Groups and topological dynamics

Volodymyr Nekrashevych

E-mail address: nekrash@math.tamu.edu

2010 *Mathematics Subject Classification.* Primary ... ;
Secondary ...

Key words and phrases. ....

The author ...

Abstract. ...

Contents

Chapter 1. Dynamical systems 1

§1.1. Introduction by examples 1

§1.2. Subshifts 21

§1.3. Minimal Cantor systems 36

§1.4. Hyperbolic dynamics 74

§1.5. Holomorphic dynamics 103

Exercises 111

Chapter 2. Group actions 117

§2.1. Structure of orbits 117

§2.2. Localizable actions and Rubin's theorem 136

§2.3. Automata 149

§2.4. Groups acting on rooted trees 163

Exercises 194

Chapter 3. Groupoids 201

§3.1. Basic definitions 201

§3.2. Actions and correspondences 214

§3.3. Fundamental groups 233

§3.4. Orbispaces and complexes of groups 238

§3.5. Compactly generated groupoids 243

§3.6. Hyperbolic groupoids 252

Exercises 252

Chapter 4. Iterated monodromy groups 255
§4.1. Iterated monodromy groups of self-coverings 255
§4.2. Self-similar groups 265
§4.3. General case 274
§4.4. Expanding maps and contracting groups 291
§4.5. Thurston maps and related structures 311
§4.6. Iterations of polynomials 333
§4.7. Functoriality 334
Exercises 340

Chapter 5. Groups from groupoids 345
§5.1. Full groups 345
§5.2. AF groupoids and torsion groups 364
§5.3. Torsion groups 381
§5.4. Homology of totally disconnected étale groupoids 410
§5.5. Almost finite groupoids 423
§5.6. Purely infinite groupoids 430
Exercises 432

Chapter 6. Growth and amenability 437
§6.1. Growth of groups 437
§6.2. Groups of intermediate growth 437
§6.3. Inverted orbits 453
§6.4. Inverted orbits and bounded automata 456
§6.5. Growth of fragmentations of D_∞ 460
§6.6. Non-uniform exponential growth 469
§6.7. Amenability 470
Exercises 475

Bibliography 477

Chapter 4

Iterated monodromy groups

4.1. Iterated monodromy groups of self-coverings

4.1.1. Definition. A *partial self-covering* of a topological space $\mathcal{X}$ is a covering map $f : \mathcal{X}_1 \longrightarrow \mathcal{X}$, where $\mathcal{X}_1$ is a subset of $\mathcal{X}$. Partial self-coverings can be iterated in the usual way (as partial maps). We denote by $f^n : \mathcal{X}_n \longrightarrow \mathcal{X}$ the nth iteration of f. Here $\mathcal{X}_n$ is the domain of f^n and is defined inductively as $\mathcal{X}_{n+1} = f^{-1}(\mathcal{X}_n)$. Note that we have $\mathcal{X}_{n+1} \subset \mathcal{X}_n$ and that $f^n : \mathcal{X}_n \longrightarrow \mathcal{X}$ are also covering maps.

Let $t \in \mathcal{X}$, and consider the formal disjoint union $T_t = \bigsqcup_{n\geqslant 0} f^{-n}(t)$, where $f^{-0}(t) = \{t\}$. The set T_t has a natural structure of a rooted tree with the root $t \in f^{-0}(t)$ in which a vertex $v \in f^{-(n+1)}(t)$ is connected to the vertex $f(v) \in f^{-n}(t)$. If $|f^{-1}(x)|$ does not depend on x (e.g., if $\mathcal{X}$ is connected), then T_t is a regular tree of degree equal to the degree of the covering f. We call the rooted tree T_t the *tree of preimages* of t.

Suppose that $\mathcal{X}$ is path connected, and let γ be a path from t_1 to $t_2 \in \mathcal{X}$. Then for every $n \geqslant 1$ and every $v \in f^{-n}(t_1)$ there exists a unique lift of γ by f^n starting at v. Denote by $S_\gamma(v)$ the end of the lift, see Figure 4.1.

Proposition 4.1.1. *The map S_γ is an isomorphism from T_{t_1} to T_{t_2}. It depends only on the homotopy class of the path γ.*

Proof. .... □

It follows directly from the definitions that $S_{\gamma_1} \circ S_{\gamma_2} = S_{\gamma_1\gamma_2}$ if the end of γ_2 is equal to the beginning of γ_1, and we multiply the paths in the same

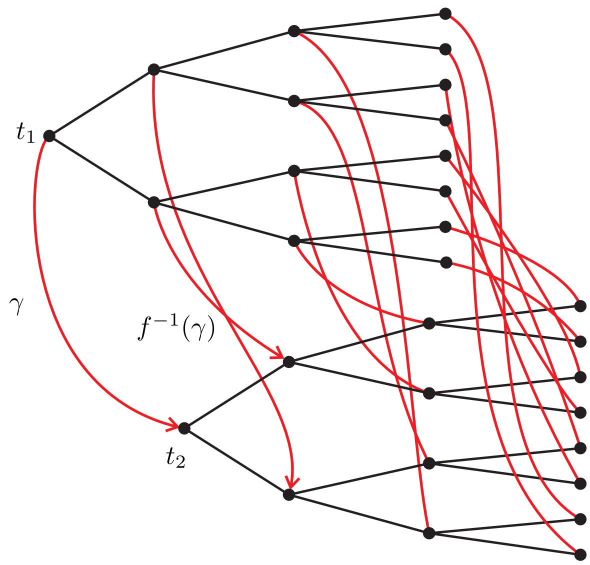

Figure 4.1. The map S_γ

order as we multiply the functions: in a product $\gamma_1\gamma_2$ the path γ_2 is traversed before the path γ_1.

In particular, we get an action of the fundamental group $\pi_1(\mathcal{X}, t)$ by automorphisms of the tree T_t, i.e., a homomorphism $[\gamma] \mapsto S_\gamma$ from $\pi_1(\mathcal{X}, t)$ to $\operatorname{Aut} T_t$. The action is called the *iterated monodromy action*, and the image of $\pi_1(\mathcal{X}, t)$ in $\operatorname{Aut} T_t$ is called the *iterated monodromy group* of the map f, denoted $\mathrm{IMG}\,(f)$.

Example 4.1.2. Consider the double self-covering $x \mapsto 2x$ of the circle $\mathbb{R}/\mathbb{Z}$, see 1.1.2. Take $0 \in \mathbb{R}/\mathbb{Z}$ as the basepoint. The fundamental group is generated by the loop a equal to the image of $[0, 1]$ with the increasing orientation. For every n the lifts of a by f^n are the images in $\mathbb{R}/\mathbb{Z}$ of the arcs of the form $\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]$ for $k = 0, 1, \ldots, 2^n - 1$. It follows that a acts on the nth level of the tree of preimages of $t = 0$ as a transitive cycle. Consequently, see Theorem 2.4.7, the action of a on T_t is conjugate to the adding machine action. The iterated monodromy group is the infinite cyclic group (together with the level-transitive action on T_t by the adding machine transformation).

We will see later that the above example is not typical in the sense that $\mathrm{IMG}\,(f)$ is usually very different from the fundamental group.

4.1.2. Standard action. At the moment the tree T_t is just an abstract rooted tree. We would like to identify it with the tree of words X^* over some finite alphabet X, see 2.4.1. Equivalently, we would like to represent T_t as the right Cayley graph of the free monoid generated by X. It is enough to choose a bijection Λ from X to the first level of the tree T_t, and define a collection of isomorphisms $S_x : T_t \longrightarrow T_{\Lambda(x)}$ from t to the subtrees rooted at the vertices of the first level. These isomorphisms will correspond to the maps $S_x : \mathsf{X}^* \longrightarrow \mathsf{X}^*$, $x \in \mathsf{X}$, acting by the rule $S_x(v) = xv$. Then the vertex $\Lambda(v)$ of T_t corresponding to a word $v = x_1 x_2 \dots x_n$ can be defined as the image of the root of the tree under the composition $S_v = S_{x_1} \circ S_{x_2} \circ \dots \circ S_{x_n}$. The map S_v is an isomorphism from T_t to $T_{\Lambda(v)}$. Since t is adjacent to each $\Lambda(x) = S_x(t)$, and S_v is an isomorphism, the vertex $\Lambda(v) = S_v(t)$ will be adjacent to the vertices $\Lambda(vx) = \Lambda(v)(S_x(t))$, hence $\Lambda : \mathsf{X}^* \longrightarrow T_t$ will be an isomorphism.

If v is a vertex of the first level of the tree of preimages T_t, then the subtree T_v coincides with the tree of preimages of v (which is also denoted by T_v). Then a natural choice of an isomorphism $S_v : T_t \longrightarrow T_v$ is the isomorphism S_ℓ for a path ℓ in $\mathcal{X}$ connecting t to v. We get the following class of natural identifications of T_t with a tree of words X^*.

Definition 4.1.3. Let X be an alphabet of cardinality $\deg f$. Let $\Lambda : \mathsf{X} \longrightarrow f^{-1}(t)$ be any bijection of X with the first level of the tree T_t. Choose for every $x \in \mathsf{X}$ a path ℓ_x starting in t and ending in $\Lambda(x)$. Define a map $\Lambda : \mathsf{X}^* \longrightarrow T_t$ setting $\Lambda(x_1 x_2 \cdots x_n)$ to be the image of t under the composition $S_{\ell_{x_1}} \circ S_{\ell_{x_2}} \circ \dots \circ S_{\ell_{x_n}}$.

The following proposition gives an alternative definition of the map Λ.

Proposition 4.1.4. *Let $\Lambda : \mathsf{X} \longrightarrow f^{-1}(t)$ and ℓ_x be as in Definition 4.1.3. Define, for $x_1 x_2 \dots x_n \in \mathsf{X}^*$, the path $\ell_{x_1 x_2 \dots x_n}$ inductively by the condition $\ell_{x_1 x_2 \dots x_n} = \gamma \ell_{x_n}$, where γ is the lift of the path $\ell_{x_1 x_2 \dots x_{n-1}}$ by f to a path starting in $\Lambda(x_n)$. Then $\ell_{x_1 x_2 \dots x_n}$ is a path starting in t and ending in $\Lambda(x_1 x_2 \dots x_n)$.*

The path $\ell_{x_1 x_2 \cdots x_n}$ is equal to the concatenation $\lambda_1 \lambda_2 \dots \lambda_n$, where λ_i is the lift of ℓ_{x_i} by f^{n-i} starting at the end of λ_{i+1}.

Proof. By definition, the isomorphism $\Lambda : \mathsf{X}^* \longrightarrow T_t$ satisfies $\Lambda(xv) = S_{\ell_x}(\Lambda(v))$. If $v \in \mathsf{X}^{n-1}$, then $\Lambda(v) \in f^{-(n-1)}(t)$, and $S_{\ell_x}(\Lambda(v))$ is defined as the end of the lift of ℓ_x by f^{n-1} starting at $\Lambda(v)$. It follows now by induction that $\Lambda(x_1 x_2 \dots x_n)$ is the end of the path of the form $\lambda_1 \lambda_2 \dots \lambda_n$, where λ_i is the lift of ℓ_{x_i} by f^{n-i}. $\square$

The lifts of the paths ℓ_x, $x \in \mathsf{X}$, by iterations of f form a tree with the same set of vertices as T_t, but connecting them in a different way. Whereas in T_t a vertex $\Lambda(x_1 x_2 \dots x_n)$ is connected to the vertex $\Lambda(x_1 x_2 \dots x_{n-1})$, in the tree formed by the lifts of ℓ_x the vertex $\Lambda(x_1 x_2 \dots x_n)$ is connected to the vertex $\Lambda(x_2 x_3 \dots x_n)$. In other words, the map Λ identifies T_t with the

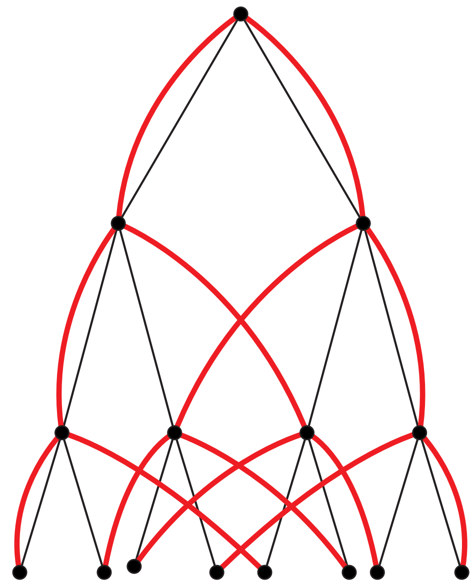

Figure 4.2. The tree of preimages and the tree of lifts of the paths ℓ_x

right Cayley graph of the free monoid, whereas the tree formed by the lifts of ℓ_x is identified by Λ with the left Cayley graph, see Figure 4.2.

Let us fix an alphabet X, a bijection $\Lambda : \mathsf{X} \longrightarrow f^{-1}(t)$, and a collection of paths ℓ_x, as in Definition 4.1.3. Let us conjugate the iterated monodromy action of the fundamental group $\pi_1(\mathcal{X}, t)$ by Λ, thus obtaining an action of the fundamental group (and of the iterated monodromy group) on the tree X^*. We call such actions *standard*.

Let γ be an element of the fundamental group $\pi_1(\mathcal{X}, t)$. Let $x \in \mathsf{X}$ and $v \in \mathsf{X}^*$ be arbitrary, and let $y \in \mathsf{X}$ be the image of x under the standard action of γ, i.e., such that $\Lambda(y) = S_\gamma(\Lambda(x))$. Suppose that $u \in \mathsf{X}^*$ is such that yu is the image of xv under the action of g.

We have then

$$S_\gamma(\Lambda(xv)) = S_\gamma S_{\ell_x}(\Lambda(v)) = S_{\ell_y}(\Lambda(u)),$$

hence $\Lambda(u) = S_{\ell_y}^{-1} S_\gamma S_{\ell_x}$. It follows from the definition of the maps S_* that $S_{\ell_y}^{-1} S_\gamma S_{\ell_x}$ is equal to S_δ, where δ is the path $\ell_y^{-1}\gamma_x \ell_x$, where γ_x is the lift of γ by f starting at $\Lambda(y)$. See Figure 4.3, where the path δ is shown.

We get hence the following description of the standard actions of $\mathrm{IMG}\,(f)$ on X^*.

Proposition 4.1.5. *Consider the standard action of* $\mathrm{IMG}\,(f)$ *on* X^* *defined by a collection of paths* ℓ_x, $x \in \mathsf{X}$. *Let* $g \in \mathrm{IMG}\,(f)$ *be an element defined by*

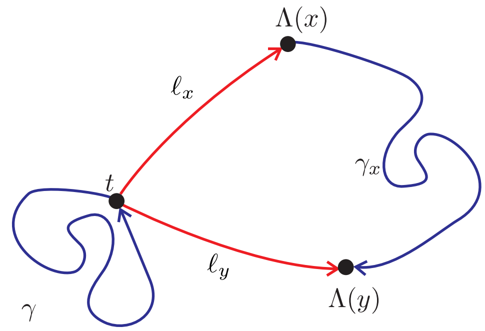

Figure 4.3. The standard action of IMG (f)

a loop $\gamma \in \pi_1(\mathcal{X}, t)$, and denote by γ_x the lift of the loop γ by f starting in the end of ℓ_x. Then, for every $v \in \mathsf{X}^*$ we have

$$g(xv) = yh(v),$$

where $y = g(x)$ and h is defined by the loop $\ell_y^{-1}\gamma_x\ell_x$.

The recurrent formula from Proposition 4.1.5 is a description of the automaton generating the standard action of the iterated monodromy group. If the automaton is in the state defined by a loop γ, and reads on the input a letter $x \in \mathsf{X}$, then we find the lift γ_x of γ by f starting in $\Lambda(x)$. The output letter is $y \in \mathsf{X}$ such that the end of γ_x is $\Lambda(y)$, and then the next state is defined by the loop $\ell_y^{-1}\gamma_x\ell_x$.

The following is a direct corollary of Propositions 4.1.5 and 4.1.4.

Proposition 4.1.6. *Let f, Λ, ℓ_x be as in Proposition 4.1.5. For $v = x_1x_2\ldots x_n$ denote by ℓ_v the path of lifts of the paths ℓ_{x_i} connecting the root $t = \Lambda(\varnothing)$ to the vertex $\Lambda(v)$, as in Proposition 4.1.4. Let γ_v be the lift of γ by f^n starting in $\Lambda(v)$, and let $\Lambda(u)$ for $u \in \mathsf{X}^n$ be the end of γ_v. Then for every $w \in \mathsf{X}^*$ we have*

$$\gamma(vw) = u(\ell_u^{-1}\gamma_v\ell_v)(w)$$

for the corresponding standard action of the iterated monodromy group on X^*.

Recall that here $\ell_{x_1x_2\ldots x_n} = \lambda_1\lambda_2\ldots\lambda_n$, where λ_i is the lift of ℓ_{x_i} by f^{n-i} starting at the end of λ_{i+1}.

