cdot t_{\delta_1} t_\beta|_\delta t_\delta^{-1} = t_{\delta_2} t_{\gamma_1}|_{\delta_1} t_\beta|_\delta t_\delta^{-1},$$

hence $\psi_{e_1}(t_{e_2,e_3})t_{e_1,e_2e_3} = t_{e_1,e_2}t_{e_1e_2,e_3}$. □

The obtained structure (the category $\mathcal{G}$, groups G_α, homomorphisms ψ_e, and the twisting elements $t_{\alpha,\beta,\gamma}$) is called a *complex of groups*. Namely, we have the following definition, see...

Definition 3.4.6. A *complex of groups* is a given by a scwol $\mathcal{G}$, groups G_α associated with every object α of $\mathcal{G}$, group homomorphisms $\psi_e : G_{\mathbf{s}(e)} \longrightarrow G_{\mathbf{r}(e)}$ associated with every morphism e of $\mathcal{G}$, and *twisting elements* $t_{e_1,e_2} \in G_{\mathbf{r}(e_1)}$ associated with every composable pair (e_1, e_2) of morphisms, such that conditions (3.4) and (3.5) are satisfied.

Geometric realization of a scwol $\mathcal{G}$ is constructed in the following way. For every sequence $(e_1, e_2, \dots, e_n)$ of composable morphisms $\mathcal{G}$ take an n-dimensional standard simplex. Its $n-1$ dimensional faces the simplices associated with the sequences $(e_1e_2, e_3, \dots, e_n)$, $(e_1, e_2e_3, \dots, e_n)$, $\dots (e_1, e_2, e_3, \dots, e_{n-1}e_n)$. (More formally, we take a disjoint union of the simplices associated with sequences of composable morphisms and then identify lower-dimensional simpleces with the corresponding faces higher-dimensional simplices.) In particular, one-dimensional cells of the geometric realization are associated with morphisms of the category $\mathcal{G}$. The vertices of the geometric realiztion are the objects of $\mathcal{G}$, so that the set of vertices of the cell associated with $(e_1, e_2, \dots, e_n)$ is $\{\mathbf{r}(e_1), \mathbf{s}(e_1) = \mathbf{r}(e_2), \mathbf{s}(e_2) = \mathbf{r}(e_3), \dots, \mathbf{s}(e_n)\}$. For example, if $\mathcal{G}$ consists of two objects a_1, a_2 and three morphisms e_i such that $\mathbf{s}(e_i) = a_1, \mathbf{r}(e_i) = a_2$, then the geometric realization is the graph with two vertices and three edges connecting them. See a more detailed discussion of scwols and their geometric realizations in [**BH99**, III.$\mathcal{C}$.1].

If $\mathcal{G}$ is the scwol associated with a groupoid simplicial complex $\mathfrak{G}$ and a $\mathfrak{G}$-transversal T, as above, then the geometric realization of $\mathcal{G}$ is naturally homeomorphic to the space of orbits of the geometric realization of $\mathfrak{G}$. Namely, consider the baricentric subdivision of the geometric realization of $\mathfrak{G}^{(0)}$. Its vertices correspond to the simplices of $\mathfrak{G}^{(0)}$, while its simplices are chains $\alpha_1 \subset \alpha_2 \subset \cdots \subset \alpha_n$ of simplices of $\mathfrak{G}^{(0)}$. We leave it to the reader as an exercise to show that the identical map from T as a subset of the baricentric subdivision of $\mathfrak{G}^{(0)}$ to T as the set of vertices of the geometric realization of the scwol $\mathcal{G}$ naturally extends to an isomorphism of the quotient of the baricentric subdivision of $\mathfrak{G}^{(0)}$ by the $\mathfrak{G}$-action to the geometric realization of $\mathcal{G}$.

One-dimensional complexes of groups are easier to define, since there are no twisting elements, as they have are no pairs of composable non-identity morphisms. We get the following definition.

Definition 3.4.7. A *graph of groups* is given by the following data.

(1) Set of vertices V;

(2) set of edges E;

(3) source and range maps $\mathbf{s}, \mathbf{r} : E \longrightarrow V$;

(4) orientation reverting map $E \longrightarrow E : e \mapsto e^{-1}$, satisfying $\mathbf{s}(e^{-1}) = \mathbf{r}(e)$ and $\mathbf{r}(e^{-1}) = \mathbf{s}(e)$;

(5) groups G_v and G_e associated with every vertex $v \in V$ and edge $e \in E$, such that $G_e = G_{e^{-1}}$;

(6) homomorphisms $\psi_e : G_e \longrightarrow G_{\mathbf{r}(e)}$ for every edge $e \in E$.

A complex of groups over the scwol $\mathcal{G}$ with the *local groups* G_α, homomorphisms ψ_e, and twisting elements t_{e_1,e_2} defines a groupoid simplicial complex in the following way...

Simple complexes of groups (without twisting elements) defined over posets...

3.4.4. Fundamental groups of complexes of groups. ...

3.4.5. Van Kampen theory in groupoid terms. Cover of orbispaces and their nerves as complexes of groups...

The general van Kampen theorem for morphisms of groupoids: consider a groupoid morphism which is locally injective on the fundamental groupoid. Then it induces an isomorhism of groups...

Particular case: the classical van Kampen theorems... Use partition of unity...

3.5. Compactly generated groupoids

We assume throughout this section that our groupoids are étale.

3.5.1. Definition. We say that a subset $\mathcal{X} \subset \mathfrak{G}^{(0)}$ is a *topological transversal* if there exists a open transversal $\mathcal{X}_0$ such that $\mathcal{X}_0 \subset \mathcal{X}$.

Definition 3.5.1. Let $\mathfrak{G}$ be a topological groupoid. A *compact generating pair* is a pair $(\mathcal{X}_1, S)$ of compact sets $\mathcal{X}_1 \subset \mathfrak{G}^{(0)}$ and $S \subset \mathfrak{G}|_{\mathcal{X}_1}$ such that $\mathcal{X}_1$ is a topological transversal and for every $g \in \mathfrak{G}|_{\mathcal{X}_1}$ there exists n such that the set $\bigcup_{k=0}^{n}(S \cup S^{-1})^k$ is a neighborhood of g in $\mathfrak{G}|_{\mathcal{X}_1}$. We say that a groupoid is compactly generated if it has a compact generating pair.

Proposition 3.5.2. *Let $(\mathcal{X}_1, S)$ be a compact generating pair of $\mathfrak{G}$. Then for every compact topological transversal $\mathcal{X}_1' \subset \mathfrak{G}^{(0)}$ there exists a compact subset $S' \subset \mathfrak{G}|_{\mathcal{X}_1'}$ such that $(\mathcal{X}_1', S')$ is a compact generating set.*

Proof. Let us assume that $S = S^{-1}$ (we can always replace S by $S \cup S^{-1}$). Moreover, we may assume that $S \supset \mathcal{X}_1$, so that $S^k \subset S^{k+1}$ for every $k \geqslant 0$.

For every $x \in \mathcal{X}_1'$ there exists $g \in \mathfrak{G}$ such that $\mathbf{s}(g) = x$ and $\mathbf{r}(g)$ belongs to the interior of $\mathcal{X}_1$. It follows that there exists an open set $U \subset \mathfrak{G}$ such that closure of U is compact, $\mathbf{s}(U) \ni x$, $\mathbf{r}(U) \subset \mathcal{X}_1$. Since $\mathcal{X}_1'$ is compact, this shows that there exists a finite set of relatively compact open bisections $U_1, U_2, \ldots, U_{m_1}$ such that $\mathcal{X}_1' \subset \bigcup_{i=1}^{m_1} \mathbf{s}(U_i)$ and $\mathbf{r}(U_i) \subset \mathcal{X}_1$ for all i. By the same argument there exists a finite collection $W_1, W_2, \ldots, W_{m_2}$ of relatively compact open bisections such that $\mathcal{X}_1 \subset \bigcup_{i=1}^{m_2} \mathbf{s}(W_i)$ and $\mathbf{r}(W_i) \subset \mathcal{X}_1'$ for every i.