4.1.3. Some examples.

4.1.3.1. *The double self-covering of the circle.* Consider the map $f : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}$ given by $f(x) = 2x$. See a discussion of its dynamical properties in 1.1.2. It is a self-covering map. Let us describe a standard action of IMG (f). Take $t = 0$ as the basepoing. We have then $f^{-1}(0) = \{0, 1/2\}$. Take $\mathsf{X} = \{0, 1\}$,

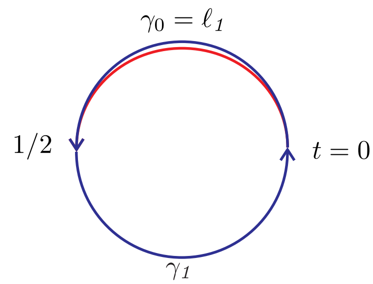

Figure 4.4. The iterated monodromy group of the double self-covering of the circle

and $\Lambda(0) = 0$, $\Lambda(1) = 1/2$. A natural choice of the connecting paths ℓ_0 and ℓ_1 is to take ℓ_0 to be the trivial path at 0, and ℓ_1 to be the image of the interval $[0, 1/2]$ in $\mathbb{R}/\mathbb{Z}$. The fundamental group $\pi_1(\mathbb{R}/\mathbb{Z}, 0)$ is generated by the loop γ equal to the image of the path $[0, 1]$ from 0 to 1 in $\mathbb{R}/\mathbb{Z}$. Let a be the image of this generator in IMG (f). The lifts of the generator by f are the paths $\gamma_0 = [0, 1/2]$ and $\gamma_1 = [1/2, 1]$. It follows that the standard action of a on the tree X^* is given by the recurrent rules

$$a(0v) = 1v, \qquad a(1v) = 0a(v),$$

since $\ell_1^{-1}\gamma_0\ell_0$ is trivial, and $\ell_0^{-1}\gamma_1\ell_1$ is homotopic to γ, see Figure 4.4.

We see that the standard action of IMG (f) coincides with the odometer action of $\mathbb{Z}$ on the binary rooted tree, see 1.1.4 and Example 2.4.12. See the Moore diagram of the automaton defining the transformation a on Figure 2.16.

The recursion defining the transformation a is written in the wreath product notation (see...) as

$$a = \sigma(1, a),$$

where σ is the transposition $(0\,1)$.

4.1.3.2. *Post-critically finite rational functions.* Let $f(z)$ be a complex rational function, seen as a map $\widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ of the Riemann sphere to itself. If z is not a critical point, then f is a homeomorphism from a neighborhood of z to a neighborhood of $f(z)$. We say that a point $p \in \widehat{\mathbb{C}}$ is *post-critical* if it is equal to $f^n(c)$ for some critical point c and $n \geqslant 1$. The function is said to be *post-critically finite* if the set of its post-critical points is finite. For dynamical properties of post-critically finite rational functions, see 1.5.4.

Let f be a post-critically finite rational function, and let P_f be its post-critical set. Since $f(P_f) \subset P_f$, and P_f contains all critical values of f, the map

$$f : \widehat{\mathbb{C}} \setminus f^{-1}(P_f) \longrightarrow \widehat{\mathbb{C}} \setminus P_f$$

is a partial self-covering of a punctured sphere. The iterated monodromy group of f is, by definition, the iterated monodromy group of this partial self-covering.

4.1.3.3. *Julia's example.* As an example, consider the polynomial $f(z) = -\frac{z^3}{2} + \frac{3z}{2}$. Its derivative is $-3z^2 + 3$, hence its critical points are ± 1 and ∞ (the latter is a critical point, since f is a polynomial). Note that all three critical points are fixed under the action of f. It follows that f is post-critically finite with the post-critical set $P_f = \{1, -1, \infty\}$. The space $\widehat{\mathbb{C}} \setminus P_f$ is the complex plane $\mathbb{C}$ with two punctures at ± 1. Let us take the basepoint equal to the third fixed point $t = 0$ of the polynomial f. The fundamental group of the punctured plane, and hence the iterated monodromy group $\mathrm{IMG}(f)$ is generated by a loop around 1 and a loop around -1, which we denote by a and b, respectively, as it is shown on the bottom half of Figure 4.5. We have $f^{-1}(0) = \{0, \sqrt{3}, -\sqrt{3}\}$.

Let us take $\mathsf{X} = \{0, 1, 2\}$, $\Lambda(0) = 0$, $\Lambda(1) = \sqrt{3}$, and $\Lambda(2) = -\sqrt{3}$. Choose the connecting path ℓ_0 to be the trivial path at t, and the paths ℓ_1, ℓ_2 as it is shown on Figure 4.5.

The preimages of the loops a and b by f are shown on the top half of Figure 4.5. Tracing the paths, we see that the corresponding standard action of $\mathrm{IMG}(f)$ is given by the recurrent formulas

$$a(0v) = 1v, \quad a(1v) = 0a(v), \quad a(2v) = 2v$$

and

$$b(0v) = 2v, \quad b(1v) = 1v, \quad b(2v) = 0b(v),$$

or, in the wreath product notation:

$$a = (0\ 1)(1, a, 1), \qquad b = (0\ 2)(1, 1, b).$$

We see that a and b act as odometers on the binary subtrees $\{0, 1\}^*$ and $\{0, 2\}^*$, and "ignore" the words containing the third letter. See the Moore diagram of the automaton describing the action of the generators a, b on Figure 4.6.

Unlike in the previous example, the iterated monodromy group $\mathrm{IMG}\left((-z^3 + 3z)/2\right)$ is not isomorphic to the fundamental group $\pi_1(\mathcal{X})$ (in other words, the corresponding iterated monodromy action of the fundamental group is not faithful). Iterated monodromy group usually possess rather exotic properties compared with the classical fundamental groups. For example, we will see later that $\mathrm{IMG}\left((-z^3 + 3z)/2\right)$ and similar iterated monodromy groups have no non-commutative free subgroups, are not finitely presented, and are non-elementary amenable.

Figure 4.7 shows the graphs of the action of the group $\mathrm{IMG}\left((-z^3 + 3z)/2\right)$ on the levels one through four of the tree.

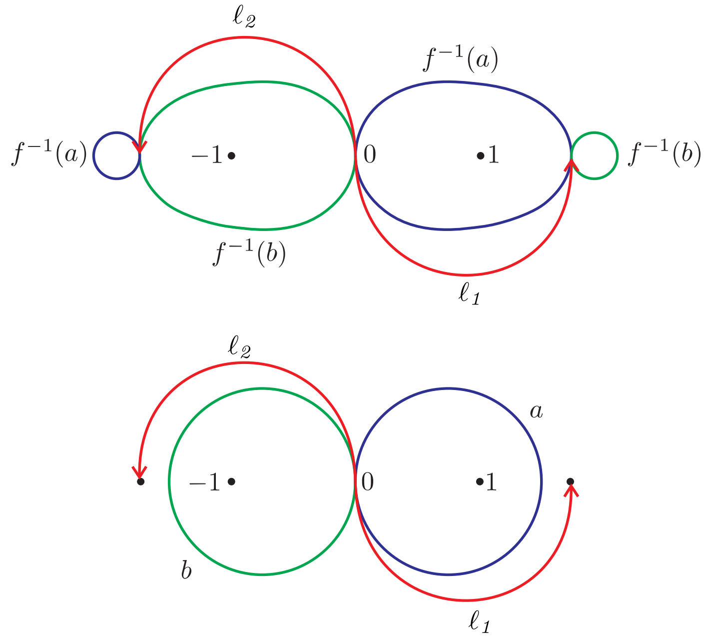

Figure 4.5. Computing IMG $\left((-z^3 + 3z)/2\right)$

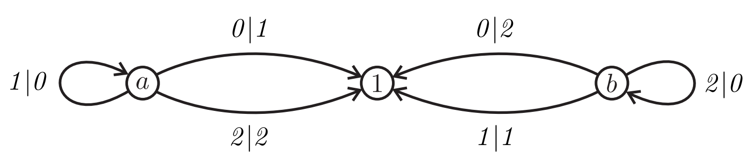

Figure 4.6. The iterated monodromy group of $(-z^3 + 3z)/2$

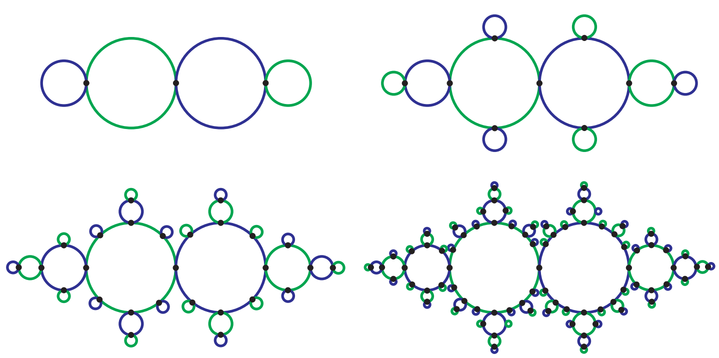

Figure 4.7. Graphs of the action of IMG $\left((-z^3 + 3z)/2\right)$ on the levels of the tree

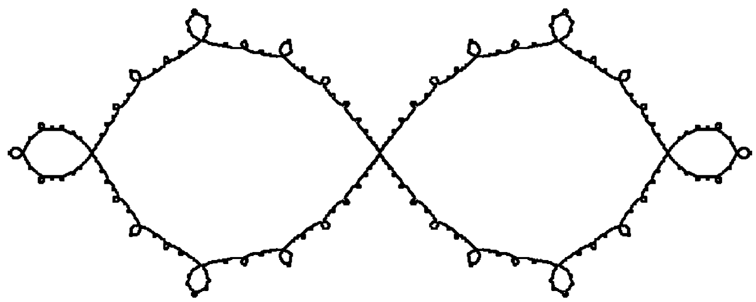

Figure 4.8. The Julia set of $(-z^3 + 3z)/2$

Compare the graphs of the action with the Julia set of the polynomial shown on Figure 4.8.

4.1.3.4. *Basilica.* Another famous example is the iterated monodromy group of the polynomial $z^2 - 1$. It is also post-critically finite: its unique finite critical point 0 belongs to a cycle of length 2:

$$0 \mapsto -1 \mapsto 0.$$

The iterated monodromy group is hence generated by the loops around 0 and -1. It is checked directly, see Figure 4.9, that a standard action of $\mathrm{IMG}\left(z^2 - 1\right)$ is generated by

$$a = \sigma(1, b), \qquad b = (1, a),$$

where $\sigma = (\mathsf{0}\ \mathsf{1})$.

The automaton defining the transformations a and b is shown on Figure 2.22.

See the graphs of the action of $\mathrm{IMG}\left(z^2 - 1\right)$ on the levels of the tree on Figurefig:basilicagraph. The Julia set of $z^2 - 1$ is shown on Figure 1.33.

4.1.3.5. *Chebyshev polynomials.* Chebyshev polynomials T_d are defined recurrently by

$$T_0(x) = 1, \qquad T_1(x) = x,$$

and

$$T_{d+1}(x) = 2xT_d(x) - T_{d-1}(x).$$

They can be also defined by

$$T_d(x) = \cos(d \arccos x) = \frac{1}{2}\left(\left(x + \sqrt{x^2 - 1}\right)^d + \left(x - \sqrt{x^2 - 1}\right)^d\right),$$

where $|x| \leqslant 1$ in the first formula, and $|x| \geqslant 1$ in the second.

They were introduced by P. Chebyshev in ... in relation to problems of approximation theory (explain more...) They were known before at least to L. Euler, see.....

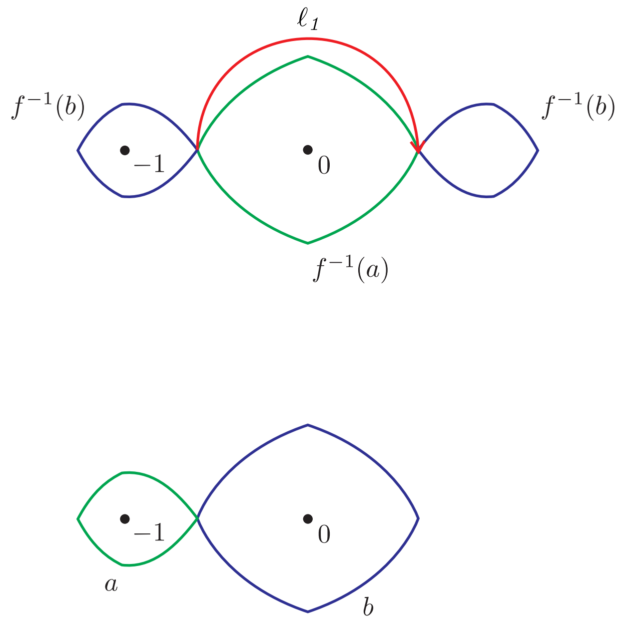

Figure 4.9. Computation of IMG $(z^2 - 1)$

The Chebyshev polynomials satisfy $T_{d_1} \circ T_{d_2} = T_{d_1+d_2}$, as it is easily seen from the formula $T_d(x) = \cos(d \arccos x)$.

We have

$$T_d'(\cos\theta) = \frac{d \sin d\theta}{\sin\theta},$$

hence the critical points of T_d are $\cos\frac{\pi m}{d}$ for $m = 1, 2, \ldots, d-1$, and the critical values are $\{\cos\pi m \; : \; m = 1, 2, \ldots, d-1\}$, which is equal to $1, -1$ for $d \geqslant 3$ and $\{-1\}$ for $d = 2$. It follows that the post-critical set of T_d for $d \geqslant 2$ is $\{1, -1\}$.

Take $t = 0$ as the basepoint, and let a, b be small loops around -1 and 1, respectively, connected to the basepoint by straight lines.

We have $T_d^{-n}(0) = T_{d^n}^{-1}(0) = \left\{\cos\frac{\pi+2l\pi}{2d^n} \; : \; l = 0, 1, \ldots, d^n - 1\right\}$. In other words, $T_d^{-n}(0)$ is the set of points obtained by projecting onto the real axis the vertices of the regular $2d^n$-gon inscribed into the unit circle so that the real axis is a non-diagonal axis of symmetry. The critical values of T_{d^n} are obtained by projecting the vertices of the regular $2d^n$-gon inscribed in the unit circle so that the real axis is a diagonal.

The preimages of the generators a and b form a chain, and we can index the vertices of the nth level of the tree by $0, 1, \ldots, d^n - 1$ (from $x = 1$ in the decreasing order to $x = -1$) so that a acts as the permutation $\alpha = (0\ 1)(2\ 3)\ldots$, and b acts as $\beta = (1\ 2)(3\ 4)\ldots$. In particular, this is true for the first level, and one can check that the standard action of $\mathrm{IMG}(T_d)$ for even d is given by the recursion

$$a = \alpha(1, 1, \dots, 1), \qquad b = \beta(b, 1, 1, \dots, 1, a)$$

and for odd d by the recursion

$$a = \alpha(1, 1, \dots, 1, a), \qquad b = \beta(b, 1, 1, \dots, 1)$$

In particular, $\mathrm{IMG}(T_2)$ is generated by

$$a = \sigma(1, a), \qquad b = (b, 1),$$

where, as usual, $\sigma = (0\,1)$.

It is easy to see from the structure of the graphs of the action of $\mathrm{IMG}(T_d)$ on the tree that $\mathrm{IMG}(T_d)$ is isomorphic to the infinite dihedral group.

4.2. Self-similar groups

4.2.1. Bisets. We have seen in Proposition 4.1.5 that the standard action of an iterated monodromy group on X^* is self-similar in the sense of Definition 2.4.24: for every $g \in \mathrm{IMG}(f)$ and $x \in \mathsf{X}$ there exist $h \in \mathrm{IMG}(f)$ and $y \in \mathsf{X}$ such that $g(xw) = yh(w)$ for all $w \in \mathsf{X}^*$.

The standard action depends on the choice of the bijection $f^{-1}(t) \longrightarrow \mathsf{X}$ and the choice of the connecting paths ℓ_x, so it is natural to seek a more canonical object. In particular, we would like to understand how the standard action changes after a change of the connecting paths.

Let G be a self-similar group acting on X^*. Consider the set $\mathsf{X} \cdot G$ of transformations $S_{x \cdot g} : v \mapsto xg(v)$ of X^*. The transformation $S_{x \cdot g}$ is an isomorphism of X^* with the subtree $x\mathsf{X}^*$ of words starting with x.

The set $\mathsf{X} \cdot G = \{S_{x \cdot g} \ : \ x \in \mathsf{X}, g \in G\}$ is invariant under the pre- and post-compositions with the action of G. Namely, for every $x \cdot g \in \mathsf{X} \cdot G$ and $h \in G$ we have $(x \cdot g) \circ h = x \cdot (gh)$ and $h \circ (x \cdot g) = y \cdot (h|_x g)$, where $y \in \mathsf{X}$ and $h|_x \in G$ are such that $h(xw) = yh|_x(w)$ for all $w \in \mathsf{X}^*$. We get two commuting left and right actions $G \curvearrowright \mathsf{X} \cdot G \curvearrowleft G$ of the group G on the set $\mathsf{X} \cdot G$.

In the case when $G \curvearrowright \mathsf{X}^*$ is the standard self-similar action of the iterated monodromy group of a partial self-covering $f : \mathcal{X}_1 \longrightarrow \mathcal{X}$, every transformation $S_{x \cdot g}$ is equal to $S_{\ell_x \gamma}$, where ℓ_x is the connecting path corresponding to the letter $x \in \mathsf{X}$, and $\gamma \in \pi_1(\mathcal{X}, t)$ is the path defining the element $g \in \mathrm{IMG}(f)$. The product $\ell_x \gamma$ is a path starting at the basepoint t and ending in the preimage $\Lambda(x)$ of t corresponding to the letter x. Conversely, if ℓ is an arbitrary path from the basepoint t to its f-preimage, then we have $\ell = \ell_x \cdot \ell_x^{-1}\ell$, where $x \in \mathsf{X}$ is such that $\Lambda(x)$ is the end of ℓ. Then $\ell_x^{-1}\ell$ is a loop at t. The isomorphism $S_\ell : T_t \longrightarrow T_{\Lambda(x)}$ coincides with the transformation $S_{x\cdot g}$, where $g \in \mathrm{IMG}\,(f)$ is the image of $\ell_x^{-1}\ell \in \pi_1(\mathcal{X}, t)$. We have proved the following fact.

Proposition 4.2.1. *The set* $\mathsf{X} \cdot \mathrm{IMG}\,(f) = \{S_{x\cdot g}\}_{x\in\mathsf{X}, g\in\mathrm{IMG}(f)}$ *of transformations of* X^* *is conjugated by the isomorphism* $\Lambda : \mathsf{X}^* \longrightarrow T_t$ *with the set of all transformations of* T_t *of the form* $S_\ell : T_t \longrightarrow T_z$*, where* $z \in f^{-1}(t)$ *and* ℓ *is a path starting in* t *and ending in* z*.*

We see that the set of isomorhisms $S_\ell : T_t \longrightarrow T_z$ is isomorphic to $\mathsf{X}\cdot\mathrm{IMG}\,(f)$, and has a natural definition purely in terms of the self-covering. The left and right actions of $\mathrm{IMG}\,(f)$ on this set is also by composition.

Definition 4.2.2. A *G-biset* is a set $\mathfrak{M}$ with commuting left and right actions $G \curvearrowright \mathfrak{M} \curvearrowleft G$. It is a *covering biset* if the right G-action is free. The number of right orbits of a covering biset is the *degree* of the biset. Two G-bisets $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are said to be *isomorphic* if there exists a bijection $\Phi : \mathfrak{M}_1 \longrightarrow \mathfrak{M}_2$ such that $\Phi(g_1 \cdot x \cdot g_2) = g_1 \cdot \Phi(x) \cdot g_2$ for all $x \in \mathfrak{M}_1$ and $g_1, g_2 \in G$.

Note that the right action (given by $(x \cdot g) \cdot h = x \cdot (gh)$ of G on $\mathsf{X} \cdot G$ is free and has $|\mathsf{X}|$ orbits labeled by the letters of X.

The terminology of G-bisets has many advantages over the usual terminology of self-similar actions and wreath recursions (as in 2.4.7 and 2.4.8). Besides being more natural in the setting of iterated monodromy groups, it lends better to generalizations, as we will see later in 4.3 (see also the notion of a groupoid biaction in 3.2.2).

Accordingly, we adopt a new definition of self-similarity of groups.

Definition 4.2.3. A *self-similar group* is a pair $(G, \mathfrak{M})$ consisting of a group G and a finite degree covering G-biset $\mathfrak{M}$. Two self-similar groups $(G_1, \mathfrak{M}_1)$ and $(G_2, \mathfrak{M}_2)$ are said to be *equivalent* if there exists an isomorphism $\phi : G_1 \longrightarrow G_2$ and a bijection $F : \mathfrak{M}_1 \longrightarrow \mathfrak{M}_2$ such that $F(g_1 \cdot x \cdot g_2) = \phi(g_1) \cdot F(x) \cdot \phi(g_2)$ for all $g_1, g_2 \in G_1$ and $x \in \mathfrak{M}_1$.

More generally, we can talk about the *category of self-similar groups*. Its objects are pairs $(G, \mathfrak{M})$, where G is a group, and $\mathfrak{M}$ is a covering G-biset. A morphism $(G_1, \mathfrak{M}_1) \longrightarrow (G_2, \mathfrak{M}_2)$ is a pair of maps $\phi : G_1 \longrightarrow G_2$ and $F : \mathfrak{M}_1 \longrightarrow \mathfrak{M}_2$ such that ϕ is a homomorphism of groups and $F(g_1 \cdot x \cdot g_2) = \phi(g_1) \cdot F(x) \cdot \phi(g_2)$ for all $g_1, g_2 \in G_1$ and $x \in \mathfrak{M}_1$. Then equivalence of self-similar groups are precisely the isomorphisms in so defined category of self-similar groups.

Definition 4.2.4. Let $f : \mathcal{X}_1 \longrightarrow \mathcal{X}$ be a partial self-covering. Let $\mathfrak{M}_{t,f}$ be the set of homotopy classes of paths from t to a point of $f^{-1}(t)$. It as a

$\pi_1(\mathcal{X}, t)$-biset with respect to the actions

$$[\ell] \cdot [\gamma] = [\ell\gamma], \qquad [\gamma] \cdot [\ell] = [\gamma'\ell],$$

where $\ell \in \mathfrak{M}$, $\gamma \in \pi_1(\mathcal{X}, t)$, and γ' is the lift of γ by f starting at the end of ℓ. We call $\mathfrak{M}_{t,f}$ the *biset associated with the covering* $f : \mathcal{X}_1 \longrightarrow \mathcal{X}$.

It is easy to see that $\mathfrak{M}_{t,f}$ is a covering $\pi_1(\mathcal{X}, t)$-biset.

Let $(G, \mathfrak{M})$ be a covering biset. A *basis* of $\mathfrak{M}$ is a transversal $\mathsf{X} \subset \mathfrak{M}$ of the orbits of the right action, i.e., such a set that every element $a \in \mathfrak{M}$ can be uniquely written as $x \cdot g$ for $x \in \mathsf{X}$ and $g \in G$. Note that we are using here the fact that the right action is free.

Example 4.2.5. A basis of $\mathfrak{M}_{t,f}$ is a collection of paths $\{\ell_z\}_{z \in f^{-1}(t)}$ connecting the basepoint t to the points of $f^{-1}(t)$. This follows from the fact that two elements of $\mathfrak{M}_{t,f}$ belong to the same right orbit if and only if their endpoints coincide.

Example 4.2.6. Consider the biset $\mathfrak{M}_{t,f}$, where $f : x \mapsto 2x$ is the double self-covering of the circle $\mathbb{R}/\mathbb{Z}$. Choose the basepoint $t = 0$. Then the fundamental group of $\mathbb{R}/\mathbb{Z}$ is naturally identified with $\mathbb{Z}$, where an integer $n \in \mathbb{Z}$ corresponds to the image in $\mathbb{R}/\mathbb{Z}$ of the path from 0 to n in $\mathbb{R}$. The biset $\mathfrak{M}_{t,f}$ consists of homotopy classes of paths from 0 to 0 or $1/2 \in \mathbb{R}/\mathbb{Z}$. Similarly to the fundamental group, $\mathfrak{M}_{t,f}$ is identified with the set $\frac{1}{2}\mathbb{Z}$ of half-integers, where a number $\frac{n}{2} \in \frac{1}{2}\mathbb{Z}$ is identified with the image in $\mathbb{R}/\mathbb{Z}$ of the path in $\mathbb{R}$ from 0 to $\frac{n}{2}$. The right action of the fundamental group on $\mathfrak{M}_{t,f}$ is by appending loops, i.e., is the natural action of $\mathbb{Z}$ on $\frac{1}{2}\mathbb{Z}$: a number n maps $\frac{m}{2}$ to $\frac{m}{2} + n$. The element of the fundamental group corresponding to $n \in \mathbb{Z}$ has two lifts by the covering f: one is the image of the path from 0 to $n/2$, the other is the image of the path from $1/2$ to $(n+1)/2$. It follows that n acts on $\mathfrak{M}_{t,f}$ by mapping a path corresponding to $\frac{m}{2}$ to the path corresponding to $\frac{m+n}{2}$.

This description gives the following natural interpretation of the biset $\mathfrak{M}_{t,f}$. We identify the fundamental group with the set of translations $x \mapsto x + n$ of $\mathbb{R}$ for $n \in \mathbb{Z}$. The biset $\mathfrak{M}_{t,f}$ is identified then with the set of affine transformation of $\mathbb{R}$ of the form $x \mapsto \frac{x+m}{2}$, where $m \in \mathbb{Z}$. Then the right action of the fundamental group on $\mathfrak{M}_{t,f}$ is by post-composition: the transformation $x \mapsto \frac{x+m}{2}$ is mapped to the transformation $x \mapsto \frac{x+m}{2} + n$. The left action is the action by pre-composition: it transforms $x \mapsto \frac{x+m}{2}$ to $x \mapsto \frac{x+n+m}{2}$.

The choice of a basis $\mathsf{X} \subset \mathfrak{M}$ defines a natural self-similar action of $G \curvearrowright \mathsf{X}^*$. For every $x \in \mathsf{X}$ and every $g \in G$ there exist unique $y \in \mathsf{X}$ and $h \in G$ such that $g \cdot x = y \cdot h$, as the right action is free, and X is its transversal. We can use the equalities $g \cdot x = y \cdot h$ as a recurrent definition of the *associated action* $G \curvearrowright \mathsf{X}^*$. Namely, we require that $g(xw) = yh(w)$ for all $w \in \mathsf{X}^*$ whenever $g \cdot x = y \cdot h$ in $\mathfrak{M}$.

It follows directly from the definition of a standard action of $\mathrm{IMG}\,(f)$ that the set of actions of $\pi_1(\mathcal{X}, t)$ on X^* associated with choices of right orbit transversals for the $\pi_1(\mathcal{X}, t)$-biset $\mathfrak{M}_{t,f}$ coincides with the set of the standard self-similar actions.

4.2.2. Trees associated with bisets. There is a more canonical approach to defining actions on rooted trees associated with bisets, not relying on the choice of a basis.

Let $\mathfrak{M}_1, \mathfrak{M}_2$ be G-bisets. We leave it as an exercise to show that

$$(x_1 \cdot g, x_2) \sim (x_1, g \cdot x_2)$$

on $\mathfrak{M}_1 \times \mathfrak{M}_2$ is an equivalence relation on $\mathfrak{M}_1 \times \mathfrak{M}_2$. Let us denote by $\mathfrak{M}_1 \otimes \mathfrak{M}_2$ the quotient $\mathfrak{M}_1 \times \mathfrak{M}_2/\sim$. The equivalence relation is invariant under the left and the right actions $g \cdot (x_1, x_2) = (g \cdot x_1, x_2)$, $(x_1, x_2) \cdot g = (x_1, x_2 \cdot g)$, hence $\mathfrak{M}_1 \otimes \mathfrak{M}_2$ is a G-biset. We denote the equivalence class of (x_1, x_2) by $x_1 \otimes x_2$. This operation of a "tensor product" of bisets is a particular case of a more general notion of a composition of biactions of groupoids, see 3.2.2.

It is also not hard to show that $(x_1 \otimes x_2) \otimes x_3 \mapsto x_1 \otimes (x_2 \otimes x_3)$ is a well defined isomorphism from $(\mathfrak{M}_1 \otimes \mathfrak{M}_2) \otimes \mathfrak{M}_3$ to $\mathfrak{M}_1 \otimes (\mathfrak{M}_2 \otimes \mathfrak{M}_3)$.

In particular, if $\mathfrak{M}$ is a G-biset, then we have well defined tensor powers $\mathfrak{M}^{\otimes n}$. We also denote by $\mathfrak{M}^{\otimes 0}$ the group G itself with the natural left and right actions of G on it by multiplication.

Denote by $\mathfrak{M}^*$ the disjoint union $\bigsqcup_{n=0}^{\infty} \mathfrak{M}^{\otimes n}$. It is also a G-biset. Note also that $\mathfrak{M}^*$ is a monoid with respect to the operation $\otimes$.

Lemma 4.2.7. *If the right action $\mathfrak{M} \curvearrowleft G$ is free, then the semigroup $\mathfrak{M}^*$ is left-cancellative, i.e., $v \otimes u_1 = v \otimes u_2$ implies $u_1 = u_2$.*

Proof. It is enough to prove the statement for the case $v \in \mathfrak{M}$. If $v \otimes u_1 = v \otimes u_2$, then there exists $g \in G$ such that $v \cdot g = v$ and $u_1 = g \cdot u_2$. Since the right action is free, the first equality implies $g = 1$. Then the second equality is $u_1 = u_2$. □

Since the left G-action commutes with the right G-action, the group G acts from the left on the set set of orbits $\mathfrak{M}^*/G$ of the right action.

Consider the left divisibility relation on $\mathfrak{M}^*$: we write $w_1 \prec w_2$ if there exists $u \in \mathfrak{M}^*$ such that $w_1 \otimes u = w_2$. It is obviously a transitive reflexive order invariant under the left action; and $w_1 \prec w_2$ for $w_i \in \mathfrak{M}^{\otimes n_i}$ implies $n_1 \leqslant n_2$. Moreover, if $n_1 < n_2 < n_3$ and $w_1 \in \mathfrak{M}^{\otimes n_1}$ and $w_3 \in \mathfrak{M}^{\otimes n_3}$ are such that $w_1 \prec w_3$, then there exists $w_2 \in \mathfrak{M}^{\otimes n_2}$ such that $w_1 \prec w_2 \prec$

w_3. Note also that the restriction of $\prec$ to any $\mathfrak{M}^{\otimes n}$ is equal to the orbit equivalence relation for the right G-action. It follows that $\prec$ induces a partial order on $\mathfrak{M}^*/G$, and that G acts on the left on $\mathfrak{M}^*/G$ by order preserving automorphisms.

Consider the Hasse diagram $T_{\mathfrak{M}}$ of the partially ordered set $\mathfrak{M}^*/G$. The set of vertices of $T_{\mathfrak{M}}$ is $\mathfrak{M}^*/G$. Two vertices $v_1, v_2 \in \mathfrak{M}^*/G$, $v_1 \prec v_2$, are connected by an edge if and only if there exist representatives $w_i \in v_i$ and $x \in \mathfrak{M}$ such that $w_2 = w_1 \otimes x$. The image of $\mathfrak{M}^{\otimes n}$ in $T_{\mathfrak{M}}$ is called the nth level of $T_{\mathfrak{M}}$. An edge connects only vertices of neighboring levels. Then $G \curvearrowright T_{\mathfrak{M}}$ is an action by level-preserving automorphisms of the graph $T_{\mathfrak{M}}$.

Lemma 4.2.8. *The Hasse diagram $T_{\mathfrak{M}}$ is a tree.*

Proof. It is sufficient to prove that for every vertex v of the nth level of $T_{\mathfrak{M}}$ there exists a unique adjacent vertex of the $(n-1)$-st level. Let $w \in \mathfrak{M}^{\otimes n}$ be a representative of v. Then there exists $w' \in \mathfrak{M}^{\otimes(n-1)}$ and $x \in \mathfrak{M}$ such that $w = w' \otimes x$, and w' represents a vertex of the $(n-1)$-st level adjacent to v. Suppose that w'' is another element of $\mathfrak{M}^{\otimes(n-1)}$ representing a vertex adjacent ot v. Then there exists $y \in \mathfrak{M}$ such that $w'' \otimes y$ represents v. This means that $w'' \otimes y$ and $w' \otimes x$ belong to the same right G-orbit, i.e., there exists $g \in G$ such that $w' \otimes x \cdot g = w'' \otimes y$. The latter means that there exists $h \in G$ such that $w' \otimes h = w''$ and $x \cdot g = h \cdot y$. But then w' and w'' represent the same vertex $T_{\mathfrak{M}}$. □

We see that every G-biset $\mathfrak{M}$ naturally defines an action of G by automorphisms of the rooted tree $T_{\mathfrak{M}} = \mathfrak{M}^*/G$. Here the root is the unique vertex of the level number 0. The following proposition describes the symbolic encoding of the tree $T_{\mathfrak{M}}$ by words over a basis $\mathsf{X}^* \subset \mathfrak{M}$.

Proposition 4.2.9. *Let $\mathfrak{M}$ be a covering G-biset, and let $\mathsf{X} \subset \mathfrak{M}$ be a basis. Then for every $n \geqslant 1$ the set $\mathsf{X}^n = \{x_1 \otimes x_2 \otimes \cdots \otimes x_n \ : \ x_i \in \mathsf{X}\}$ is a basis of $\mathfrak{M}^{\otimes n}$. In particular, the set $\mathsf{X}^* = \bigsqcup_{n\geqslant 0} \mathsf{X}^n$ is a basis of $\mathfrak{M}^*$, and the identical embedding $\mathsf{X}^* \hookrightarrow \mathfrak{M}^*$ induces an isomorphism of trees $\mathsf{X}^* \longrightarrow T_{\mathfrak{M}}$.*

Proof. It is enough to show that if X_i is a basis of $\mathfrak{M}_i$, then the set $\mathsf{X}_1 \otimes \mathsf{X}_2$ is a basis of $\mathfrak{M}_1 \otimes \mathfrak{M}_2$. Every element of $\mathfrak{M}_1 \otimes \mathfrak{M}_2$ can be written in the form $x_1 \cdot g_1 \otimes x_2 \cdot g_2$ for some $x_i \in \mathsf{X}_i$ and $g_i \in G$. We can rewrite $g_1 \cdot x_2$ as $y \cdot h$ for some $y \in \mathsf{X}_2$ and $h \in G$, so that $x_1 \cdot g_1 \otimes x_2 \cdot g_2 = x_1 \otimes y \cdot hg_2$. This shows that $\mathsf{X}_1 \otimes \mathsf{X}_2$ intersects every orbit. Suppose that $x_1 \otimes x_2$ and $y_1 \otimes y_2$ belong to the same right orbit. Then there exists $g \in G$ such that $x_1 \otimes x_2 \cdot g = y_1 \otimes y_2$, i.e., there exists $h \in G$ such that $x_1 \cdot h = y_1$ and $x_2 \cdot g = h \cdot y_2$. The first equality implies $x_1 = y_1$ and $h = 1$, since X_1 is a basis of $\mathfrak{M}_1$. Then we have $x_2 \cdot g = y_2$, which implies $x_2 = y_2$. It follows that $\mathsf{X}_1 \otimes \mathsf{X}_2$ intersects each right orbit exactly one time, i.e., is a basis of $\mathfrak{M}_1 \otimes \mathfrak{M}_2$. □

The action $G \curvearrowright T_{\mathfrak{M}}$ is not faithful in general. Let K be its kernel. It is equal to the set of elements $g \in G$ such that for every $v \in \mathfrak{M}^*$ the elements v and $g \cdot v$ belong to the same right G-orbit.