Let $g \in \mathfrak{G}|_{\mathcal{X}_1'}$. Then there exists U_i and U_j such that $\mathbf{s}(g) \in \mathbf{s}(U_i)$ and $\mathbf{r}(g) \in \mathbf{s}(U_j)$. The element $U_j g U_i^{-1}$ belongs to $\mathfrak{G}$. Consequently, there exists n such that S^n is a neighborhood of $U_j g U_i^{-1}$. Let $s_1 s_2 \cdots s_n \in S^n$ be an arbitrary element of this neighborhood. Then there exist $W_0, W_1, \ldots, W_n$ such that $\mathbf{r}(s_1) \in \mathbf{s}(W_0)$, $\mathbf{s}(s_1) \in \mathbf{s}(W_1)$, $\mathbf{s}(s_2) \in \mathbf{s}(W_2)$, $\ldots$, $\mathbf{s}(s_n) \in \mathbf{s}(W_n)$. Then

$$s_1 s_2 \cdots s_n = W_0^{-1} \cdot (W_0 s_1 W_1^{-1})(W_1 s_2 W_2^{-1}) \cdots (W_{n-1} s_n W_n^{-1}) W_n,$$

and $W_0^{-1} \cdot (W_0 S W_1^{-1})(W_1 S W_2^{-1}) \cdots (W_{n-1} S W_n^{-1}) W_k$ is a neighborhood of $s_1 s_2 \cdots s_n$. It follows that

$$U_j^{-1} W_0^{-1} \cdot (W_0 S W_1^{-1})(W_1 S W_2^{-1}) \cdots (W_{n-1} S W_n^{-1}) W_n U_i$$

is a neighborhood of g. Consequently, if we take S' to be the intersection of the closure of $\bigcup_{k=1,\ldots,m_2, l=1,\ldots,m_1} (W_k U_l \cup U_l^{-1} W_k) \cup \bigcup_{1 \leqslant k,l \leqslant m_2} W_k S W_l^{-1}$ with $\mathfrak{G}|_{\mathcal{X}_1'}$, then $(\mathcal{X}_1', S')$ is a generating pair of $\mathfrak{G}$. $\square$

Proposition 3.5.3. *Let $\mathfrak{G}_1$ and $\mathfrak{G}_2$ be equivalent topological groupoids. If $\mathfrak{G}_1$ is compactly generated, then so is $\mathfrak{G}_2$.*

Proof. Follows directly from Propositions 3.5.2 and 3.2.26. $\square$

Proposition 3.5.4. *Let $\mathfrak{G}$ be an étale groupoid with compact totally disconnected space of units. Then the following conditions are equivalent.*

(1) *The groupoid $\mathfrak{G}$ is compactly generated.*
(2) *There exists a compact open subset $S \subset \mathfrak{G}$ such that $\mathfrak{G} = \bigcup_{n \geqslant 0} S^n$.*
(3) *There exists a finite set $\mathcal{S}$ of compact open bisections such that the set of all products of the elements of $\mathcal{S}$ is a cover of $\mathfrak{G}$.*

Proof. By Proposition 3.5.2, $\mathfrak{G}$ is compactly generated if and only if there exists a compact generated pair $(\mathfrak{G}^{(0)}, S)$. Since every compact subset of $\mathfrak{G}$ is contained in an open compact set, we may increase S to a symmetric compact open set, which proves the implication (1)$\Longrightarrow$(2). The converse implication is obvious.

Every compact open subset of $\mathfrak{G}$ is a union of a finite number of compact open bisections, which proves the equivalence (2)$\Longleftrightarrow$(3). $\square$

3.5.2. Cayley graphs of groupoids.

Let $(\mathcal{X}_1, S)$ be a compact generating pair of a groupoid $\mathfrak{G}$. Let $x \in \mathcal{X}_1$. The *Cayley graph* $\mathfrak{G}_x(S)$ is the graph with the set of vertices $\{g \in \mathfrak{G} \ : \ \mathbf{s}(g) = x, \mathbf{r}(g) \in \mathcal{X}_1\}$ in which there is an arrow from g_1 to g_2 if $g_2 g_1^{-1} \in S$, i.e., if there exists $s \in S$ such that $g_2 = s g_1$.

Proposition 3.5.5. *Let $(\mathcal{X}_1, S)$ be a compact generating pair of $\mathfrak{G}$, and let $\mathcal{X}_2 \subset \mathcal{X}_1$ be a compact $\mathfrak{G}$-transversal. Then the set of vertices g of $\mathfrak{G}_x(S)$ such that $\mathbf{r}(g) \in \mathcal{X}_2$ is a net in the Cayley graph $\mathfrak{G}_x(S)$ for every $x \in \mathcal{X}_2$.*

Here we say that a subset N of a metric space (X, d) is a *net* if there exists $R > 0$ such that for every $x \in X$ there exists $y \in N$ such that $d(x, y) \leqslant R$.

Proof. As in the proof of Proposition 3.5.2, there exists a finite collection of relatively compact bisection $U_1, U_2, \ldots U_k$ such that $\mathbf{s}(U_i) \subset \mathcal{X}_2$ and $\mathbf{r}(U_i)$ cover $\mathcal{X}_1$. Consequently, there exists a compact set C such that $\mathbf{s}(C) \subset \mathcal{X}_2$ and $\mathbf{r}(C) = \mathcal{X}_1$. Then there exists n such that $C \subset \bigcup_{k=0}^{n}(S \cup S^{-1})^k$. This proves that every vertex $g \in \mathfrak{G}_x(S)$ is on the distance at most n from a vertex $h \in \mathfrak{G}_x$ such that $\mathbf{r}(h) \in \mathcal{X}_2$. □

Proposition 3.5.6. *The quasi-isometry class of the Cayley graph $\mathfrak{G}_x(S)$ depends only on the groupoid $\mathfrak{G}$ and the point x, and does not depend on the choice of the generating pair.*

Proof. Let $(\mathcal{X}_1, S_1)$ and $(\mathcal{X}_2, S_2)$ be compact generating pairs of $\mathfrak{G}$, and let $x \in \mathcal{X}_1 \cap \mathcal{X}_2$. Then, by Proposition 3.5.2, there exists a generating pair $(\mathcal{X}_1 \cup \mathcal{X}_2, S)$. We may increase S so that $S \supset S_1 \cup S_2$. It is enough to prove that the identical embedding of the Cayley graph $\mathfrak{G}_x(S_1)$ into $\mathfrak{G}_x(S)$ is a quasi-isometry. We know that the set of vertices of $\mathfrak{G}_x(S_1)$ is a net in $\mathfrak{G}_x(S)$, by Proposition 3.5.5. The identity map is distance non-increasing (i.e., 1-Lipschitz), since $S_1 \subset S$. It remains to bound the distance in $\mathfrak{G}_x(S)$ between vertices of $\mathfrak{G}_x(S_1)$ in terms of the distance in $\mathfrak{G}_x(S_1)$. Let, as in the proof of Proposition 3.5.2, $\mathcal{U}$ be a finite set of relatively compact open $\mathfrak{G}$ bisections such that $\mathbf{s}(U)$ for $U \in \mathcal{U}$ cover $\mathcal{X}_1 \cup \mathcal{X}_2$ and $\mathbf{r}(U) \subset \mathcal{X}_1$ for every $U \in \mathcal{U}$. Then for every product $s_1 s_2 \cdots s_n$ of elements of $S \cup S^{-1}$ such that $\mathbf{s}(s_n), \mathbf{r}(s_1) \in \mathcal{X}_1$ there exists a sequence $U_1, U_2, \ldots, U_{n-1}$ such that

$$s_1 s_2 \cdots s_n = s_1 U_1^{-1} \cdot U_1 s_2 U_2^{-1} \cdot U_2 s_3 U_3^{-1} \cdots U_{n-1} s_n$$