For every $v \in \mathfrak{M}^*$ the set of elements $g \in G$ such that $g \cdot v$ and v belong to the same right G-orbit is a subgroup. Let us denote it G_v. For every $g \in G_v$ there exists $g|_v \in G$ such that $g \cdot v = v \cdot g|_v$. If the right action is free, then such $g|_v$ is unique.

Proposition 4.2.10. *Let $\mathfrak{M}$ be a covering biset. Then the map $g \mapsto g|_v$ is a homomorphism from G_v to G. We have $G_{h\cdot v} = hG_vh^{-1}$, $G_{v\cdot h} = G_v$ for all $v \in \mathfrak{M}^*$ and $h \in G$. We also have $g|_{h\cdot v} = (h^{-1}gh)|_v$, for all $g \in G_{h\cdot v}$ and $g|_{v\cdot h} = h^{-1}(g|_v)h$ for all $g \in G_v$.*

The group $G_{v_1\otimes v_2}$ is equal to the set of elements $g \in G_{v_2}$ such that $g|_{v_2} \in G_{v_1}$, and we have $g|_{v_1\otimes v_2} = g|_{v_1}|_{v_2}$.

Proof. .... □

Proposition 4.2.11. *Let $\mathfrak{M}$ be a covering G-biset, and let K be the kernel of the action of G on $\mathfrak{M}^*/G = T_{\mathfrak{M}}$. Then the set $\mathfrak{M}/K$ is a covering G/K-biset. The identity map induces an isomorphism of the associated trees $T_{\mathfrak{M}}$ and $T_{\mathfrak{M}/K}$.*

Proof. ... □

Definition 4.2.12. Let K be the kernel of the action of G on the tree of right orbits associated with a covering biset $\mathfrak{M}$. Then the G/K-biset $\mathfrak{M}/K$ is called the *faithful quotient* of the G-biset $\mathfrak{M}$.

Example 4.2.13. The $\mathrm{IMG}\,(f)$-biset associated with the standard self-similar actions is precisely the faithful quotient of the $\pi_1(\mathcal{X}, t)$-biset $\mathfrak{M}_{t,f}$.

4.2.3. Wreath recursion. Given a biset $(G, \mathfrak{M})$ the associated *wreath recursion* (see 2.4.8) can be defined in the following way. The right G-set $\mathfrak{M}_G$ is isomorphic to a disjoint union of $|\mathsf{X}|$ copies of G, where X is a right orbit transversal (i.e., a basis). It follows that the automorphism group $\mathrm{Aut}\,\mathfrak{M}_G$ of the right G-set $\mathfrak{M}$, defined as the set of all permutations of $\mathfrak{M}_G$ commuting with the right G-action, is isomorphic to the wreath product $\mathsf{S}(\mathsf{X}) \ltimes G^{\mathsf{X}}$. Namely, if ψ is an automorphism of the right G-space $\mathfrak{M}$, then there exists a permutation π of X and a collection $(h_x)_{x\in\mathsf{X}}$ of elements of G such that $\psi(x) = \pi(x) \cdot h_x$. We set then $\pi \cdot (h_x)_{x\in\mathsf{X}}$ to be the element of $\mathsf{S}(\mathsf{X}) \ltimes G^{\mathsf{X}}$ corresponding to ψ. It is easy to check that this correspondence is an isomorphism. Note that it depends on the choice of the basis X, while the automorphism group of the right G-space $\mathfrak{M}_G$ is canonically associated with $(G, \mathfrak{M})$.

Since the left action of G on $\mathfrak{M}$ commutes with the right action, every element $g \in G$ defines an automorphism $x \mapsto g \cdot x$ of the right G-space $\mathfrak{M}_G$. We get a homomorphism from G to $\operatorname{Aut} \mathfrak{M}_G$, which we call the *wreath recursion*. If we choose a basis of $\mathfrak{M}$, then we get a concrete realization $\phi_{\mathsf{X}} : G \longrightarrow \mathsf{S}(\mathsf{X}) \rtimes G^{\mathsf{X}}$ of the wreath recursion, which will define (as in 2.4.8) the associated standard action of G on X^*.

Let $\mathsf{X} = \{x_1, x_2, \ldots, x_d\}$ and $\mathsf{Y} = \{y_1, y_2, \ldots, y_d\}$ be two bases of $\mathfrak{M}$. Let us use the chosen indexing of their elements to identify both groups $\mathsf{S}(\mathsf{X}) \rtimes G^{\mathsf{X}}$ and $\mathsf{S}(\mathsf{Y}) \rtimes G^{\mathsf{Y}}$ with $\mathsf{S}_d \rtimes G^d$. Let $\phi_{\mathsf{X}}, \phi_{\mathsf{Y}} : G \longrightarrow \mathsf{S}_d \rtimes G^d$ be the corresponding wreath recursions. If $\phi_{\mathsf{X}}(g) = \sigma(g_1, g_2, \ldots, g_d)$ and $\phi_{\mathsf{Y}}(g) = \pi(h_1, h_2, \ldots h_d)$ for $g \in G$, then we have $g \cdot x_i = x_{\sigma(i)} \cdot g_i$ and $g \cdot y_i = y_{\pi(i)} \cdot h_i$ for all $i = 1, 2, \ldots, d$. There exists a permutation $\alpha \in \mathsf{S}_d$ and a collection $f_1, f_2, \ldots, f_d$ of elements of G such that

$$y_i = x_{\alpha(i)} \cdot f_i. \tag{4.1}$$

Note also that for any $\alpha(i) \in \mathsf{S}_d$ and $(f_1, f_2, \ldots, f_d) \in G^d$, the sequence $x_{\alpha(i)} \cdot f_i$ is a basis of $\mathfrak{M}$.

Then we have $x_{\alpha\pi(i)} \cdot f_{\pi(i)} h_i = y_{\pi(i)} \cdot h_i = g \cdot y_i = g \cdot x_{\alpha(i)} \cdot f_i = x_{\sigma\alpha(i)} \cdot g_{\alpha(i)} f_i$ for every i, which implies $\alpha\pi = \sigma\alpha$ and $f_{\pi(i)} h_i = g_{\alpha(i)} f_i$ for every i. Consequently, we have $f_{\pi\alpha^{-1}(i)} h_{\alpha^{-1}(i)} = g_i f_{\alpha^{-1}(i)}$, and hence

$$\sigma(g_1, g_2, \ldots, g_d) = \alpha(f_1, f_2, \ldots, f_d)\pi(h_1, h_2, \ldots, h_d)(\alpha(f_1, f_2, \ldots, f_d))^{-1}.$$

We see that changing the basis of $\mathfrak{M}$ is equivalent to post-composing the wreath recursion by an inner automorphism of $\mathsf{S}_d \rtimes G^d$. We proved the following description of equivalence of self-similar groups.

Proposition 4.2.14. *Two degree d covering G-bisets $\mathfrak{M}_1, \mathfrak{M}_2$ are isomorphic if and only if the associated wreath recursions $G \longrightarrow \mathsf{S}_d \rtimes G^d$ are obtained from each other by post-composing with an inner automorphism of $\mathsf{S}_d \rtimes G^d$.*

Note that in the case of the bisets $\mathfrak{M}_{t,f}$ associated with a partial self-covering $f : \mathcal{X}_1 \longrightarrow \mathcal{X}$, the relation (4.1) between a pair of bases has a natural interpretation. A basis of $\mathfrak{M}_{t,f}$ is a collection of paths $\ell_1, \ell_2, \ldots, \ell_d$ connecting the basepoint t with the points of $f^{-1}(t)$. If $\ell_1', \ell_2', \ldots, \ell_d'$ is another such a collection, then there exists a unique permutation $\alpha \in \mathsf{S}_d$ such that ℓ_i' and $\ell_{\alpha(i)}$ end in the same point of $f^{-1}(t)$ for every i. Then $\ell_i' = \ell_{\alpha(i)} \cdot (\ell_{\alpha_i}^{-1} \ell_i')$, and $\ell_{\alpha_i}^{-1} \ell_i' = f_i$ is a loop. (Remember that here ℓ_i' is traversed before $\ell_{\alpha_i}^{-1}$.)

Example 4.2.15. ...

4.2.4. Virtual endomorphisms of groups. We have seen two ways of defining self-similar actions: bisets and wreath recursions. Another approach uses the notion of a *virtual endomorphism* of a group. It is best suited for self-similar actions that are transitive on the first level of the tree, i.e., for bisets that do not contain proper sub-bisets.

Let $(G, \mathfrak{M})$ be a covering biset. Let $x \in \mathfrak{M}$, and let G_x, as above be the set of elements $g \in G$ such that there exists $g|_x \in G$ such that $g \cdot x = x \cdot g|_x$. Then, by Proposition 4.2.10 the map $g \mapsto h$ is a homorphism from G_x to G. Since the number of right orbits is finite, the subgroup G_x has finite index in G. So, we adopt the following definition.

Definition 4.2.16. A *virtual endomorphism* $\phi : G \dashrightarrow G$ is a homomorphism from a subgroup of finite index of G to G. If $(G, \mathfrak{M})$ is a covering biset, and $x \in \mathfrak{M}$, then the *associated virtual endomorphism* ϕ_x is given by the condition

$$g \cdot x = x \cdot \phi_x(g).$$

Its domain is the set G_x of elements of G fixing (with respect to the left action) the right orbit of x.

If the biset $(G, \mathfrak{M})$ *irreducible*, i.e., that the left action of G is transitive on the set of the right G-orbits, then the biset $\mathfrak{M}$ can be reconstructed from the virtual endomorphism ϕ in the following way. Consider the set, denoted $\phi(G)G$, of all partially defined maps from G to G of the form

$$x \mapsto \phi(x \cdot h) \cdot g,$$

where $h, g \in G$. Note that if ϕ is onto, then we can write any such a map as $\phi(x \cdot h)$. Two maps $x \mapsto \phi(x \cdot h_1) \cdot g_1$ and $x \mapsto \phi(x \cdot h_2) \cdot g_2$ are equal if and only if $h_1^{-1}h_2 \in \operatorname{Dom}\phi$ and $\phi(h_1^{-1}h_2) = g_1 g_2^{-1}$. We will formally write the transformation $x \mapsto \phi(x \cdot h) \cdot g$ by $\phi(h)g$.

Consider the action of G on itself by right multiplication, and consider the action of G on the set $\phi(G)G$ by pre- and post-compositions with the action of G on itself. These will be the left and the right actions on the biset, respectively. They are given by

$$f \cdot \phi(h)g = \phi(fh)g, \qquad \phi(h)g \cdot f = \phi(h)gf.$$

It is easy to check that the map $\phi_x(h)g \mapsto h \cdot x \cdot g$ induces an isomorphism of $\phi_x(G)G$ with $\mathfrak{M}$, if ϕ_x is the virtual endomorphism associated with $\mathfrak{M}$ and $x \in \mathfrak{M}$.

A basis of $\mathfrak{M}$ in terms of $\phi_x(G)G$ is a set $\{\phi_x(h_1)g_1, \phi_x(h_2)g_2, \ldots, \phi_x(h_d)g_d\}$, where $D = \{h_1, h_2, \ldots, h_d\}$ is a left coset transversal of G modulo G_x. If ϕ_x is onto, then we can assume that $g_i = 1$ for all i. Even if ϕ_x is not onto, we still can choose $g_i = 1$ (but then we will restrict the set of bases of $\mathfrak{M}$ that we are considering). We call D the *digit set* for ϕ_x.

Let $D = \{h_1, h_2, \ldots, h_d\}$ be a digit set for a virtual endomorphism ϕ. Then the associated wreath recursion is obtained as follows. For every $g \in G$ and $i \in \{1, 2, \ldots, d\}$ there exists a unique $h_j \in D$ such that $gh_i \in h_j \operatorname{Dom} \phi$. Then $h_j^{-1} gh_i \in \operatorname{Dom} \phi$, and we gen an element $\phi(h_j^{-1} gh_i)$ of G. This defines us the element $\sigma(g_1, g_2, \ldots, g_d)$ of the wreath product $\mathsf{S}_d \ltimes G^d$, where σ and g_i are defined by the conditions $g_i = \phi(h_{\sigma(i)}^{-1} gh_i)$.

In other words, we get the following description of the corresponding standard action of G on X^*.

Proposition 4.2.17. *Let $(G, \mathfrak{M})$ be an irreducible covering biset, and let $\phi_x : G_x \longrightarrow G$ be a virtual endormorphism associated with it. Choose a left coset transversal $D = \{h_1, h_2, \ldots, h_d\}$ of G modulo G_x. Then a standard action of G is defined by the recursion*

$$g \cdot x_i = x_j \cdot \phi_x(h_j^{-1} gh_i),$$

where j is defined by the condition $h_j^{-1} gh_i \in G_x$.

Two virtual endomorphisms $\phi_1, \phi_2 : G \dashrightarrow G$ define isomorphic G-bisets if and only if they are equal up to inner outomorphisms of G, i.e., if there exist $g, h \in G$ such that $\phi_1(x) = g^{-1}\phi_2(h^{-1}xh)h$ for all x in the domain of ϕ_1...

We leave the following as an exercise, see also ...

Proposition 4.2.18. *Let $\phi : G \dashrightarrow G$ be virtual endomorphism. The kernel of the self-similar action of G associated with ϕ is equal to the maximal subgroup N normal in G such that N is contained in the domain of ϕ and $\phi(N) \leqslant N$. Equivalently, it is the subgroup $N = \bigcap_{g \in G, n \geqslant 1} g^{-1} \operatorname{Dom} \phi^n \cdot g$.*

In particular, the action is faithful if and only if G has no non-trivial normal subgroup N such that N is contained in the domain of ϕ and $\phi(N) \leqslant N$.

Example 4.2.19. The virtual endomorphism of $\mathbb{Z}$ associated with the binary odometer action is $n \mapsto n/2$ with the domain equal to the group of even integers. If we choose the classical coset representatives $\{0, 1\}$ of $\mathbb{Z}$ modulo $2\mathbb{Z}$, we will get the classical binary odometer action ...

Example 4.2.20. Abelian self-similar groups...

Example 4.2.21. It is known (see...) that the free group F_2 is isomorphic to the group generated by the matrices $a = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Consider the virtual endomorphism of this group given by

$$\phi \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12}/2 \\ 2a_{21} & a_{22} \end{pmatrix}.$$

Let us show that there is no normal subgroup of F_2 invariant under this virtual endomorphism. The matrices in $\bigcap_{n\geqslant 1}\mathrm{Dom}\,\phi^n$ must have zero above the diagonal. We have

$$\begin{pmatrix} 1 & -2m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1 & 2m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11}-2ma_{21} & 2m(a_{11}-a_{22})-4m^2a_{21} \\ a_{21} & a_{22}+2ma_{21} \end{pmatrix}.$$

Unless $a_{21}=0$ and $a_{11}=a_{22}$, we can always find $m\in\mathbb{Z}$ such that the number in the upper left corner of this matrix is not equal to 0. As the determinant of every matrix in our group is equal to 1, this implies that only the identity matrix belongs to $\bigcap_{g\in F_2,n\geqslant 1} g^{-1}\cdot\mathrm{Dom}\,\phi^n\cdot g$.

Let us choose $\{1,a\}$ as the coset transversal of F_2 modulo $\mathrm{Dom}\,\phi$. Since $\phi(a^2)=a$, $\phi(b)=b^2$, and $\phi(a^{-1}ba)=(b^{-1}a)^2$, the associated wreath recursion is

$$a=\sigma(1,a),\qquad b=(b^2,(b^{-1}a)^2).$$

This gives a faithful (though not finite state) self-similar action of the free group F_2 on the binary rooted tree.

4.3. General case

4.3.1. Correspondences. A partial self-covering $f:\mathcal{X}_1\longrightarrow\mathcal{X}$ is in fact a pair of maps between two topological spaces: the covering f and the identical embedding $\iota:\mathcal{X}_1\longrightarrow\mathcal{X}$. The formula for the standard action of the iterated monodromy group is written in these terms as

$$\gamma(xv)=\ell_y^{-1}\iota(\gamma_x)\ell_x(v),$$

where γ_x is the lift of γ by f to a path starting in $\Lambda(x)$, $y\in\mathsf{X}$ is such that $\Lambda(y)$ is the end of γ_x, and ℓ_x and ℓ_y are paths connecting t to $\iota(\Lambda(x))$ and $\iota(\Lambda(y))$, respectively.

There is no need to assume that ι is a homeomorphic embedding. The above formula of the standard action makes sense for any pair of maps $f:\mathcal{X}_1\longrightarrow\mathcal{X}$, $\iota:\mathcal{X}_1\longrightarrow\mathcal{X}$, where f is a covering map and ι is continuous.

Definition 4.3.1. A *covering correspondence* (or a *topological virtual endomorphism*) is a pair $f,\iota:\mathcal{X}_1\longrightarrow\mathcal{X}$ of continuous maps (or morphisms of orbispaces) such that $f:\mathcal{X}_1\longrightarrow\mathcal{X}$ is a finite degree covering.

Ever covering correspondence $f,\iota:\mathcal{X}_1\longrightarrow\mathcal{X}$, where $\mathcal{X}$ is path connected and locally path connected, naturally defines a $\pi_1(\mathcal{X})$-biset in the following way. Choose a basepoint $t\in\mathcal{X}$, and define $\mathfrak{M}_{t,f,\iota}$ as the set of all pairs $(z,[\ell])$, where $z\in f^{-1}(t)$ and $[\ell]$ is the homotopy class of a path ℓ in $\mathcal{X}$

starting in t and ending in $\iota(z)$. Note that if ι is injective on $f^{-1}(t)$, then z is uniquely determined by $[\ell]$.

The left and right actions of $\pi_1(\mathcal{X}, t)$ on $\mathfrak{M}_{t,f,\iota}$ are given by

$$(z, [\ell]) \cdot \gamma = (z, [\ell\gamma]), \qquad \gamma \cdot (z, [\ell]) = (z', [\iota(\gamma_z)\ell]),$$

where γ_z is the f-lift of γ starting in z, and z' is its end.

Suppose that $\mathcal{X}$ is semi-locally simply connected, and let $P : \widetilde{\mathcal{X}} \longrightarrow \mathcal{X}$ and $P_1 : \widetilde{\mathcal{X}}_1 \longrightarrow \mathcal{X}_1$ be universal covering maps. Choose basepoints $t \in \mathcal{X}$, $z \in \widetilde{\mathcal{X}}$, $t_1 \in \mathcal{X}_1$, $z_1 \in \widetilde{\mathcal{X}}_1$ such that $f(t_1) = t$, $P(z) = t$, and $P_1(z_1) = t_1$. Then there is a unique homeomorphism $\epsilon : \widetilde{\mathcal{X}}_1 \longrightarrow \widetilde{\mathcal{X}}$ making the diagram

$$\begin{array}{ccc} \widetilde{\mathcal{X}}_1 & \xrightarrow{\epsilon} & \widetilde{\mathcal{X}} \\ \downarrow{\scriptstyle P_1} & & \downarrow{\scriptstyle P} \\ \mathcal{X}_1 & \xrightarrow{f} & \mathcal{X} \end{array}$$

commutative and $\epsilon(z_1) = z$. Let us identify $\widetilde{\mathcal{X}}_1$ with $\widetilde{\mathcal{X}}$ by such a homeomorphism, and identify $\pi_1(\mathcal{X}_1, t_1)$ with the subgroup $f_*(\pi_1(\mathcal{X}_1, t_1))$ of $\pi_1(\mathcal{X}, t)$. Then $\mathcal{X}$ and $\mathcal{X}_1$ are the quotients of $\widetilde{\mathcal{X}}$ by the actions of $\pi_1(\mathcal{X}, t)$ and $\pi_1(\mathcal{X}_1, t_1)$ on $\widetilde{\mathcal{X}}$ by the deck transformations, and f is the map $\widetilde{\mathcal{X}}/\pi_1(\mathcal{X}_1, t_1) \longrightarrow \widetilde{\mathcal{X}}/\pi_1(\mathcal{X}, t)$ induced by the identity map on $\widetilde{\mathcal{X}}$.

There exists a continuous map $I : \widetilde{\mathcal{X}} \longrightarrow \widetilde{\mathcal{X}}$ making the diagram

$$\begin{array}{ccc} \widetilde{\mathcal{X}} & \xrightarrow{I} & \widetilde{\mathcal{X}} \\ \downarrow{\scriptstyle P_1} & & \downarrow{\scriptstyle P} \\ \mathcal{X}_1 & \xrightarrow{\iota} & \mathcal{X} \end{array}$$

commutative. We will write both the action of I and the action of $\pi_1(\mathcal{X}, t)$ on $\widetilde{\mathcal{X}}$ from the right: as $x \mapsto x \cdot I$ and $x \mapsto x \cdot g$.

Proposition 4.3.2. *The biset associated with the correspondence $f, \iota : \mathcal{X}_1 \longrightarrow \mathcal{X}$ is isomorphic to the biset consisting of maps $\widetilde{\mathcal{X}} \longrightarrow \widetilde{\mathcal{X}}$ of the form $x \mapsto x \cdot g_1 I g_2$, where $g_i \in \pi_1(\mathcal{X}, t)$ are deck transformations of the covering $P : \widetilde{\mathcal{X}} \longrightarrow \mathcal{X}$, with the natural left and right actions of $\pi_1(\mathcal{X}, t)$ by compositions.*

Proof. .... □

Definition 4.3.3. The iterated monodromy group of the covering correspondence $f, \iota : \mathcal{X}_1 \longrightarrow \mathcal{X}$ is the faithful quotient of the self-similar group $(\pi_1(\mathcal{X}, t), \mathfrak{M}_{t,f,\iota})$.

The associated virtual endomorphism can be written as $\iota_* \circ f^*$, where f^* is the isomorphism of the subgroup $f_*(\pi_1(\mathcal{X}_1)) < \pi_1(\mathcal{X})$ with $\pi_1(\mathcal{X}_1)$ inverse to the monodmorphism $f_*: \pi_1(\mathcal{X}_1) \hookrightarrow \pi_1(\mathcal{X})$ induced by f. Everything is defined here up to inner automorphisms of $\pi_1(\mathcal{X})$.

Example 4.3.4. Every partial self-covering $f : \mathcal{X}_1 \longrightarrow \mathcal{X}$ together with the identical embedding $\iota : \mathcal{X}_1 \longrightarrow \mathcal{X}$ is a covering correspondence.

Example 4.3.5. Let $\mathcal{A} = (\mathsf{X}, Q, \pi, \lambda)$ be a synchronous non-inital automaton over the same input-output alphabet X, see 2.3.1. Let $\mathcal{X}_1$ be its dual Moore diagram, i.e., the graph with the set of vertices X and the set of edges $Q \times \mathsf{X}$, where each edge (q, x) starts in x, ends in $\lambda(q, x)$, and is labeled by $q|\pi(q, x)$. Let $\mathcal{X}$ be the graph consisting of one vertex and the set of $|Q|$ loops labeled bijectively by the set of states Q of $\mathcal{A}$. Consider two maps $f, \iota : \mathcal{X}_1 \longrightarrow \mathcal{X}$ mapping all vertices of $\mathcal{X}_1$ to the unique vertex of $\mathcal{X}$, and acting on the edges by the rules

$$f(q, x) = q, \qquad \iota(q, x) = \pi(q, x),$$

i.e., we interpret the labeling of the edges of $\mathcal{X}_1$ as instructions describing two maps f and ι. We get a covering correspondence (if we identify the graphs $\mathcal{X}_1$ and $\mathcal{X}$ with their topological realizations) *associated* with the automaton $\mathcal{A}$. In some sense the general definition of the topological correspondence is a generalization of this situation. Sometimes we call topological correspondences *topological automata* to stress this analogy (see [**Nek14**]).

Example 4.3.6. Let G be a Lie group, let $\Gamma < G$ be its lattice, and let $\phi : G \longrightarrow G$ be an automorphism. Suppose that $\Gamma_1 = \phi^{-1}(\Gamma) \cap L$ has finite index in Γ, and consider the spaces $\mathcal{X}_0 = G/\Gamma$ and $\mathcal{X}_1 = G/\Gamma_1$. Then the inclusion $\Gamma_1 < \Gamma$ induces a covering map $\pi : \mathcal{X}_1 \longrightarrow \mathcal{X}_0$, and the inclusion $\phi(\Gamma_1) < \Gamma$ shows that the homeomorphism $\phi : G \longrightarrow G$ induces a covering map $\iota : \mathcal{X}_1 \longrightarrow \mathcal{X}_0$. The iterated monodromy group of the pair $\pi, \iota : \mathcal{X}_1 \longrightarrow \mathcal{X}_0$ coincides with the self-similar group defined by the virtual endomorphism of Γ induced by ϕ. Iterated monodromy groups in this class were used to prove the following theorem of M. Kapovich, cite...

Theorem 4.3.7. *Let Γ be an irreducible lattice in a semisimple algebraic Lie group G. Then the following are equivalent.*

(1) Γ *is virtually isomorphic to an arithmetic lattice in G, i.e., contains a finite index subgroup isomorphic to such arithmetic lattice.*

(2) Γ *admits a faithful self-similar action which is transitive on the first level.*

Example 4.3.8. Consider the map $(a, b) \mapsto ((a + b)/2, \sqrt{ab})$. It is well defined on the set $[0, +\infty)^2$, and was studied by Gauss and Lagrange in relation to *arithmetic-geometric mean*, see...

The map $(a, b) \mapsto ((a + b)/2, \sqrt{ab})$ is not well defined on $\mathbb{C}^2$, since there are two choices for $\sqrt{ab}$. But we can consider it as a correspondence. Note that $((a+b)/2, \sqrt{ab})$ is homogeneous, so we get a correspondence $[z_1 : z_2] \mapsto [(z_1 + z_2)/2 : \sqrt{z_1 z_2}]$ of the complex projective line with itself. It is written in affine coordinates as $w \mapsto \frac{1+w}{2\sqrt{w}}$. It is natural to model this correspondence as a pair of maps f, ι given by

$$f(w) = \frac{(1+w)^2}{4w}, \qquad \iota(w) = w^2.$$

Let $\mathcal{X} = \mathbb{C} \setminus \{0,1\}$ and $\mathcal{X}_1 = \mathbb{C} \setminus \{0,1,-1\}$. Then $f, \iota : \mathcal{X}_1 \longrightarrow \mathcal{X}$ are covering maps. It follows that we can consider the iterated monodromy group of this correspondence, and that it will be a homomorphic image of the fundamental group of $\mathcal{X}$, i.e., of a free group of rank two.

It is shown in... that the iterated monodromy group is equivalent to the self-similar action of the free group described in Example 4.2.21.

4.3.2. Orbispaces. The next level of generality is to consider the case when $\mathcal{X}$ and $\mathcal{X}_1$ are orbispaces, so that f is a covering of orbispaces, and $\iota : \mathcal{X}_1 \longrightarrow \mathcal{X}$ is a morphism of orbispaces, see 3.4.2 for definitions. This is especially useful in the study of the iterated monodromy groups of sub-hyperbolic rational functions, when the Julia set contains critical points. The associated $\pi_1(\mathcal{X})$-biset is defined in the same way as when $\mathcal{X}$ and $\mathcal{X}_1$ are topological spaces, but this time the corresponding paths groupoid paths in the atlases of the orbispaces. We will discuss a more general setting and more rigorously in 4.3.5, while here we will just consider several examples.

4.3.2.1. *The tent map.* Consider the graph of groups I consisting of two vertex groups of order two connected by a segment (with trivial edge group). It is equivalent to the orbispace of the action of the dihedral group D_∞ generated by the transformations $x \mapsto -x$ and $x \mapsto 2-x$ of $\mathbb{R}$. The segment $(0,1)$ is the fundamental domain of the action, and we can identify the orbispace with the interval $[0,1]$ with groups of order two at its ends.

Consider the map $F : x \mapsto 2x$ on $\mathbb{R}$. It is a homeomorphism satisfying $FD_\infty F^{-1} < D_\infty$ for the action of D_∞ defined above. It induces the tent map

$$f(x) = \begin{cases} 2x & \text{for } x \in [0,1/2], \\ 2-2x & \text{for } x \in [1/2,1] \end{cases}$$

on the underlying space of the orbifold $\mathbb{R}/D_\infty$. Moreover, it is easy to see that it induces a double self-covering of the orbifold I. (So that we can take $\iota : I \longrightarrow I$ to be the identity morphism.)

Let us compute the iterated monodromy group of the self-covering $f : I \longrightarrow I$. Consider the atlas of I coming from the action of D_∞ on $\mathbb{R}$ restricted to an open neighborhood U of $[0,1]$ in $\mathbb{R}$. It will be the atlas for the target orbispace for the covering map $f : I \longrightarrow I$. The atlas for the domain of the covering map is constructed in the usual way, as it is described in 3.4.2.

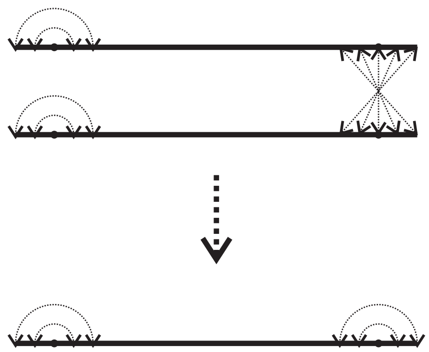

Figure 4.10. The tent map

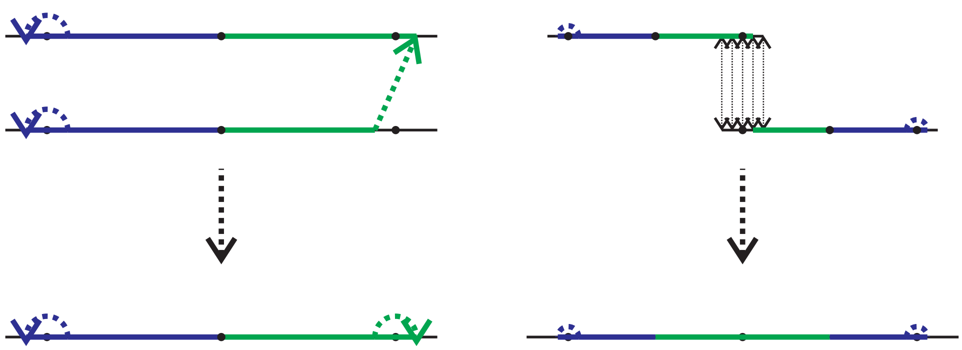

Figure 4.11. The iterated monodromy group of the tent map (put labels...)

See Figure 4.10 for the description of the covering map and the atlas of the covering orbifold. It is easy to see that the covering atlas is equivalent to the atlas of the original orbifold, i.e., that the map f is a self-covering.

Take the basepoint $t = 1/2$. The fundamental group of I is generated by two loops a and b consiting of the non-trivial elements of the isotropy groups at the endpoints 0 and 1 connected to the basepoint by simple paths inside the interval $[0, 1]$.

The lifts of the paths a and b by f are shown on Figure 4.11. Using the natural choice of the connecting paths (inside the neighborhood U), we get the following standard action:

$$a = \sigma(1, 1), \quad b = (a, b).$$

Note that this iterated monodromy group is the same as $\mathrm{IMG}\,(T_2) \cong \mathrm{IMG}\,\left(z^2 - 2\right)$, see 4.1.3.5.

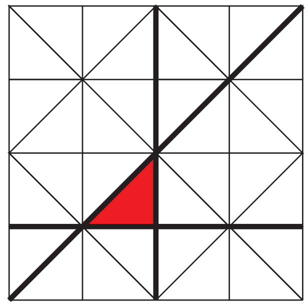

Figure 4.12. A fundamental domain of the action of G on $\mathbb{R}^2$

4.3.2.2. *Folding an isosceles right triangle.* Consider the group G of all isometries of $\mathbb{Z}^2$ with its natural action on $\mathbb{R}^2$. It is generated by the transformations

$$a : (x, y) \mapsto (x, -y), \quad b : (x, y) \mapsto (y, x), \quad c : (x, y) \mapsto (1 - x, y).$$

The action is proper, and its fundamental domain is, for example, the triangle Δ with the vertices $(0, 0)$, $(1/2, 1/2)$, and $(1/2, 0)$, see Figure 4.12. Note that each of the transformations a, b, c leaves one side the sides of the triangle Δ invariant.

The underlying space of the orbispace $\mathbb{R}^2/G$ can be naturally identified with the triangle Δ. The orbispace structure is a complex of groups described by identical embeddings between groups from the following set of subgroups of G: the identity group $\{1\}$ (for the interior of Δ), groups of order two $\langle a\rangle$, $\langle b\rangle$, $\langle c\rangle$ (for the sides of Δ), and dihedral groups $\langle a, b\rangle \cong D_4$, $\langle b, c\rangle \cong D_4$, $\langle a, c\rangle \cong D_2$ (for the vertices). We will denote this complex of groups also by Δ.

A natural self-covering of the orbispace Δ folds the triangle Δ along the bisector of the right angle in two, and then identifies the result with Δ by a similarity. There are two choices for the identification. Let us assume that the identification is such that the vertex $(0, 0)$, i.e., the vertex with isotropy group $\langle a, b\rangle$, is fixed under the obtained self-covering map. Figure 4.13 shows the computation of the iterated monodromy group of this self-covering.

We see that the iterated monodromy group of this covering map is generated by

$$a = (b, b), \qquad b = (a, c), \qquad c = \sigma.$$

4.3.2.3. *Dynamical systems with symmetries.* If a dynamical system $f \circlearrowright \mathcal{X}$ has a finite group of symmetries G, then it is sometimes natural to consider the induced dynamical system f/G on the orbispace $\mathcal{X}/G$. The iterated monodromy group of the quotient f/G is in some cases easier to compute than for the original map. The iterated monodromy group of the quotient will

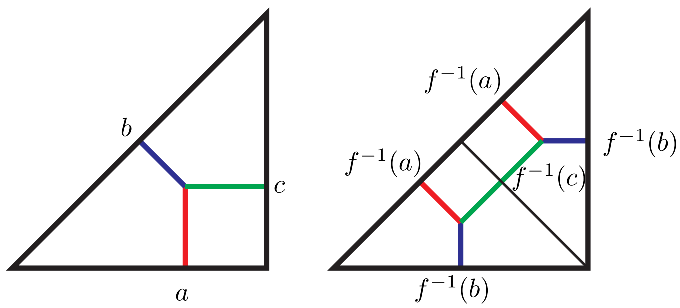

Figure 4.13. The iterated monodromy group of a triangle folding

contain IMG (f) as a normal subgroup such that the quotient of IMG (f/G) by the image of IMG (f) is isomorphic to G.