Note that $s_1 U_1^{-1}, U_{n-1} s_n$, and all $U_i s_{i+1} s_{i+1}^{-1}$ belong to $\mathfrak{G}|_{\mathcal{X}_1}$. The closures of the sets of the form $(S \cup S^{-1})U^{-1}$, $U(S \cup S^{-1})W^{-1}$ and $U(S \cup S^{-1})$ are compact for all $U, W \in \mathcal{U}$, and there are finitely many of them. It follows that there exists a compact set $C \subset \mathfrak{G}$ such that all the factors $s_1 U_1^{-1}$, $U_i s_{i+1} U_{i+1}^{-1}$, and $U_{n-1} s_n$ belong to it. Moreover, they belong to $C \cap \mathfrak{G}|_{\mathcal{X}_1}$. By the definition of a compact generating pair, for every $h \in C \cap \mathfrak{G}|_{\mathcal{X}_1}$ there exists m such that $\bigcup_{k=0}^{m}(S_1 \cup S_1^{-1})^k$ is a neighborhood of h in $\mathfrak{G}|_{\mathcal{X}_1}$. Since $C \cap \mathfrak{G}|_{\mathcal{X}_1}$ is compact, there exists m such that $C \cap \mathfrak{G}|_{\mathcal{X}_1} \subset \bigcup_{k=0}^{m}(S_1 \cup S_1^{-1})^k$. This proves that every product of length n of elements of $S \cup S^{-1}$ belonging to $\mathfrak{G}|_{\mathcal{X}_1}$ can be written as a product of elements of length at most mn of elements of $S_1 \cup S_1^{-1}$. This finishes the proof of the proposition. □

Proposition 3.5.6 implies that the quasi-isometry class of the Cayley graph depends only on the equivalence class of the groupoid, since any two equivalent groupoids can be embedded into one groupoid. More precisely, if

$\mathfrak{G}$ and $\mathfrak{H}$ are equivalent, and $x \in \mathfrak{G}^{(0)}$, then for every unit $y \in \mathfrak{H}^{(0)}$ related by an equivalence $\mathfrak{G} \curvearrowright \mathcal{E} \curvearrowleft \mathfrak{H}$ with x the Cayley graphs $\mathfrak{G}_x(S_1)$ and $\mathfrak{H}_y(S_2)$ are quasi-isometric.

The quasi-isometry class of $\mathfrak{G}_x(S)$, however, depends on x, see for example...

3.5.3. Examples of compactly generated groupoids. It is easy to see that our notion of a compact generating set coincides with the usual notion of a compact generating set of a group in the case when $\mathfrak{G}$ has a unique unit (i.e., is a group). The notion of the Cayley graph also coincides in this case with the classical notion of a Cayley graph.

Let us describe several other examples.

Example 3.5.7. Suppose that G is a discrete group acting on a compact space $\mathcal{X}$. The action groupoid $G \ltimes \mathcal{X}$ is compactly generated if and only if G is finitely generated. The corresponding compact generating set is $S \times \mathcal{X}$, where S is a finite generating set of G. The Cayley graphs of $G \ltimes \mathcal{X}$ are naturally isomorphic to the Cayley grah of G.

Example 3.5.8. If the action of G on a space $\mathcal{X}$ is proper and co-compact, then the action groupoid $G \ltimes \mathcal{X}$ is compactly generated. The Cayley graphs are finite.

Example 3.5.9. Consider the action of $\mathbb{Z}$ on $\mathbb{R}/\mathbb{Z}$ generated by the rotation $R_\theta : x \mapsto x + \theta$ by an irrational angle θ. The corresponding groupoid of the action is equivalent to the groupoid of the action of $\mathbb{Z}^2$ on $\mathbb{R}$ generated by the transformations $x \mapsto x+1$ and $x \mapsto x+\theta$. Both groupoids are compactly generated. The first one is generated by the $\{R_\theta\} \times (\mathbb{R}/\mathbb{Z}) \subset \mathbb{Z} \ltimes (\mathbb{R}/\mathbb{Z})$, which is compact. The corresponding Cayley graphs of the groupoid of the action do not depend on the basepoint and are just the Cayley graphs of $\mathbb{Z}$.

A compact generating set of of the second groupoid is obtained by taking any compact topological transversal, e.g.., any closed interval $[a, b] \subset \mathbb{R}$, and considering a sufficiently big subset $S \subset \mathbb{Z}^2$, so that the set of elements $(g, x) \in \mathbb{Z}^2 \ltimes \mathbb{R}$ such that $g \in S$ and $x, g \cdot x \in [a, b]$ is a generating set of the restriction of $\mathbb{Z}^2 \ltimes \mathbb{R}$ onto $[a, b]$. For example, if $[a, b] = [0, 1]$, then we can take S consisting of the transformations $x \mapsto x + \theta$ and $x \mapsto 1 - \theta$, if $\theta \in (0, 1)$. The Cayley graphs $\mathfrak{G}_x(S)$ for such generating sets are equal to the graph spanned by the set $\{(m, n) \in \mathbb{Z}^2 \ : \ x + m + n\theta \in [a, b]\}$ in the Cayley graph of $\mathbb{Z}^2$ with respect to the generating set S, see Figure 1.1.

Example 3.5.10. The groupoid of the stable equivalence relation for a Ruelle-Smale system... Compact generation is equivalent to local connectivity of the unstable leaves, quasi-isometry of the Cayley graphs with the unstable leaves...

3.5.4. The space of well labeled graphs and the associated groupoid. Let X be a finite set. A *well labeled graph* with the set of labels X is a connected graph Γ with edges labeled by elements of the set X so that for every vertex v of Γ and every label $x \in \mathsf{X}$ there exists at most one edge starting in v and labeled by x, and at most one edge ending in v and labeled by x. (Compare with the definition of a *perfect labeling* of a graph in 2.1.1.)

Denote by $\mathcal{G}_{\mathsf{X}}$ the set of all *rooted* well labeled by X graphs. We consider it with the usual topology, as in 2.1.4. Two graphs (Γ_1, v_1) and (Γ_2, v_2) are close in this topology if for a big radius $R > 0$ the balls of radius R with centers in v_1 and v_2 are isomorphic as labeled rooted (with roots v_i) graphs.

Denote by $\mathfrak{G}_{\mathsf{X}}$ the set of *bi-rooted* well labeled graphs. Its elements are triples (Γ, v_1, v_2), where Γ is a well labeled graph, and v_1, v_2 are two vertices of Γ. We topologize $\mathfrak{G}_{\mathsf{X}}$ in the same way as $\mathcal{G}_{\mathsf{X}}$: two elements (Γ_1, v_1, v_2) and (Γ_2, u_1, u_2) are close if for a big $R > 0$ (in particular, bigger than the distances $d(v_1, v_2)$ and $d(u_1, u_2)$) there exists an isomorphism of the labeled graphs $B_{v_1}(R) \longrightarrow B_{u_1}(R)$ mapping v_1 to u_1 and v_2 to u_2.

The space $\mathfrak{G}_{\mathsf{X}}$ is an étale groupoid in a natural way. The source and range maps are $\mathbf{s}(\Gamma, v_1, v_2) = (\Gamma, v_1, v_1)$ and $\mathbf{r}(\Gamma, v_1, v_2) = (\Gamma, v_2, v_2)$. The multiplication is given by the rule

$$(\Gamma, v_2, v_3)(\Gamma, v_1, v_2) = (\Gamma, v_1, v_3).$$

Note that the fact that this multiplication is well defined (that the isomorphism class of (Γ, v_1, v_3) depends only on the isomorphism classes of (Γ, v_1, v_2) and (Γ, v_2, v_3)) follows from the fact that the automorphism group of a rooted well labeled graph is trivial (there is only one isomorphism $\Gamma \longrightarrow \Gamma$ mapping v_2 to v_2). We will identify the space of units of $\mathfrak{G}_{\mathsf{X}}$ with the space of rooted graphs $\mathcal{G}_{\mathsf{X}}$, where (Γ, v, v) is identified with (Γ, v).