For example, suppose that f is a rational function with real coefficients. Then it satisfies $f(\overline{z}) = \overline{f(z)}$, i.e., the action of f on $\widehat{\mathbb{C}}$ is invariant under the action of the group of order two generated by the complex conjugation. The quotient will be a map acting on an orbispace which is a disc with groups of order two on the boundary. The iterated monodromy group of the quotient will contain IMG (f) as a subgroup of index two.

Consider, for example the rational function $f(z) = \frac{z^2-c}{z^2+c}$ for $c \approx 0.2956$ from Exercise 4.2. It has real coefficients. Moreover, all its critical and hence post-critical points are real. Consider the corresponding quotient by the complex conjugation. It is the disc orbifold, but since we have to remove the post-critical set, we have to take $\mathcal{X}_0$ to be the disc orbifold with small neighborhoods of the post-critical points removed. Let $\mathcal{X}_1$ be the preimage of $\mathcal{X}_0$ under f. The orbifold $\mathcal{X}_0$ is isomorphic to an octagon in which every other side is singular with isotropy groups of order 2, while the remaining sides are regular. The orbispace $\mathcal{X}_1$ is 12-gon with analogously defined orbispace structure.

The fundamental group of $\mathcal{X}_0$ is generated by four paths from the base-point to an internal point of one of the four singular sides and then back. Let $\alpha, \beta, \gamma, \delta$ be such generators, as it is shown on the bottom half of Figure 4.14.

Then considering the lifts of the generators to $\mathcal{X}_1$, we see from the top half of Figure 4.14 that the iterated monodromy group is generated by the wreath recursion

$$\alpha = \sigma, \quad \beta = (\alpha, \alpha), \quad \gamma = (\delta, \beta), \quad \delta = (\delta, \gamma).$$

The iterated monodromy group of $f(z)$ can be recovered now as the group generated by the products $\alpha\beta, \beta\gamma, \gamma\delta$. They satisfy $\alpha\beta = \sigma(\alpha, \alpha), \beta\gamma = (\alpha\delta, \alpha\beta)$, and $\gamma\delta = (1, \beta\delta)$. Post-conjugating the recursion by $(\alpha, 1)$, we get

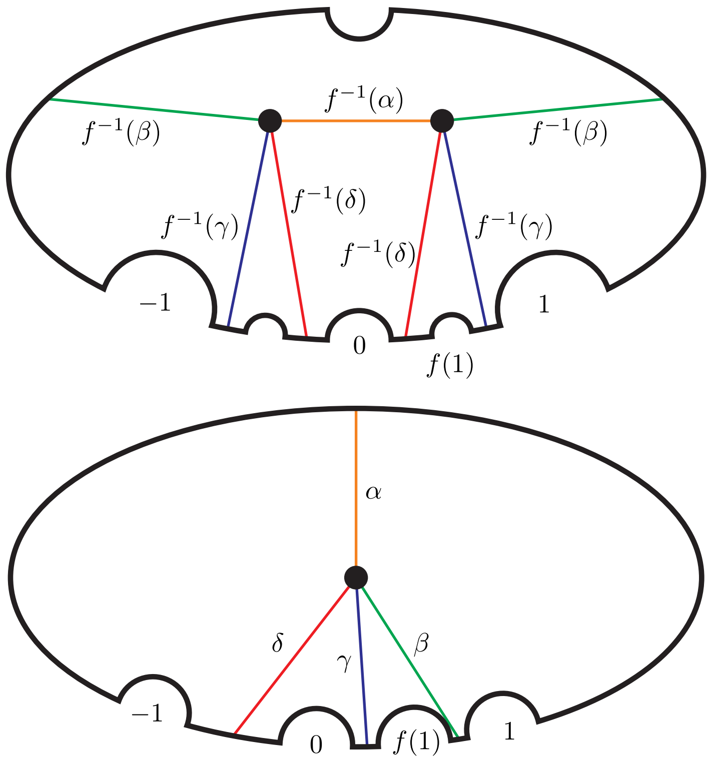

Figure 4.14.

an equivalent recursion $\alpha\beta = \sigma, \beta\gamma = (\delta\alpha, \alpha\beta)$, and $\gamma\delta = (1, \beta\delta)$, which is equivalent to the wreath recursion from Exercise 4.2.

4.3.3. Thurston orbifold of a sub-hyperbolic rational function. Considering orbispaces and their virtual endomorphisms is necessary even in the case of dynamical systems on topological spaces. For example, if f is a post-critically finite rational function such that the Julia set contains critical points, then f can not be expanding on the Julia set. It is, however, expanding on a naturally defined orbispace (or orbitfold on a neighborhood of the Julia set). It is a classical construction known as *Thurston orbifold*, see....

Let f be a post-critically finite rational function, or more generally, a *Thurston map*, i.e., a post-critically finite orientation-preserving branched covering of the two dimensional sphere. Here an *orientation-preserving branched covering* is a continuous map $f : S^2 \longrightarrow S^2$ such that for every $t \in S^2$ there exist homeomorphisms of neighborhoods of t and $f(t)$ with neighborhoods of $0 \in \mathbb{C}$ conjugating the action of f with the action of $z \mapsto z^n$ for some $n \geqslant 1$. The number n is called then the *local degree* of f, and is denoted by $\deg_t f$. One can show that $\deg_t f = 1$ for all but finitely many points $t \in S^2$. The points t such that $\deg_t f > 1$ are called *critical*, and the corresponding values $f(t)$ are called *critical values*. The map f is said to be *post-critically finite* if the union P_f of forward orbits of the critical values of f is finite. Then P_f is the post-critical set.

Let $f : S^2 \longrightarrow S^2$ be a Thurston map. Let $\nu(t) \in \mathbb{N} \cup \{\infty\}$ be the least common multiple of the local degrees $\deg_z f^n$ for all $z \in f^{-n}(t)$ and $n \geqslant 1$. It follows directly from the definitions that $\nu(t) = 1$ if and only if $t \notin P_f$. It is also not hard to see that $\nu(t) = \infty$ if and only if t belongs to a cycle containing a critical point.

If $\nu : S^2 \longrightarrow \mathbb{N} \cup \{\infty\}$ is a map equal to 1 for all but finitely many points, then the corresponding orbifold S^2_ν is defined as the orbifold with the underlying space $S^2 \setminus \nu^{-1}(\infty)$ where a neighborhood of $t \in S^2$ is uniformized by the action of a cyclic group of order $\nu(t)$ of rotations of a disc. More explicitly, it is defined by the following atlas. Take small disjoint neighborhoods U_t of the points $t \in \nu^{-1}(\mathbb{N} \cap [2, \infty))$ homeomorphic to discs, and represent them as quotients $D_t/G_t \approx U_t$, where $D_t \subset \mathbb{C}$ is the open unit disc, and $G_t \cong \mathbb{Z}/\nu(t)\mathbb{Z}$ is the group of rotations $z \mapsto e^{\frac{2\pi i}{\nu(t)}k} z$ of the disc, so that $0 \in D_t$ is mapped to $t \in U_t$. Consider the pseudogroup $\mathcal{G}$ acting on the disjoint union of $\mathcal{X} = S^2 \setminus \nu^{-1}(\mathbb{N} \cap [2, \infty])$ and the discs D_t generated by the groups $G_t \curvearrowright D_t$ and the germs of the quotient map

$$D_t \longrightarrow (D_t \setminus \{0\})/G_t \approx U_t \setminus \{t\} \hookrightarrow \mathcal{X}.$$

The pseudogroup $\mathcal{G}$ is then an atlas of the orbifold S^2_ν. One can show that this pseudogroup depends, up to equivalence of groupoids of germs, only on the function ν.

If f is a Thurston map, and ν is the above defined least common multiple of local degrees, then S^2_ν is its *Thurston orbifold*. It follows directly from the definition that $\nu(f(t))$ is divisible by $\nu(t) \cdot \deg_t f$. Denote $\nu_0(t) = \frac{\nu(f(t))}{\deg_t f}$. Then $\nu_0(t)$ is divisible by $\nu(t)$.

The condition $\nu_0(t) = \frac{\nu(f(t))}{\deg_t f}$ implies that $f : S^2 \longrightarrow S^2$ naturally induces a covering morphism $f : S^2_{\nu_0} \longrightarrow S^2_\nu$. The condition that $\nu_0(t)$ is divisible by $\nu(t)$ implies that the identity map $S^2 \longrightarrow S^2$ can be extended to a morphism of orbifolds $\iota : S^2_{\nu_0} \longrightarrow S^2_\nu$ acting as the identical embedding $S^2 \setminus f^{-1}(P_f) \longrightarrow S^2 \setminus P_f$ on the underlying spaces. The obtained virtual endomorphism of the orbifold S^2_ν is called the *natural uniformization* of f.

Example 4.3.9. Consider the rational function $z^2 + i$. Its critical points are 0 and ∞, both of local degree 2, which have orbits $0 \mapsto i \mapsto -1 + i \mapsto -i \mapsto -1 + i$ and $\infty \mapsto \infty$. It follows that $\nu(\infty) = \infty$, and $\nu(i) = \nu(-1+i) =$

$\nu(-i) = 2$. Hence the Thurston orbifold of $z^2 + i$ is the plain $\mathbb{C}$ with three singular points $i, -1+i, -i$ both uniformized by cyclic groups of order 2.

Example 4.3.10. Consider the function $f(z) = \left(1 - \frac{2}{z}\right)^2$. It has two critical points $z = 2$ and $z = 0$. Their orbit is $2 \mapsto 0 \mapsto \infty \mapsto 1 \mapsto 1$. Both critical points are of local degree 2. It follows that the corresponding function ν is $\nu(0) = 2$, $\nu(\infty) = 4$, $\nu(1) = 4$. Note also that $\nu_0(0) = \frac{\nu(\infty)}{2} = 2$, $\nu_0(\infty) = \frac{\nu(1)}{1} = 4$, and $\nu_0(1) = \frac{\nu(1)}{1} = 4$, i.e., that $\nu_0 = \nu$. We see that $S^2_{\nu_0}$ and S^2_ν coincide, and that f induces a self-covering of the orbifold S_ν. The orbifold S_ν coincides with the orbifold of the action on $\mathbb{R}^2$ of the index two subgroup H generated by ba and ca of the group from 4.3.2.2. In other words, it is the orbifold of the action on $\mathbb{C}$ of the group H generated by the transformations $z \mapsto iz$ and $z \mapsto 1 - z$. The fundamental domain of this group is, for example, the triangle with the vertices 0, $(1+i)/2$, and 1. The quotient map from $\mathbb{C}$ to the orbifold $\mathbb{C}/H$ folds this triangle along the line connecting $(1+i)/2$ to $1/2$, so that one gets a "triangular pillow", whose corners are the singular points of the orbifold: the isotropy groups of the image of $(1+i)/2$ is cyclic of order 4 (generated by the transformation $z \mapsto iz + 1$), the isotropy group of the common image of 0 and 1 is cyclic of order 4 (generated by the transformation $z \mapsto iz$ acting on a neighborhood of 0), and the isotropy group of the image of $1/2$ is cyclic of order 2 (generated by the action of $z \mapsto 1 - z$). Figure... We will see later ... that the self-covering f of the orbifold S^2_ν is topologically conjugate to the map induced by $z \mapsto (i-1)z$ on $\mathbb{C}/H$.

4.3.4. Uniformizations of the tent map. Consider again the tent map $f : [0,1] \longrightarrow [0,1]$ from 4.2.10. It was transformed into a virtual endomorphism of an orbifold by converting $[0,1]$ into the graph of groups with two copies of $\mathbb{Z}/2\mathbb{Z}$ at the endpoints of the segment. Let us consider a more general case: a covering and a morphism $f, \iota : \mathcal{X}_1 \longrightarrow \mathcal{X}$, where $\mathcal{X}$ and $\mathcal{X}_1$ are graphs of two groups connected by an edge with trivial edge group.

Let G_0 and G_1 be the vertex groups of $\mathcal{X}$ at the endpoints 0 and 1, respectively. The map f must be a degree two covering, hence in the covering orbispace $\mathcal{X}_1$ the vertex groups are isomorphic to G_0, and $G_1 \cong \mathbb{Z}/2\mathbb{Z}$. The morphism $\iota : \mathcal{X}_1 \longrightarrow \mathcal{X}$ will induce homomorphisms $\iota_0 : G_0 \longrightarrow G_0$ and $\iota_1 : G_0 \longrightarrow G_1 \cong \mathbb{Z}/2\mathbb{Z}$. (See Figures 4.10 and 4.11, where 0 and 1 correspond to the left and the right endpoints of the orbispace). The fundamental group of $\mathcal{X}$ is isomorphic to the free product $G_0 * G_1$. In the iterated monodromy group of $f, \iota : \mathcal{X}_1 \longrightarrow \mathcal{X}$, the non-trivial element of $G_1 \cong \mathbb{Z}/2\mathbb{Z}$ acts as $\sigma = (0\ 1) \in \mathsf{S}_2$, while an element $g \in G_0$ satisfies the wreath recursion $g = (\iota_0(g), \iota_1(g))$.

Note that since $G_1 \cong \mathbb{Z}/2\mathbb{Z}$, the wreath recursion $g = (\iota_0(g), \iota_1(g))$ implies that the image of G_0 in the itereated monodromy group is an abelian group of exponent 2 invariant under ι_1. (We have $g^2 = (1, (\iota_1(g))^2)$ for all $g \in G_0$.) Therefore, we may assume without change the class of the corresponding iterated monodromy groups, that this is true for G_1 itself. Consequently (as we consider only the case of finite vertex groups here) we may assume that $G_0 \cong (\mathbb{Z}/2\mathbb{Z})^n$ for some n. Then the iterated monodromy group is determined by an automorphism $\iota_1 : (\mathbb{Z}/2\mathbb{Z})^n \longrightarrow (\mathbb{Z}/2\mathbb{Z})^n$ and an epimorphism $\iota_0 : (\mathbb{Z}/2\mathbb{Z})^n \longrightarrow \mathbb{Z}/2\mathbb{Z}$. Furthermore, one can assume (see Exercise ...) that ι_0 and ι_1 are defined by the matrices

$$(0, 0, \ldots, 0, 1)\,, \text{ and } \begin{pmatrix} 0 & 0 & \ldots & 0 & a_1 \\ 1 & 0 & \ldots & 0 & a_2 \\ 0 & 1 & \ldots & 0 & a_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & a_n \end{pmatrix},$$

for some $a_i \in \mathbb{Z}/2\mathbb{Z}$, respectively.

Example 4.3.11. Take $n = 2$ and the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Denote the elements of $G_1 = (\mathbb{Z}/2\mathbb{Z})^2$ by $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $c = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $d = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Denote the unique non-trivial element of G_0 by a. Then ι_1 acts by $b \mapsto c$, $c \mapsto d$, and $d \mapsto b$. The epimorphism ι_0 maps b and c to a, and d to 1. It follows that the wreath recursion for the corresponding iterated monodromy group is the same as for the Grigorchuk group, see the automaton on Figure 2.21 and Subsection 6.2.1.

These groups (including their generalizations to alphabets of more than two letters) were defined and studied by Z. Šunić in [**Šun07**].

4.3.5. Virtual morphisms of groupoids. The most natural and general definition of the iterated monodromy groups, in particular in the setting of orbispaces, is to via groupoids theory and biactions. See Section 3.2 for the definitions and properties of biactions.

Definition 4.3.12. Let $\mathfrak{G}, \mathfrak{H}$ be topological groupoids. A *virtual morphism* from $\mathfrak{G}$ to $\mathfrak{H}$ is a biaction $\mathfrak{G} \curvearrowright \mathcal{M} \curvearrowleft \mathfrak{H}$ such that the action $\mathcal{M} \curvearrowleft \mathfrak{H}$ is free and proper, and the ancor $P_{\mathfrak{G}} : \mathcal{M} \longrightarrow \mathfrak{G}^{(0)}$ induces a finite-to-one covering map $\mathcal{M}/\mathfrak{H} \longrightarrow \mathfrak{G}^{(0)}$. We sometimes denote the anchors of the left and the right actions by P_l and P_r, respectively.

A *virtual endomorphism* of $\mathfrak{G}$ is a virtual morphism from $\mathfrak{G}$ to $\mathfrak{G}$.

Recall that a biaction $\mathfrak{G} \curvearrowright \mathcal{M} \curvearrowleft \mathfrak{H}$ is a *morphism* if the action of $\mathfrak{H}$ is free and proper, and the map $P_{\mathfrak{G}}$ induces a homeomorphism $\mathcal{M}/\mathfrak{H} \longrightarrow \mathfrak{G}^{(0)}$. Thus, a virtual morphism can be seen as a multi-valued morphism, where the covering $P_{\mathfrak{G}}/\mathfrak{H}$ describes branches of the multivalued map.