The element (Γ, v_1, v_2) of the groupoid $\mathfrak{G}_{\mathsf{X}}$ can be seen as the act of moving the root from v_1 to v_2.

Definition 3.5.11. The groupoid $\mathfrak{G}_{\mathsf{X}}$ is called the *full graph shift* over the alphabet X. Restrictions of $\mathfrak{G}_{\mathsf{X}}$ onto closed $\mathfrak{G}_{\mathsf{X}}$-invariant subsets of the unit space are called *graph sub-shifts*.

We leave it as an exercise to show that the full graph shift is an étale groupoid.

A *graph subshift* is the restriction of the full graph shift $\mathfrak{G}_{\mathsf{X}}$ to a closed $\mathfrak{G}_{\mathsf{X}}$-invariant (i.e., equal to a union of orbits) subset of $\mathcal{G}_{\mathsf{X}} = \mathfrak{G}_{\mathsf{X}}^{(0)}$.

Example 3.5.12. The groupoid $\mathbb{Z} \ltimes \mathsf{X}^{\mathbb{Z}}$ of the full $\mathbb{Z}$-shift $\mathsf{X}^{\mathbb{Z}}$ is naturally identified with a graph sub-shift: with the restriction of $\mathfrak{G}_{\mathsf{X}}$ to the subset of $\mathcal{G}_{\mathsf{X}}$ consisting of all X-labelings of the graph with the set of vertices $\mathbb{Z}$ with arrows from n to $n+1$. It follows that the groupoids of all $\mathbb{Z}$-subshifts are also graph sub-shifts.

Example 3.5.13. Every quotient G of the free group generated by X is naturally identified with the sub-shift equal to the set of all pairs $(\Gamma, 1, g)$, where Γ is the Cayley graph of G with the natural edge labeling by the elements of X.

Another important example is the *tree shift groupoid*.

Definition 3.5.14. The *tree shift* $\mathfrak{F}_{\mathsf{X}}$ *generated by* X is the restriction of the full graph shift $\mathfrak{G}_{\mathsf{X}}$ to the set trees well labeled by X.

The tree shift $\mathfrak{F}_{\mathsf{X}}$ will play a role of the *free étale groupoid* generated by X.

Proposition 3.5.15. *Let $\mathfrak{H} \subset \mathfrak{G}_{\mathsf{X}}$ be a graph subshift. Denote for $x \in \mathsf{X}$ by S_x the subset of $\mathfrak{H}$ consisting of all elements $(\Gamma, v_1, v_2) \in \mathfrak{H}$ such that there exists an edge e of Γ labeled by x such that $\mathbf{s}(e) = v_1$ and $\mathbf{r}(e) = v_2$. Then S_x is a clopen bisection and the set $\bigcup_{x \in \mathsf{X}} S_x$ generates $\mathfrak{H}$.*

Proof. ... □

3.5.5. Expansive groupoids.

Definition 3.5.16. Let $\mathfrak{G}$ be an étale groupoid, and let $(\mathcal{X}, S)$ be its compact generating pair. A finite cover $\mathcal{S}$ of S by open $\mathfrak{G}$-bisections is called *expansive* if for every $g \in \mathfrak{G}|_{\mathcal{X}}$ and every neighborhood U of g in $\mathfrak{G}$ there exist sequences $s_1, s_2, \ldots, s_n \in S \cup S^{-1}$ and $F_1, F_2, \ldots, F_n \in \mathcal{S} \cup \mathcal{S}^{-1}$ such that $s_i \in F_i$, $s_1 s_2 \cdots s_n = g$, and $F_1 F_2 \cdots F_n \subset U$.

We say that $\mathfrak{G}$ is *expansive* if there exists an expansive cover of for a compact generating pair of $\mathfrak{G}$.

Proposition 3.5.17. *If $\mathcal{S}$ is an expansive cover of a compact generating pair, then any subordinate cover is expansive.*

If $\mathfrak{G}$ is expansive, then for every compact generating pair $(\mathcal{X}, S)$ of $\mathfrak{G}$ there exists a finite expansive cover $\mathcal{S}$ of S.

In particular, the property of being expansive is invariant under equivalence of étale groupoids.

Proof. The first statement follows directly from the definitions. Note also that if $\mathcal{S}$ is an expansive cover, then cover containing it is also expansive.

The second statement of the proposition is proved by the same argument as in proof of Proposition 3.5.2. Namely, in the notation of the proof, if $\mathcal{S}$

is an expansive cover of S, then for every $g \in \mathfrak{G}|_{\mathcal{X}_1'}$ there exist $F_i \in \mathcal{S}$ such that

$$g \in U_j^{-1} W_0^{-1} \cdot (W_0 F_1 W_1^{-1})(W_1 F_2 W_2^{-1}) \cdots (W_{n-1} F_n W_n^{-1}) W_n.$$

Since $\mathcal{S}$ is expansive, we can make $F_1 F_2 \cdots F_n$ an arbitrarily small neighborhood of $U_j g U_i^{-1}$. Then

$$U_j^{-1} W_0^{-1} \cdot (W_0 F_1 W_1^{-1})(W_1 F_2 W_2^{-1}) \cdots (W_{n-1} F_n W_n^{-1}) W_n U_i \subset U_j^{-1} F_1 F_2 \cdots F_n U_i$$

is arbitrarily small neighborhood of g. It follows that the collection of the sets of the form $W_k U_l$, $U_l^{-1} W_k$, $W_k F W_l^{-1}$ for $F \in \mathcal{S}$ is an expansive cover of S. □

If $\mathfrak{G}$ is a compactly generated groupoid with a compact space of units, then it is sufficient to use finite sets of open bisections without specifying a compact generating set of $\mathfrak{G}$. Namely, we have the following characterization of expansivity.

Lemma 3.5.18. *Let $\mathfrak{G}$ be a compactly generated groupoid with compact space of units. Then $\mathfrak{G}$ is expansive if and only if there exists a finite set $\mathcal{S}$ of open bisections such that the set $\bigcup_{n \geqslant 0} (\mathcal{S} \cup \mathcal{S}^{-1})^n$ is a basis of topology of $\mathfrak{G}$.*

Proof. If $\mathfrak{G}$ is expansive, then such a set exists by definition. Suppose that $\mathfrak{G}$ is compactly generated, and such a set $\mathcal{S}$ of bisections exists. Let $\mathcal{S}_1$ be a finite cover of a compact generating set S of $\mathfrak{G}$ by open bisections. Then $\mathcal{S} \cup \mathcal{S}_1$ is an expansive cover of S, hence $\mathfrak{G}$ is expansive. □

We say that a finite set $\mathcal{S}$ of open bisections is *expansive* if it satisfies the conditions of Lemma 3.5.18.

Proposition 3.5.19. *Let $\mathfrak{G}$ be an étale groupoid with compact Hausdorff space of units. Let $\mathcal{S}$ be a finite set of relatively compact (...) open $\mathfrak{G}$-bisections such that $\mathcal{S}^{-1} = \mathcal{S}$. Denote by $\mathcal{S}^*$ the set of all finite products of elements of $\mathcal{S}$. Then the following conditions are equivalent.*

(1) *The set $\mathcal{S}$ is expansive.*
(2) *The set $\mathbf{s}(\mathcal{S}^*) = \{\mathbf{s}(F) : F \in \mathcal{S}^*\}$ is a basis of topology of $\mathfrak{G}^{(0)}$.*
(3) *For any two different points $x, y \in \mathfrak{G}^{(0)}$ there exist $A, B \in \mathbf{s}(\mathcal{S}^*)$ such that $x \in A$, $y \in B$, and $A \cap B = \varnothing$.*
(4) *For every $x \in \mathfrak{G}^{(0)}$ the intersection of closures of the sets $A \in \mathbf{s}(\mathcal{S}^*)$ containing x is equal to $\{x\}$.*

Proof. We have $\mathbf{s}(F_1 F_2 \cdots F_n) = F_n^{-1} \cdots F_2^{-1} F_1^{-1} F_1 F_2 \cdots F_n$. Consequently, $\mathbf{s}(\mathcal{S}^*) \subset \mathcal{S}^*$. The unit space $\mathfrak{G}^{(0)}$ is open in $\mathfrak{G}$, hence (1) implies (2).

Note also that for every bisection $F \subset \mathfrak{G}$ and any open set $U \subset \mathfrak{G}^{(0)}$ the restriction of F to U is equal to the product FU. It follows that (2) implies (1).

The implication (2)$\Longrightarrow$(3)$\Longrightarrow$(4) are obvious. Let us prove (3)$\Longrightarrow$(2). Let $x \in \mathfrak{G}^{(0)}$ and let U be an open neighborhood of x. For every $y \in \mathfrak{G}^{(0)} \setminus U$ there exist a pair A_y, B_y of elements of $\mathbf{s}(\mathcal{S}^*)$ such that $x \in A_y$, $y \in B_y$, and $A_y \cap B_y = \varnothing$. Since $\mathfrak{G}^{(0)} \setminus U$ is compact, and the sets B_y cover it, we can find two finite sequences $A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n \in \mathbf{s}(\mathcal{S}^*)$ such that $x \in A_i$, $\bigcup_{i=1}^n B_i \supset U$, and $A_i \cap B_i = \varnothing$ for all i. Then $\bigcap_{i=1}^n A_i = A_1 A_2 \cdots A_n$ is an element of $\mathbf{s}(\mathcal{S}^*)$ contained in U.

Let us prove (4)$\Longrightarrow$(3). Denote, for $n \geqslant 1$ and $x \in \mathfrak{G}^{(0)}$, by $T_n(x)$ the closure of the intersection of the sets of the form $\mathbf{s}(F_1 F_2 \cdots F_n)$ containing x, where $F_i \in \mathcal{S}$. Recall that the set $\mathbf{s}(\mathcal{S}^*)$ is closed under finite intersections, hence $T_n(x)$ is closure of an element of $\mathbf{s}(\mathcal{S}^*)$. In particular, condition (4) is equivalent to $\bigcap_{n \geqslant 1} T_n(x) = \{x\}$. If (3) is not true for a pair $x, y \in \mathfrak{G}^{(0)}$, then $T_n(x) \cap T_n(y)$ is non-empty for all n. But then $T_n(x) \cap T_n(y)$ is a decreasing sequence of closed sets, hence by compactness of $\mathfrak{G}^{(0)}$, we get that $\bigcap_{n \geqslant 1} T_n(x) \cap T_n(y)$ is non-empty, which is a contradiction. □

Corollary 3.5.20. *Let $G \curvearrowright \mathcal{X}$ be an action of a finitely generated group on a compact Hausdorff space. Then the following conditions are equivalent.*

(1) *The groupoid of the action $G \ltimes \mathcal{X}$ is expansive.*

(2) *The groupoid of the germs of the action is expansive.*

(3) *The action $G \curvearrowright \mathcal{X}$ is expansive in the sense of Subsection 1.2.2.*

Proof. ... □

3.5.6. Cayley graphs of expansive groupoids. Let $\mathfrak{G}$ be an expansive groupoid such that $\mathfrak{G}^{(0)}$ is compact Hausdorff. Let $\mathcal{S}$ be a finite set of relatively compact open bisections satisfying the equivalent conditions (1)–(4) of Proposition 3.5.19. Denote by $\mathfrak{G}_x(\mathcal{S})$ the labeled Cayley graph of $\mathfrak{G}$ based at x, i.e., the graph with the set of vertices $\mathbf{s}^{-1}(x)$ in which for every $g \in \mathbf{s}^{-1}(x)$ and $F \in \mathcal{S}$ such that $\mathbf{r}(g) \in \mathbf{s}(F)$ there is an arrow from g to Fg labeled by F. Note that $\mathfrak{G}_x(\mathcal{S})$ with the root x is an element of the space $\mathcal{G}_{\mathcal{S}}$ of well labeled graphs.

Denote by $\widetilde{\mathfrak{G}}_x(\mathcal{S})$ the universal covering of the CW-complex $\mathfrak{G}_x(\mathcal{S})$. It is an $\mathcal{S}$-labeled tree with the root x, and is an element of the tree shift $\mathfrak{F}_{\mathcal{S}}^{(0)}$.

Theorem 3.5.21. *Let $\mathfrak{G}$ be a compactly generated groupoid with compact space of units. It is expansive if and only if there exists a finite set of open bisections $\mathcal{S}$ generating $\mathfrak{G}$ and such that for every pair $x, y \in \mathfrak{G}^{(0)}$ the rooted graphs $\widetilde{\mathfrak{G}}_x(\mathcal{S})$ and $\widetilde{\mathfrak{G}}_y(\mathcal{S})$ are not isomorphic.*

Proof. If $\mathcal{S}$ is an expansive cover of a generating set, then for any two different points $x, y \in \mathfrak{G}^{(0)}$ there exist finite products A and B of the elements of $\mathcal{S} \cup \mathcal{S}^{-1}$ such that $x \in \mathbf{s}(A), y \in \mathbf{s}(B)$, and $A \cap B = \varnothing$, see Proposition 3.5.19, condition (3). But this means that the tree $\widetilde{\mathfrak{G}}_x(\mathcal{S})$ has a path labeled by the word corresponding to A and starting in the root, while $\widetilde{\mathfrak{G}}_y(\mathcal{S})$ does not contain such a word. In particular, the universal covers of the Cayley graphs are not isomorphic.

Suppose now that $\mathcal{S}$ is such that the universal coverings $\widetilde{\mathfrak{G}}_x(\mathcal{S})$ are pairwise non-isomorphic for all $x \in \mathfrak{G}^{(0)}$. Denote by $U_n(x)$ the intersection of the closures of the domains containing x of products of length at most n of elements of $\mathcal{S} \cup \mathcal{S}^{-1}$. Note that for every fixed n the number of sets of the form $U_n(x)$ is finite, and $\bigcap_{n \geqslant 1} U_n(x) = \{x\}$ by condition (4) of Proposition 3.5.19. Since the number of sets of the form $U_1(x)$ is finite, we can replace $\mathcal{S}$ by a refinement $\mathcal{S}_1$ so that the new set $\mathcal{S}_1$ satisfies the condition that for every $F \in \mathcal{S}_1$ and $x \in \mathbf{s}(F)$ we have $\mathbf{s}(F) \subset U_1(x)$... Note that then the Cayey graphs $\mathfrak{G}_x(\mathcal{S}_1)$ are obtained from $\mathfrak{G}_x(\mathcal{S})$ just by relabeling the edges (by incorporating into the label the information about the 1-ball of the vertices connected by the arrow). The same is true about their universal coverings.