Proposition 4.3.13. *A composition of virtual morphisms is a virtual morphism.*

Proof. ... □

Example 4.3.14. If $\mathfrak{G}$ is a group, then Definition 4.3.12 coincides with the definition of a covering biset, see Definition 4.2.2.

Example 4.3.15. If the groupoids $\mathfrak{G}$ and $\mathfrak{H}$ are trivial, i.e., are topological spaces, then the left and the right actions are trivial, hence the virtual endomorphism is a pair of maps $P_{\mathfrak{G}} : \mathcal{M} \longrightarrow \mathfrak{G}^{(0)} = \mathfrak{G}$ and $P_{\mathfrak{H}} : \mathcal{M} \longrightarrow \mathfrak{H}^{(0)} = \mathfrak{H}$. Then, by Definition 4.3.12, $P_{\mathfrak{G}}$ is a finite-to-one covering map. We see that we recover the original definition of a covering correspondence, see Definition 4.3.1. We sometimes the virtual morphism as the multivalued map $P_{\mathfrak{H}} \circ P_{\mathfrak{G}}^{-1} : \mathfrak{G} \longrightarrow \mathfrak{H}$.

The following lemma will be needed later.

Lemma 4.3.16. *Let $\mathfrak{G} \curvearrowright \mathcal{M} \curvearrowleft \mathfrak{H}$ be a biaction such that $\mathfrak{H}$ is étale and the action $\mathcal{M} \curvearrowleft \mathfrak{H}$ is free and proper. Then the map $P_{\mathfrak{G}} : \mathcal{M} \longrightarrow \mathfrak{G}^{(0)}$ is a local homeomorphism.*

Proof. By Proposition 3.2.31, the quotient map $\mathcal{M} \longrightarrow \mathcal{M}/\mathfrak{H}$ is a local homeomorphism, i.e., for every point $x \in \mathcal{M}$ there exists a neighborhood U such that the quotient map is a homeomorphism from U to its image. The map $P_{\mathfrak{G}}/\mathfrak{H} : \mathcal{M}/\mathfrak{H} \longrightarrow \mathfrak{G}^{(0)}$ is a covering, hence a local homeomorphism. It follows that $P_{\mathfrak{G}}$ is a composition of two local homeomorphisms $\mathcal{M} \longrightarrow \mathcal{M}/\mathfrak{H} \longrightarrow \mathfrak{G}^{(0)}$, hence is a local homeomorphism. □

4.3.6. Groupoid automata. Let us describe a class of virtual morphisms of groupoids analogous to automata and wreath recursions.

Definition 4.3.17. Let $\mathfrak{G}$ and $\mathfrak{H}$ be groupoids, and let X be an alphabet. A *groupoid automaton* is a continuous map $(g, x) \mapsto (g(x), g|_x)$ from $\mathfrak{G} \times \mathsf{X}$ to $\mathsf{X} \times \mathfrak{H}$ satisfying the following conditions:

(1) if $g \in \mathfrak{G}^{(0)}$, then for all $x \in \mathsf{X}$ we have $g|_x \in \mathfrak{H}^{(0)}$;
(2) for all $(g_1, g_2) \in \mathfrak{G}^{(2)}$ and $x \in \mathsf{X}$ we have

$$(g_1 g_2)(x) = g_1(g_2(x)), \qquad (g_1 g_2)|_x = g_1|_{g_2(x)} g_2|_x.$$

Note that condition (2) implies that if $g \in \mathfrak{G}^{(0)}$, then $g(x) = x$ for all $x \in \mathsf{X}$. The map $(g, x) \mapsto (g(x), g|_x)$ induces a continuous map $\iota : \mathfrak{G}^{(0)} \times \mathsf{X} \longrightarrow \mathfrak{H}^{(0)}$ by the rule $\iota(u, x) = u|_x$. Note that for every $g \in \mathfrak{G}$ we have $\mathbf{s}(g)|_x = (g^{-1}g)|_x = g^{-1}|_{g(x)}g|_x$, hence $\mathbf{s}(g)|_x = \mathbf{s}(g|_x)$. It follows that $\mathbf{s}(g|_x) = \iota(g, x)$ for all $g \in \mathfrak{G}$.

Definition 4.3.18. The *virtual morphism* associated with the groupoid automaton is the biaction $\mathfrak{G} \curvearrowright \mathcal{M} \curvearrowleft \mathfrak{H}$, where

$$\mathcal{M} = \{(t, x, h) \in \mathfrak{G}^{(0)} \times \mathsf{X} \times \mathfrak{H} \ : \ \mathbf{r}(h) = \iota(t, x)\},$$

$P_{\mathfrak{G}}(t, x, h) = t$, $P_{\mathfrak{H}}(t, x, h) = \mathbf{s}(h)$, and

$$g \cdot (t, x, h) = (\mathbf{r}(g), g(x), g|_x h), \qquad (t, x, h_1) \cdot h_2 = (t, x, h_1 h_2)$$

for all $(t, x, h), (t, x, h_1) \in \mathcal{M}$, $g \in \mathbf{s}^{-1}(t)$, and $h_2 \in \mathbf{r}^{-1}(\mathbf{s}(h_1))$.

It is checked directly that the action $\mathcal{M} \curvearrowleft \mathfrak{H}$ in the above definition is free and proper, and that the quotient $\mathcal{M}/\mathfrak{H}$ is naturally homeomorphic to $\mathfrak{G}^{(0)} \times \mathsf{X}$, so that the map $P_{\mathfrak{G}}$ induces the covering map $\mathfrak{G}^{(0)} \times \mathsf{X} \longrightarrow \mathfrak{G}^{(0)}$ equal to the projection onto the first coordinate. Consequently, $\mathfrak{G} \curvearrowright \mathcal{M} \curvearrowleft \mathfrak{H}$ is a virtual morphism.

Note that the map $x \mapsto g(x)$ is a permutation of X, and that this way we get a cocycle $\sigma : \mathfrak{G} \longrightarrow \mathsf{S}(\mathsf{X})$. The map $(g, x) \mapsto g|_x$ is a functor from $\mathfrak{G} \ltimes \sigma$ to $\mathfrak{H}$ (see...). It follows that the structure of a groupoid automaton can be described as a cocycle $\sigma : \mathfrak{G} \longrightarrow \mathsf{S}(\mathsf{X})$ and a functor $I : \mathfrak{G} \ltimes \sigma \longrightarrow \mathfrak{H}$. This description is a generalization of wreath recursions for self-similar groups.

Proposition 4.3.19. *Let $\mathfrak{G} \curvearrowright \mathcal{M} \curvearrowleft \mathfrak{H}$ be a virtual morphism of an étale groupoid such that the covering map $P_{\mathfrak{G}}/\mathfrak{G} : \mathcal{M}/\mathfrak{G} \longrightarrow \mathfrak{G}^{(0)}$ is d-to-one, and let X be a set of cardinality d. Then there exists an equivalence $\mathfrak{G}_1 \curvearrowright \mathcal{E} \curvearrowleft \mathfrak{G}$ such that the biaction $\mathfrak{G}_1 \curvearrowright \mathcal{E} \otimes \mathcal{M} \curvearrowleft \mathfrak{H}$ is isomorphic to the biaction associated with a groupoid automaton $\mathfrak{G}_1 \times \mathsf{X} \longrightarrow \mathsf{X} \times \mathfrak{H}$.*

Proof. .... Let $\mathfrak{G} \curvearrowright \mathcal{M} \curvearrowleft \mathfrak{G}$ be a virtual morphism of an étale groupoid. Suppose that the covering map $P_l/\mathfrak{G} : \mathcal{M}/\mathfrak{G} \longrightarrow \mathfrak{G}^{(0)}$ is d-to-one. Let X be an alphabet of cardinality d. For every point $t \in \mathfrak{G}^{(0)}$ choose a right $\mathfrak{G}$-orbit transversal $x_1, x_2, \ldots, x_d \in P_l^{-1}(x)$, and choose a bijection $\Lambda : \mathsf{X} \longrightarrow P_l^{-1}(t)$. Since the map $P_l : \mathcal{M} \longrightarrow \mathfrak{G}^{(0)}$ is étale by Lemma 4.3.16, there exist neighborhoods U_x, $x \in \mathsf{X}$, of $\Lambda(x)$ and an open neighborhood U of t such that $P_l : U_x \longrightarrow U$ are homeomorphisms.

The set $P_l^{-1}(U)$ is invariant under the right $\mathfrak{G}$-action, and every point $a \in P_l^{-1}(U)$ is uniquely written in the form $\Lambda(x) \cdot g$ for some $x \in \mathsf{X}$ and $g \in \mathfrak{G}$. The right action in this notation is given just by multiplication in $\mathfrak{G}$: an action $(\Lambda(x) \cdot g) \cdot h$ is defined if and only if $\Lambda(x) \cdot g$ and gh are defined, and then it is equal to $\Lambda(x) \cdot (gh)$.

Consider any subset $\mathcal{U}$ of the set of all such neighborhoods U covering an open $\mathfrak{G}$-transvesal, and consider the localization $\mathfrak{G}|_{\mathcal{U}}$. For every $U \in \mathcal{U}$ we have the corresponding sets U_x, $x \in \mathsf{X}$. We will identify the disjoint union of the sets U_x as the direct product $U \times \mathsf{X}$. Then the homeomorphisms $P_l : U_x \longrightarrow U$ are identified with the projection of the direct product $U \times \mathsf{X}$ onto the first coordinate. The disjoint union of the sets U_x for all $x \in \mathsf{X}$ and $U \in \mathcal{U}$ is identified therefore with the direct product $\mathfrak{G}|_{\mathcal{U}}^{(0)} \times \mathsf{X}$.

Consider the disjoint union of the sets $P_l^{-1}(U)$ for $U \in \mathcal{U}$. By the above, it is isomorphic as a right $\mathfrak{G}$-action to the subspace of $\mathfrak{G}|_{\mathcal{U}}^{(0)} \times \mathsf{X} \times \mathfrak{G}$ consisting of triples (t, x, g) such that ... □

4.3.7. Lifting paths by virtual morphisms. Let us show how a virtual morphism of groupoids induces virtual morphisms of the fundamental groupoids and fundamental groups. Let $\mathfrak{G} \curvearrowright \mathcal{M} \curvearrowleft \mathfrak{H}$ be a virtual morphism.

At first let us show how to lift $\mathfrak{G}$-paths to $\mathcal{M} \rtimes \mathfrak{G}$-paths. Let γ be a path in $\mathfrak{G}^{(0)}$. For every point $x \in \mathcal{M}$ such that $P_{\mathfrak{G}}(x)$ is the beginning of γ there exists a unique lift γ_x of γ by $P_{\mathfrak{G}}/\mathfrak{H}$ to a path in $\mathcal{M}/\mathfrak{H}$ starting at the orbit $x\mathfrak{H}$. The groupoid $\mathcal{M} \rtimes \mathfrak{H}$ is equivalent to the trivial groupoid on the space $\mathcal{M}/\mathfrak{H}$, hence the path γ_x can be lifted to a $(\mathcal{M} \rtimes \mathfrak{H})$-path. The lift is unique up to isomorphism except for the choice of the endpoints. It is equal to $h_m\delta_m \cdots \delta_1 h_0$, where $h_i \in \mathcal{M} \rtimes \mathfrak{H}$ and δ_i are paths in $\mathcal{M}$. We can define its *image in* $\mathfrak{H}$ as the $\mathfrak{H}$-path obtained by applying the natural projection $\mathcal{M} \rtimes \mathfrak{H} \longrightarrow \mathfrak{H}$. We will denote it $P_{\mathfrak{H}}(\gamma_x)$. The path $P_{\mathfrak{H}}(\gamma_x)$ is unique up to a choice of the endpoints inside particular $\mathfrak{H}$-orbits.

We see that every path in $\mathfrak{G}^{(0)}$ can be lifted to a $(\mathcal{M} \rtimes \mathfrak{H})$-path, and then mapped to a $\mathfrak{H}$-path.

Let now $\gamma = g_n\gamma_n \cdots \gamma_1 g_0$ be a $\mathfrak{G}$-path, and let $x \in \mathcal{M}$ be such that $P_{\mathfrak{G}}(x)$ is equal to $\mathbf{s}(\gamma)$. Let $x_1 = g_0 \cdot x$. Let γ_1' be a lift of γ_1 to a $(\mathcal{M} \rtimes \mathfrak{H})$-path starting in x_1. Let x_2 be the image under g_1 of the end of γ_1'. Then there exists a lift of γ_2 to an $(\mathcal{M} \rtimes \mathfrak{H})$-path starting in x_2. Continue the lifting process inductively. Note that the end of $P_{\mathfrak{H}}(\gamma_i')$ is equal to the beginning of $P_{\mathfrak{H}}(\gamma_{i+1}')$. It follows that concatenation of the paths $P_{\mathfrak{H}}(\gamma_i')$ is an $\mathfrak{H}$-path.

We see that also every $\mathfrak{G}$-path can be lifted and then mapped to an $\mathfrak{H}$-path. This procedure coincides, in the case of morphisms with the composition of a path and a morphism described at the beginning of this section. And in the same way, the corresponding $\mathfrak{H}$-path is unique, up to isomorphism, only as a morphism, and not as an $\mathfrak{H}$-path.

Let us describe now the $\pi_1(\mathfrak{G})$-biset associated with the virtual endomorphism $\mathfrak{G} \curvearrowright \mathcal{M} \curvearrowleft \mathfrak{G}$. Define $\widetilde{\mathcal{M}}$ as the set of triples (x, γ, y), where γ is a morphism from $[0,1]$ to $\mathfrak{H}$, $y \in \mathfrak{H}^{(0)}$ is a beginning of γ, and $x \in \mathcal{M}/\mathfrak{H}$ is the $\mathfrak{H}$-orbit of a point $x_0 \in \mathcal{M}$ such that $P_{\mathfrak{H}}(x_0)$ is the end of γ.

We have a natural structure of a biaction $\pi_1(\mathfrak{G}) \curvearrowright \widetilde{\mathcal{M}} \curvearrowleft \pi_1(\mathfrak{H})$ over the anchors $P_{\pi_1(\mathfrak{G})}(x,\gamma,y) = P_{\mathfrak{G}}(x)$, $P_{\pi_1(\mathfrak{H})}(x,\gamma,y) = y$. The right action is just by concatenation:

$$(x,\gamma,y)\cdot\beta = (x,\gamma\beta,\mathbf{s}(\beta)).$$

For the left action of a $\mathfrak{G}$-path α, we find the lift α_x of α to a $\mathfrak{G} \ltimes (\mathcal{M}/\mathfrak{H})$-path starting in x, finding its end x', mapping α_x to a $\mathfrak{H}$-path $P_{\mathfrak{H}}(\alpha_x)$, and then setting

$$\alpha\cdot(x,\gamma,y) = (x', P_{\mathfrak{H}}(\alpha_x)\gamma, y).$$

Restricting it to the fundamental group $\pi_1(\mathfrak{G},t)$ defines a group biset.

The case of virtual morphisms defined by groupoid automata is more explicit, so let us describe the induced biset over the fundamental groups in this case.

Let $\sigma : \mathfrak{G} \longrightarrow \mathsf{S}(\mathsf{X})$ be a cocycle, and let $I : \mathfrak{G} \ltimes \sigma \longrightarrow \mathfrak{H}$ be functor, and let $(g,x) \mapsto (g(x), g|_x)$ be the corresponding groupoid automaton. (Recall that this means that $\sigma(g)$ is the permutation $x \mapsto g(x)$ and that $I(g,x) = g|_x$.)

Let $\gamma = g_n\gamma_n\cdots\gamma_1 g_0$ be a $\mathfrak{G}$-path, and let $x \in \mathsf{X}$. Then γ has a unique lift to a $\mathfrak{G} \ltimes \sigma$-path starting at $(\mathbf{s}(\gamma), x)$, namely

$$\begin{aligned}(g_n, g_{n-1}g_{n-2}\cdots g_1(x))(\gamma_n, g_{n-1}g_{n-2}\cdots g_0(x))\cdots \\ (g_2, g_1g_0(x))(\gamma_1, g_1g_0(x))(g_1, g_0(x))(\gamma_1, g_0(x))(g_0, x),\end{aligned}$$

where (α, y), for a path α in $\mathfrak{G}^{(0)}$ and a letter $y \in \mathsf{X}$ is the path $t \mapsto (\alpha(t), y)$ in $\mathfrak{G}^{(0)} \times \mathsf{X}$. We will denote this lift by (γ, x), which agrees with the notation (g,x) for the elements of $\mathfrak{G}\ltimes\sigma$. In fact, we see that the fundamental groupoid of $\mathfrak{G} \ltimes \sigma$ is naturally isomorphic to $\pi_1(\mathfrak{G}) \ltimes \tilde{\sigma}$, where $\tilde{\sigma} : \pi_1(\mathfrak{G}) \longrightarrow \mathsf{S}(\mathsf{X})$ is the cocycle

$$\tilde{\sigma}(g_n\gamma_n g_{n-1}\cdots g_1\gamma_1 g_0) = \sigma(g_n g_{n-1}\cdots g_1 g_0).$$

The functor $I : \mathfrak{G} \ltimes \sigma \longrightarrow \mathfrak{H}$ induces then the functor $\tilde{I} : \pi_1(\mathfrak{G} \ltimes \sigma) = \pi_1(\mathfrak{G}) \times \tilde{\sigma} \longrightarrow \pi_1(\mathfrak{H})$. We see that the virtual morphism from $\mathfrak{G}$ to $\mathfrak{H}$ defined by σ and I induces the virtual morphism of the fundamental groupoids defined by $\tilde{\sigma}$ and $\tilde{I}$.