Suppose that $\mathcal{S}_1$ is not expansive. Then, by part (4) of Proposition 3.5.19, there exist two different points $x, y \in \mathfrak{G}^{(0)}$ such that for every finite product A of the elements $\mathcal{S}_1 \cup \mathcal{S}_1^{-1}$ such that $x \in \mathbf{s}(A)$ we have $y \in \mathbf{s}(A)$. It follows that there is a morphism of the labeled rooted tree $\widetilde{\mathfrak{G}}_x(\mathcal{S}_1) \longrightarrow \widetilde{\mathfrak{G}}_y(\mathcal{S}_1)$. Such a morphism is necessarily injective (since the trees are well labeled). It is also locally bijective, since the labels contain the information about the unit balls of the adjacent vertices. Consequently, it is an isomorphism, which is a contradiction. □

Let now $\mathcal{S}$ be a finite generating set of $\mathfrak{G}$ satisfying the conditions of Theorem 3.5.21. Let $\mathcal{T}$ be the set of all universal coverings of the Cayley graphs $\mathfrak{G}_x(\mathcal{S})$. It is a subset of the space of units of the tree shift $\mathfrak{F}_{\mathcal{S}}$.

Show that every expansive groupoid is a quotient of a tree shift by an open isotropical sub-groupoid... Expansive graph subshifts...

It is often easier to prove that the Cayley graphs $\mathfrak{G}_x(\mathcal{S})$ are pairwise non-isomorphic than to understand their universal coverings. It is possible that the Cayley graphs are pairwise non-isomorphic, but the action is not expansive. See, for example....

On the other hand, it is possible sometimes to deduce non-isomorphism of the universal coverings from non-isomorphism of the Cayley graphs.

Definition 3.5.22. The *Rips complex* $\Delta_n(\Gamma)$ of a graph Γ is the simplicial complex with the same set of vertices as Γ in which a set of vertices is a simplex if and only if its diameter in Γ is less than or equal to n.

We say that a graph Γ is *large-scale simply connected* if there exists n such that the Rips complex $\Delta_n(\Gamma)$ is simply connected.

It is known that the propery of large-scale simple connectivity is invariant under quasi-isometries. Consequently, if Cayley graphs of a compactly generated groupoid are large-scale simply connected with respect to one compact generating set, then they are large-scale simply connected with respect to every compact generating set, see Proposition 3.5.6.

The following theorem is proved in [**?**, Theorem 6.6].

Theorem 3.5.23. *Let $\mathfrak{G}$ be a Hausdorff compactly generated groupoid. Suppose that its Cayley graphs are large-scale simply connected. Then $\mathfrak{G}$ is expansive if and only if there exists a finite generating set $\mathcal{S}$ of bisections such that the rooted labeled Cayley graphs $\mathfrak{G}_x(\mathcal{S})$ are pairwise non-isomorphic for all $x \in \mathfrak{G}^{(0)}$.*

3.6. Hyperbolic groupoids

An overview of the theory: definition, construction of the Smale flow and dual groupoid (with few proofs), examples...

Exercises

3.1. Prove that every abstract groupoid is isomorphic to a groupoid described in Example 3.1.8.

3.2. Prove that if F_1 and F_2 are bisections, then F_1^{-1} and F_1F_2 are also bisections.

3.3. Prove that the source and range maps are open in Example 3.1.15.

3.4. Prove that $\mathcal{M}_1 \otimes (\mathcal{M}_2 \otimes \mathcal{M}_3)$ is isomorphic to $(\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \mathcal{M}_3$.

3.5. Prove Proposition 3.2.13.

3.6. Let $\mathfrak{G} \curvearrowright \mathcal{M} \curvearrowleft \mathfrak{H}$ be a biaction. Show that the natural action of $\mathfrak{H}$ on the groupoid $\mathfrak{G} \rtimes \mathcal{M}$ given by $(g, x) \cdot h = (g, x \cdot h)$ is an action by automorphisms of $\mathfrak{G} \ltimes \mathcal{M}$ and therefore the set of orbits $(\mathfrak{G} \ltimes \mathcal{M})/\mathfrak{H}$ is a groupoid. Prove that $(\mathfrak{G} \ltimes \mathcal{M})/\mathfrak{H}$ is naturally isomorphic to the action groupoid $\mathfrak{G} \ltimes (\mathcal{M}/\mathfrak{H})$.

3.7. Let $\mathfrak{G} \curvearrowright \underrightarrow{\mathcal{M}} \curvearrowleft \mathfrak{H}$ be a morphism. Show that the projection $(g, x) \mapsto g$ from $\mathfrak{G} \ltimes \mathcal{M}$ to $\mathfrak{G}$ induces an isomorphism of groupoids $\mathfrak{G} \ltimes \mathcal{M}/\mathfrak{H} \longrightarrow \mathfrak{G}$, where $\mathfrak{G} \ltimes \mathcal{M}/\mathfrak{H}$ is the groupoid $(\mathfrak{G} \ltimes \mathcal{M})/\mathfrak{H} \cong \mathfrak{G} \ltimes (\mathcal{M}/\mathfrak{H})$ from the previous problem.

3.8. Prove Proposition 3.2.19.

3.9. Let $\mathfrak{G} \curvearrowright \mathcal{M} \curvearrowleft \mathfrak{H}$ be a biaction. Transform the right action $\mathfrak{H}$ into a left action $h \cdot x := x \cdot h^{-1}$, so that we get two commuting left actions of $\mathfrak{G}$ and $\mathfrak{H}$ on $\mathcal{M}$, i.e., an action $(\mathfrak{G} \times \mathfrak{H}) \curvearrowright \mathcal{M}$ of the obviously defined direct product of two groupoids. Denote by $\mathfrak{G} \ltimes \mathcal{M} \rtimes \mathfrak{H}$ the groupoid $(\mathfrak{G} \times \mathfrak{H}) \ltimes \mathcal{M}$. Check that an equivalent definition of the groupoid $\mathfrak{G} \ltimes \mathcal{M} \rtimes \mathfrak{H}$ is as the set $\{(g, x, h) \in \mathfrak{G} \times \mathcal{M} \times \mathfrak{H} : \mathbf{s}(g) = P_{\mathfrak{G}}(x), \mathbf{r}(h) = P_{\mathfrak{H}}(x)\}$ with the source and range maps

$$\mathbf{s}(g, x, h) = x \cdot h, \qquad \mathbf{r}(g, x, h) = g \cdot x$$

and multiplication

$$(g_1, g_2 \cdot x \cdot h_1^{-1}, h_1)(g_2, x, h_2) = (g_1 g_2, x \cdot h_1^{-1}, h_1 h_2).$$

3.10. Let $\mathfrak{G} \curvearrowright \underrightarrow{\mathcal{M}} \curvearrowleft \mathfrak{H}$ be a morphism. Prove that the projection $\mathfrak{G} \ltimes \mathcal{M} \rtimes \mathfrak{H} \longrightarrow \mathfrak{G} : (g, x, h) \mapsto g$ is an equivalence of groupoids. This shows that every morphism $\mathfrak{G} \curvearrowright \underrightarrow{\mathcal{M}} \curvearrowleft \mathfrak{H}$ is a composition of the equivalence $\mathfrak{G} \approx \mathfrak{G} \ltimes \mathcal{M} \rtimes \mathfrak{H}$ with the projection $\mathfrak{G} \ltimes \mathcal{M} \rtimes \mathfrak{H} \longrightarrow \mathfrak{H}$.

3.11. Let $\mathcal{X}$ be a connected and semi-locally simply connected space, and let $F : \tilde{\mathcal{X}} \longrightarrow \mathcal{X}$ be its universal covering. For a group action $G \curvearrowright \mathcal{X}$, let $\tilde{G}$ be the group of all lifts of the homeomorphisms $g \in G$ of $\mathcal{X}$ to $\tilde{\mathcal{X}}$. Prove that action groupoids $G_i \ltimes \mathcal{X}_i$ are equivalent as groupoids if and only if the lifts $\tilde{G}_i \curvearrowright \tilde{\mathcal{X}}_i$ to the universal coverings are topologically conjugate.

3.12. Let $\mathfrak{G}$ be a topological groupoid. Show that the set of functors $\{\cdot\} \longrightarrow \mathfrak{G}$ can be identified with $\mathfrak{G}^{(0)}$ so that the category of isomorphisms between the corresponding morphisms is isomorphic to $\mathfrak{G}$.

3.13. Let $f \circlearrowleft \mathcal{X}$ be a minimal homeomorphism of a Cantor set. Show that the mapping torus $\mathcal{T}$ of $f \circlearrowleft \mathcal{X}$ is a connected topological space, and that $\mathbb{R}$-orbits of the associated flow on $\mathcal{T}$ coincide with the path connected components of $\mathcal{T}$.

3.14. Prove that two minimal homeomorphisms of Cantor sets are Kakutani equivalent if and only if there is an orientation-preserving homeomorphisms of the associated mapping tori. Here a homeomorphism is said to be orientation preserving if it preserves the positive direction on the $\mathbb{R}$-orbits (which coincide with the path connected components of the mapping tori, see the previous problem).

3.15. Find the fundamental group of the groupoid generated by the set of germs of the angle doubling map on the circle.

3.16. Let $\mathfrak{G}$ be a path connected and locally simply connected étale groupoid, and let $\mathcal{X}_t$ be the space of homotopy classes of paths starting in $t \in \mathfrak{G}^{(0)}$.

Let $\mathfrak{G} \curvearrowright \mathcal{X}_t \curvearrowleft \pi_1(\mathfrak{G}, t)$ be the natural actions. Show that the groupoid $\mathfrak{G} \ltimes \mathcal{X}_t \curvearrowleft \pi_1(\mathfrak{G}, t)$ is equivalent to $\mathfrak{G}$. (See Exercise 3.10 for the definition of $\mathfrak{G} \ltimes \mathcal{X}_t \curvearrowleft \pi_1(\mathfrak{G}, t)$.)

3.17. Prove that if $\mathfrak{G}$ is a second-countable étale groupoid, then every closed transversal $T \subset \mathfrak{G}^{(0)}$ contains an open transversal. (Check...)

3.18. Show that the full graph shift is an étale groupoid.

3.19. Show that the isotropy group of (Γ, v) in the full graph shift is isomorphic to the automorphism group of Γ as a (non-rooted) labeled graph.

3.20. Let $S_x \subset \mathfrak{F}_{\mathsf{X}}$ be as in Proposition 3.5.15 for the tree subshift $\mathfrak{H} = \mathfrak{F}_{\mathsf{X}}$. Then the inverse semigroup generated by the elements S_x is free (as an inverse semigroup). (See ... for classical descriptions of free inverse semigroups.)

3.21. Let $\mathcal{X}$ be a compact topological space, and let $G \curvearrowright \mathcal{X}$ be a finitely generated group acting on $\mathcal{X}$. Prove that the following conditions are equivalent.

(a) The action $G \curvearrowright \mathcal{X}$ is expansive in the sense of Definition 1.2.5.
(b) The groupoid of the action $G \ltimes \mathcal{X}$ is expansive.
(c) The groupoid of the germs of the action $G \curvearrowright \mathcal{X}$ is expansive.

3.22. Groupoid of the action of a group on its Stone-Čech compactification: show that quasi-isometric groups have equivalent groupoids... Same for the coarse groupoid of a metric space...

Bibliography

[Abé05] Miklós Abért, *Group laws and free subgroups in topological groups*, Bull. London Math. Soc. **37** (2005), no. 4, 525–534.

[AH03] Valentin Afraimovich and Sze-Bi Hsu, *Lectures on chaotic dynamical systems*, AMS/IP Studies in Advanced Mathematics, vol. 28, American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2003.

[Bau93] Gilbert Baumslag, *Topics in combinatorial group theory*, Lectures in Mathematics, ETH Zürich, Birkhäuser Verlag, Basel, 1993.

[BB17] James Belk and Collin Bleak, *Some undecidability results for asynchronous transducers and the Brin-Thompson group* $2V$, Trans. Amer. Math. Soc. **369** (2017), no. 5, 3157–3172.

[BBM17] J. Belk, C. Bleak, and F. Matucci, *Rational embeddings of hyperbolic groups*, (preprint, arXiv:1711.08369), 2017.

[BCM$^+$16] Collin Bleak, Peter Cameron, Yonah Maissel, Andrés Navas, and Feyishayo Olukoya, *The further chameleon groups of richard thompson and graham higman: Automorphisms via dynamics for the higman groups* $g_{n,r}$, (preprint, arXiv:1605.09302), 2016.

[Bea91] Alan F. Beardon, *Iteration of rational functions. Complex analytic dynamical systems*, Graduate Texts in Mathematics, vol. 132, Springer-Verlag. New York etc., 1991.

[BGŠ03] Laurent Bartholdi, Rostislav I. Grigorchuk, and Zoran Šunić, *Branch groups*, Handbook of Algebra, Vol. 3, North-Holland, Amsterdam, 2003, pp. 989–1112.

[BH99] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften, vol. 319, Springer, Berlin, 1999.

[BMH17] J. Belk, F. Matucci, and James Hyde, *On the asynchronous rational group*, (preprint, arXiv:1711.01668), 2017.

[Bow70] Rufus Bowen, *Markov partitions for Axiom* A *diffeomorphisms*, Amer. J. Math. **92** (1970), 725–747.

[Bra72] Ola Bratteli, *Inductive limits of finite-dimensional* C^**-algebras*, Transactions of the American Mathematical Society **171** (1972), 195–234.

[BŠ01] Laurent Bartholdi and Zoran Šunić, *On the word and period growth of some groups of tree automorphisms*, Comm. Algebra **29** (2001), no. 11, 4923–4964.

[BS02] Michael Brin and Garrett Stuck, *Introduction to dynamical systems*, Cambridge University Press, Cambridge, 2002.

[CFP96] John W. Cannon, William I. Floyd, and Walter R. Parry, *Introductory notes on Richard Thompson groups*, L'Enseignement Mathematique **42** (1996), no. 2, 215–256.

[CN10] Julien Cassaigne and François Nicolas, *Factor complexity*, Combinatorics, automata and number theory, Encyclopedia Math. Appl., vol. 135, Cambridge Univ. Press, Cambridge, 2010, pp. 163–247.

[Dev89] Robert L. Devaney, *An introduction to chaotic dynamical systems*, second ed., Addison-Wesley Studies in Nonlinearity, Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1989.

[DHS99] F. Durand, B. Host, and C. Skau, *Substitutional dynamical systems, Bratteli diagrams and dimension groups*, Ergod. Th. and Dynam. Sys. **19** (1999), 953–993.

[DL06] David Damanik and Daniel Lenz, *Substitution dynamical systems: characterization of linear repetitivity and applications*, J. Math. Anal. Appl. **321** (2006), no. 2, 766–780.

[Eil74] Samuel Eilenberg, *Automata, languages and machines*, vol. A, Academic Press, New York, London, 1974.

[Fek23] M. Fekete, *über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten*, Math. Z. **17** (1923), no. 1, 228–249.

[Fer99] Sébastien Ferenczi, *Complexity of sequences and dynamical systems*, Discrete Math. **206** (1999), no. 1-3, 145–154, Combinatorics and number theory (Tiruchirappalli, 1996).

[Fer02] S. Ferenczi, *Substitutions and symbolic dynamical systems*, Substitutions in dynamics, arithmetics and combinatorics, Lecture Notes in Math., vol. 1794, Springer, Berlin, 2002, pp. 101–142.

[Fri83] David Fried, *Métriques naturelles sur les espaces de Smale*, C. R. Acad. Sci. Paris Sér. I Math. **297** (1983), no. 1, 77–79.

[Fri87] ______, *Finitely presented dynamical systems*, Ergod. Th. Dynam. Sys. **7** (1987), 489–507.

[GN05] R. I. Grigorchuk and V. V. Nekrashevych, *Amenable actions of nonamenable groups*, Zapiski Nauchnyh Seminarov POMI **326** (2005), 85–95.

[GNS00] Rostislav I. Grigorchuk, Volodymyr V. Nekrashevich, and Vitaliĭ I. Sushchanskii, *Automata, dynamical systems and groups*, Proceedings of the Steklov Institute of Mathematics **231** (2000), 128–203.

[GPS95] Thierry Giordano, Ian F. Putnam, and Christian F. Skau, *Topological orbit equivalence and C^*-crossed products*, Journal für die reine und angewandte Mathematik **469** (1995), 51–111.

[Gri80] Rostislav I. Grigorchuk, *On Burnside's problem on periodic groups*, Functional Anal. Appl. **14** (1980), no. 1, 41–43.

[Gri85] ______, *Degrees of growth of finitely generated groups and the theory of invariant means*, Math. USSR Izv. **25** (1985), no. 2, 259–300.

[Gri73] Christian Grillenberger, *Constructions of strictly ergodic systems. I. Given entropy*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **25** (1972/73), 323–334.

[Gro81] Mikhael Gromov, *Groups of polynomial growth and expanding maps*, Publ. Math. I. H. E. S. **53** (1981), 53–73.

[Hir70] Morris W. Hirsch, *Expanding maps and transformation groups*, Global Analysis, Proc. Sympos. Pure Math., vol. 14, American Math. Soc., Providence, Rhode Island, 1970, pp. 125–131.

[HP09] Peter Haïssinsky and Kevin M. Pilgrim, *Coarse expanding conformal dynamics*, Astérisque (2009), no. 325, viii+139 pp.

[HPS92] Richard H. Herman, Ian F. Putnam, and Christian F. Skau, *Ordered Bratteli diagrams, dimension groups, and topological dynamics*, Intern.J̃. Math. **3** (1992), 827–864.

[Kai01] Vadim A. Kaimanovich, *Equivalence relations with amenable leaves need not be amenable*, Topology, Ergodic Theory, Real Algebraic Geometry. Rokhlin's Memorial, Amer. Math. Soc. Transl. (2), vol. 202, 2001, pp. 151–166.

[Kro84] L. Kronecker, *Näherunsgsweise ganzzahlige auflösung linearer gleichungen*, Monatsberichte Königlich Preussischen Akademie der Wissenschaften zu Berlin (1884), 1179–1193, 1271–1299.

[Ku03] Petr K ˚ urka, *Topological and symbolic dynamics*, Cours Spécialisés [Specialized Courses], vol. 11, Société Mathématique de France, Paris, 2003.

[LN02] Yaroslav V. Lavreniuk and Volodymyr V. Nekrashevych, *Rigidity of branch groups acting on rooted trees*, Geom. Dedicata **89** (2002), no. 1, 155–175.

[LY75] T. Y. Li and James A. Yorke, *Period three implies chaos*, Amer. Math. Monthly **82** (1975), no. 10, 985–992.

[Lys85] Igor G. Lysionok, *A system of defining relations for the Grigorchuk group*, Mat. Zametki **38** (1985), 503–511.

[Mat12] Hiroki Matui, *Homology and topological full groups of étale groupoids on totally disconnected spaces*, Proc. Lond. Math. Soc. (3) **104** (2012), no. 1, 27–56.

[Mat16] ______, *Étale groupoids arising from products of shifts of finite type*, Adv. Math. **303** (2016), 502–548.

[MH38] M. Morse and G. A. Hedlund, *Symbolic dynamics*, American Journal of Mathematics **60** (1938), no. 4, 815–866.

[Mil06] John Milnor, *Dynamics in one complex variable*, third ed., Annals of Mathematics Studies, vol. 160, Princeton University Press, Princeton, NJ, 2006.

[Mor21] Harold Marston Morse, *Recurrent geodesics on a surface of negative curvature*, Trans. Amer. Math. Soc. **22** (1921), no. 1, 84–100.

[MRW87] Paul S. Muhly, Jean N. Renault, and Dana P. Williams, *Equivalence and isomorphism for groupoid C^*-algebras*, J. Oper. Theory **17** (1987), 3–22.

[Nek05] Volodymyr Nekrashevych, *Self-similar groups*, Mathematical Surveys and Monographs, vol. 117, Amer. Math. Soc., Providence, RI, 2005.

[Nek14] ______, *Combinatorial models of expanding dynamical systems*, Ergodic Theory and Dynamical Systems **34** (2014), 938–985.

[Nek18] ______, *Palindromic subshifts and simple periodic groups of intermediate growth*, Annals of Math. **187** (2018), no. 3, 667–719.

[Ply74] R. V. Plykin, *Sources and sinks of* A-*diffeomorphisms of surfaces*, Mat. Sb. (N.S.) **94(136)** (1974), 243–264, 336.

[Pro51] E. Prouhet, *Mémoire sur quelques relations entre les puissances des nombres*, C. R. Acad. Sci. Paris **33** (1851), 225.

[PS72] G. Polya and G. Szego, *Problems and theorems in analysis, volume I*, Springer, 1972.

[Que87] Martine Queﬄec, *Substitution dynamical systems - spectral analysis*, Lecture Notes in Mathematics, vol. 1294, Berlin etc.: Springer-Verlag, 1987.

[Röv99] Claas E. Röver, *Constructing finitely presented simple groups that contain Grigorchuk groups*, J. Algebra **220** (1999), 284–313.

[Roz86] A. V. Rozhkov, *On the theory of groups of Aleshin type*, Mat. Zametki **40** (1986), no. 5, 572–589, 697. MR 886178

[Rub89] Matatyahu Rubin, *On the reconstruction of topological spaces from their groups of homeomorphisms*, Trans. Amer. Math. Soc. **312** (1989), no. 2, 487–538.

[Rue78] D. Ruelle, *Thermodynamic formalism*, Addison Wesley, Reading, 1978.

[Shu69] Michael Shub, *Endomorphisms of compact differentiable manifolds*, Am. J. Math. **91** (1969), 175–199.

[Shu70] ———, *Expanding maps*, Global Analysis, Proc. Sympos. Pure Math., vol. 14, American Math. Soc., Providence, Rhode Island, 1970, pp. 273–276.

[Sid00] Said N. Sidki, *Automorphisms of one-rooted trees: growth, circuit structure and acyclicity*, J. of Mathematical Sciences (New York) **100** (2000), no. 1, 1925–1943.

[Sma67] S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–817.

[Šun07] Zoran Šunić, *Hausdorff dimension in a family of self-similar groups*, Geometriae Dedicata **124** (2007), 213–236.

[Tho80] Richard J. Thompson, *Embeddings into finitely generated simple groups which preserve the word problem*, Word Problems II (S. I. Adian, W. W. Boone, and G. Higman, eds.), Studies in Logic and Foundations of Math., 95, North-Holand Publishing Company, 1980, pp. 401–441.

[Thu12] A. Thue, *über die gegenseitige lage gleicher teile gewisser zeichenreihen*, Norske vid. Selsk. Skr. Mat. Nat. Kl. **1** (1912), 1–67.

[Vor12] Yaroslav Vorobets, *Notes on the Schreier graphs of the Grigorchuk group*, Dynamical systems and group actions (L. Bowen et al., ed.), Contemp. Math., vol. 567, Amer. Math. Soc., Providence, RI, 2012, pp. 221–248.

[Wie14] Susana Wieler, *Smale spaces via inverse limits*, Ergodic Theory Dynam. Systems **34** (2014), no. 6, 2066–2092.

[Wil67] R. F. Williams, *One-dimensional non-wandering sets*, Topology **6** (1967), 473–487.

[Wil74] ———, *Expanding attractors*, Inst. Hautes Études Sci. Publ. Math. (1974), no. 43, 169–203.

[Yi01] Inhyeop Yi, *Canonical symbolic dynamics for one-dimensional generalized solenoids*, Trans. Amer. Math. Soc. **353** (2001), no. 9, 3741–3767.