4.3.8. Iterating a virtual endomorphism. By Proposition 4.3.13, composition of two virtual morphisms of groupoids is also a virtual morphism. It follows that if $\mathfrak{G} \curvearrowright \mathcal{M} \curvearrowleft \mathfrak{G}$ is a virtual endomorphism of a groupoid $\mathfrak{G}$, then we can iterated it, and get a sequence $\mathcal{M}^{\otimes n}$ of virtual endomorphism of $\mathfrak{G}$.

4.3.8.1. *Trivial groupoids.* Let consider at first the case of trivial groupoids (i.e., topological spaces). Since every action of a trivial groupoid is free and proper, we can compose any two biactions of trivial groupoids. A biaction in this case is just a topological correspondence, i.e., a pair of maps $P_{\mathcal{Y}_1} : \mathcal{M} \longrightarrow \mathcal{Y}_1$ and $P_{\mathcal{Y}_2} : \mathcal{M} \longrightarrow \mathcal{Y}_2$. A correspondence is a virtual morphism if $P_{\mathcal{Y}_1}$ is a covering map.

If $(P_{\mathcal{Y}_1} : \mathcal{M}_1 \longrightarrow \mathcal{Y}_1, P'_{\mathcal{Y}_2} : \mathcal{M}_1 \longrightarrow \mathcal{Y}_2)$ and $(P''_{\mathcal{Y}_2} : \mathcal{M}_2 \longrightarrow \mathcal{Y}_2, P_{\mathcal{Y}_3} : \mathcal{M}_2 \longrightarrow \mathcal{Y}_3)$ are correspondences, then their composition is (see 3.2.2) the space $\{(x_1, x_2) \in \mathcal{M}_1 \times \mathcal{M}_2 \ : \ P'_{\mathcal{Y}_2}(x_1) = P''_{\mathcal{Y}_2}(x_2)\}$ together with the maps $(x_1, x_2) \mapsto P_{\mathcal{Y}_1}(x_1)$ and $(x_1, x_2) \mapsto P_{\mathcal{Y}_3}(x_2)$.

In other words, the composition is constructed by taking the the *pull-back* (or *fiber product*) of the maps

$$\begin{array}{ccc} & & \mathcal{M}_2 \\ & & \Big\downarrow {\scriptstyle P''_{\mathcal{Y}_2}} \\ \mathcal{M}_1 & \xrightarrow{P'_{\mathcal{Y}_2}} & \mathcal{Y}_2 \end{array}$$

and thus getting the diagram

$$\begin{array}{ccccc} \mathcal{M}_1 \otimes \mathcal{M}_2 & \longrightarrow & \mathcal{M}_2 & \xrightarrow{P_{\mathcal{Y}_3}} & \mathcal{Y}_3 \\ \Big\downarrow & & \Big\downarrow {\scriptstyle P''_{\mathcal{Y}_2}} & & \\ \mathcal{M}_1 & \xrightarrow{P'_{\mathcal{Y}_2}} & \mathcal{Y}_2 & & \\ \Big\downarrow {\scriptstyle P_{\mathcal{Y}_1}} & & & & \\ \mathcal{Y}_1 & & & & \end{array},$$

The compositions $\mathcal{M}_1 \otimes \mathcal{M}_2 \longrightarrow \mathcal{Y}_1$ of the left-hand vertical arrows and the composition $\mathcal{M}_1 \otimes \mathcal{M}_2 \longrightarrow \mathcal{Y}_3$ of the top horizontal arrows form the composition of the correspondences.

The following description of the iteration of one correspondence is proved directly by induction.

Proposition 4.3.20. *Let $F, I : \mathcal{M} \longrightarrow \mathcal{X}$ be a topological correspondence. Denote by $\mathcal{M}_n$ the subspace of $\mathcal{M}^n$ consisting of sequences $(x_1, x_2, \ldots, x_n)$ such that $I(x_i) = F(x_{i+1})$ for every $i = 1, 2, \ldots, n-1$. Define*

$$F^n(x_1, x_2, \ldots, x_n) = F(x_1), \quad I^n(x_1, x_2, \ldots, x_n) = I(x_n).$$

Then the correspondence $F^n, I^n : \mathcal{M}_n \longrightarrow \mathcal{X}$ is isomorphic to the nth iteration of $F, I : \mathcal{M} \longrightarrow \mathcal{X}$.

More explicitly, the nth iteration is inductively by constructing the following pull-back diagrams:

$$\begin{tikzcd} \mathcal{M}_{n+1} \arrow[r, "I_n"] \arrow[d, "F_n"] & \mathcal{M}_n \arrow[d, "F_{n-1}"] \\ \mathcal{M}_n \arrow[r, "I_{n-1}"] & \mathcal{M}_{n-1} \end{tikzcd},$$

where we can set $\mathcal{M}_0 = \mathcal{X}$, $F_0 = F$, and $I_0 = I$. The maps F_n and I_n are given by

$$\begin{aligned} F_n(x_1, x_2, \ldots, x_{n+1}) &= (x_1, x_2, \ldots, x_n), \\ I_n(x_1, x_2, \ldots, x_{n+1}) &= (x_2, x_3, \ldots, x_{n+1}) \end{aligned}$$

for $n \geqslant 1$.

We will usually denote a virtual endomorphism by $f, \iota : \mathcal{X}_1 \longrightarrow \mathcal{X}$, where $f : \mathcal{X}_1 \longrightarrow \mathcal{X}$ is a finite degree covering map. Our usual dynamical interpretation is that ι as an approximation of the identity map (e.g., it is the identical embedding in the case of correspondences defined by partial self-coverings). Then the space $\mathcal{X}_n$ of sequences $(x_1, x_2, \ldots, x_n)$ such that $\iota(x_i) = f(x_{i+1})$ constructed above is interpreted as the space of *orbits* of length n. Note that it is a *backward* orbit if we interpret ι as an approximation of the identity map, but it is a *forward* orbit if we interpret the virtual endomorphism as a multivalued map. Unfortunately, both approaches are convenient in different situations (we will see this ambiguity later, when the same phenomenon will be called "contraction" and "expansion" at the same time). To avoid confusion, we will call a sequence $(x_1, x_2, \ldots, x_n)$ a *forward f-orbit* if $f(x_i) = \iota(x_{i+1})$ for all i. If we have $f(x_{i+1}) = \iota(x_i)$, then we call it a *backward f-orbit.*

Example 4.3.21. Suppose that $f : \mathcal{X}_1 \longrightarrow \mathcal{X}$ is a partial self-covering, and let $\iota : \mathcal{X}_1 \longrightarrow \mathcal{X}$ be the identical embedding. Then a sequence $(x_1, x_2, \ldots, x_n)$ is a forward f-orbit of the correspondence if and only if $f(x_i) = x_{i+1}$ for every i, i.e., if it is an orbit of length n of the partial map f. It follows that the space $\mathcal{X}_n$ of orbits of length n is naturally identified with the domain of the nth iteration f^n of the partial self-covering. Moreover, our definition of iteration of the topological correspondence agrees with the natural notion of iteration of a partial map.

Example 4.3.22. Let $f, \iota : \mathcal{X}_1 \longrightarrow \mathcal{X}$ be the topological correspondence describing the dual Moore diagram of an automaton $\mathcal{A} = (\mathsf{X}, Q, \pi, \lambda)$, as in Example 4.3.5. Suppose that $(t_1, t_2, \ldots, t_n)$ is a forward f-orbit of this correspondence, such that the points t_i are not vertices (i.e., belong to the interiors of some edges). Let $(q_i, x_i) \ni t_i$ be the corresponding edges. Then $f(t_i) = \iota(t_{i+1})$ implies that $q_i = \pi(q_{i-1}, x_{i-1})$. The letters x_i are the beginnings of the edges (q_i, x_i). Their ends are $y_i = \lambda(q_i, x_i)$.

It follows that the space $\mathcal{X}_n$ of orbits of length n is isomorphic to the dual Moore diagram of the automaton describing the action of $\mathcal{A}$ on words of length n. An edge of $\mathcal{X}_n$ containing the point $(t_1, t_2, \dots, t_n)$ is uniquely determined by $(q_1, x_1x_2\dots x_n)$. The set of vertices of $\mathcal{X}_n$ is X^n, and the edge $(q_1, x_1x_2\dots x_n)$ starts in $x_1x_2\dots x_n$, ends in $y_1y_2\dots y_n$, and corresponds to the orbit $((q_1, x_1), (q_2, x_2), \dots, (q_n, x_n))$ of edges of $\mathcal{X}_1$, where $q_i = \pi(q_{i-1}, x_{i-1})$. The iteration $f^n, \iota^n : \mathcal{X}_n \longrightarrow \mathcal{X}$ is given by $f^n(q_1, x_1x_2\dots x_n) = q_1$ and $\iota^n(q_1, x_1x_2\dots x_n) = \pi(q_n, x_n)$.

Example 4.3.23. A *simplicial topological correspondence* $f, \iota : \mathcal{X}_1 \longrightarrow \mathcal{X}$ is a pair of simplicial maps between simplicial complexes $\mathcal{X}_1$ and $\mathcal{X}$ such that f is a finite degree covering map.

If $f, \iota : \mathcal{X}_1 \longrightarrow \mathcal{X}$ is a simplicial topological correspondence, then the spaces of orbits $\mathcal{X}_n$ and the maps f_n, ι_n between them are simplicial complexes and maps, respectively. Namely, we can define the set of simplices of $\mathcal{X}_n$ as the set of sequences $(\Delta_1, \Delta_2, \dots, \Delta_n)$ such that $f(\Delta_i) = \iota(\Delta_{i+1})$ with the natural incidence relations. Note that since the map ι is not required to be dimension-preserving, the dimensions of the simplices Δ_i may be different (but non-decreasing with i).

4.3.8.2. *Groupoid automata.* Iteration of groupoid automata is very similar to iterations of wreath recursions and transducers.

Proposition 4.3.24. *Let* $\mathsf{A} : \mathfrak{G} \times \mathsf{X} \longrightarrow \mathsf{X} \times \mathfrak{G} : (g, x) \mapsto (g(x), g|_x)$ *be a groupoid automaton. Define the* $n$the iterate $\mathsf{A}^{\otimes n} : \mathfrak{G} \times \mathsf{X}^n \longrightarrow \mathsf{X}^n \times \mathfrak{G}$ *of the automaton* A *inductively by the rule*

$$g(x_1x_2\dots x_n) = g(x_1)g|_{x_1}(x_2x_3\dots x_n), \quad g|_{x_1x_2\dots x_n} = g|_{x_1}|_{x_2x_3\dots x_n}.$$

Then the virtual endomorphism associated with $\mathsf{A}^{\otimes n}$ *is isomorphic to the* n*th iteration of the virtual endomorphisms associated with* A.

Proof. ... □

The approach via to iterating via groupoid automata may be convenient even in the case of self-coverings of topological spaces.

As an example, consider the basilica map $z^2 - 1$ on its Julia set. Let us localize the trivial groupoid to the cover by open subsets shown on .... Then ....

4.4. Expanding maps and contracting groups

4.4.1. Iterated monodromy groups of expanding maps. Relation between the iterated monodromy groups and dynamical systems is the closest in the case of expanding maps....

Let us recall the main definitions and properties of expanding maps. We say that a map $f \circlearrowright \mathcal{X}$, where $\mathcal{X}$ is a compact metric space, is *expanding* if there exist $\epsilon > 0$ and $L > 1$ such that $d(f(x), f(y)) \geqslant Ld(x, y)$ for all $x, y \in \mathcal{X}$ such that $d(x, y) \leqslant \epsilon$, see Definition 1.4.1.

If $f \circlearrowright \mathcal{X}$ is an expanding covering map, then there exists $\delta > 0$ such that $\delta < \epsilon$, and for every set $U \subset \mathcal{X}$ of diameter $\leqslant \delta$ the set $f^{-1}(U)$ is a disjoint union of finitely many sets $U_1, U_2, \ldots, U_d$ such that $f : U_i \longrightarrow U$ are homeomorphisms, and the distance between any two points belonging to different sets U_i is greater than δ, see Lemma 1.4.37. We call U_i the *components* of $f^{-1}(U)$. Note that then U_i are also of diameter less than δ, so for every n we can inductively define components of $f^{-n}(U)$. We will say that δ is a *strong injectivity constant* of the map.

The disjoint union of the sets of components of $f^{-n}(U)$ for $n \geqslant 0$ form the preimage tree T_U. Theorem 1.4.36 shows that the inverse limit $\hat{\mathcal{X}}$ of the maps $\mathcal{X} \xleftarrow{f} \mathcal{X} \xleftarrow{f} \mathcal{X} \cdots$ is a fiber bundle over $\mathcal{X}$: it is naturally locally homeomorphic to the direct product $U \times \partial T_U$. Our goal is to understand how different pieces $U \times \partial T_U$ are glued together to produce $\hat{\mathcal{X}}$.

For every $t \in U$, we have a natural isomorphism of the tree T_U with the tree of preimages T_t defined at the beginning of 4.1.1. It maps a vertex $v \in f^{-n}(t)$ of T_t to the unique component $V \subset f^{-n}(U)$ such that $v \in V$.

Definition 4.4.1. Let $\delta > 0$ be a strong injectivity constant of $f \circlearrowright \mathcal{X}$. Suppose that $A, B \subset \mathcal{X}$ are sets of diameters less than δ such that $A \cap B \neq \varnothing$. Then for every $x \in A \cap B$ we have natural isomorphisms $T_A \longrightarrow T_x \longrightarrow T_B$. Denote by $S_{A,B} : T_A \longrightarrow T_B$ their composition. We call $S_{A,B}$ the *elementary holonomy*.

The induced map $S_{A,B} : \partial T_A \longrightarrow \partial T_B$ describes how the pieces $A \times \partial T_A$ and $B \times \partial T_B$ are attached to each other in $\hat{\mathcal{X}}$. The isomorphism $S_{A,B}$ maps a component A_n of $f^{-n}(A)$ to the unique component B_n of $f^{-n}(B)$ such that $A_n \cap B_n \neq \varnothing$.

Lemma 4.4.2. *If U_1, U_2, U_3 be subset of diameter less than δ such that $U_1 \cap U_2 \cap U_3 \neq \varnothing$, then $S_{U_2,U_3} \circ S_{U_1,U_2} = S_{U_1,U_3}$.*

Proof. Choose a point $x \in U_1 \cap U_2 \cap U_3$. Then S_{U_i,U_j} is equal to the composition of the natural isomorphisms $T_{U_i} \longrightarrow T_x \longrightarrow T_{U_j}$, which implies the statement of the lemma. □

Let $\mathcal{U}$ be a finite cover of $\mathcal{X}$ by sets of diameter less than δ. Recall that a *nerve* of the cover $\mathcal{U}$ is the simplicial complex with the set of vertices equal to $\mathcal{U}$ in which a subset $\mathcal{C} \subset \mathcal{U}$ is a simplex if and only if $\bigcap_{A \in \mathcal{C}} A$ is non-empty.

Let $\Gamma_{\mathcal{U}}$ be the nerve of the cover $\mathcal{U}$. For every edge (U_1, U_2) of $\Gamma_{\mathcal{U}}$ we have the isomorphism $S_{U_1,U_2} : T_{U_1} \longrightarrow T_{U_2}$.

The maps S_{U_1,U_2} generate a small category of isomorphisms between the trees T_{U_i}, i.e., a groupoid. Let us denote this groupoid by $\mathrm{IMG}\,(f,\mathcal{U})$.

For every path $\gamma = (U_1, U_2, \ldots, U_n)$, we get the composition $S_\gamma : T_{U_1} \longrightarrow T_{U_n}$ of the isomorphisms $S_{U_i,U_{i+1}}$. Lemma 4.4.2 implies that the map $\gamma \longrightarrow S_\gamma$ is a homomorphism from the fundamental groupoid of the nerve $\Gamma_{\mathcal{U}}$ to the groupoid $\mathrm{IMG}\,(f,\mathcal{U})$.

Instead of the abstract (i.e., discrete) groupoid $\mathrm{IMG}\,(f,\mathcal{U})$, we can consider a naturally defined étale groupoid acting on the boundaries of the trees T_U. Namely, consider the disjoint union $\bigsqcup_{U\in\mathcal{U}} \partial T_U$. Then every isomorphism S_{U_1,U_2} induces a homeomorphism $S_{U_1,U_2} : \partial T_{U_1} \longrightarrow \partial T_{U_2}$ between two clopen subsets of the union. Denote by $\mathfrak{G}(f,\mathcal{U})$ the groupoid of germs generated by these local homeomorphisms. This is the groupoid associated with the local product structure, as defined in 3.1.4.2. Note that in this case a more natural groupoid is the groupoid defined in 3.1.4.3 for the natural extension of f (the one acting on the stable leaves), which may be bigger than $\mathfrak{G}(f,\mathcal{U})$ if the space $\mathcal{X}$ is not connected.

Suppose now that $\mathcal{X}$ is connected and locally connected. Then for any finite cover $\mathcal{U}$ of $\mathcal{X}$ by open connected subsets the nerve $\Gamma_{\mathcal{U}}$ is connected and the groupoid $\mathrm{IMG}\,(f,\mathcal{U})$ is equivalent to the isotropy group of an element $U \in \mathcal{U}$. This is the *iterated monodromy group* of f. It is the group of all elements of the form S_γ, where γ is an element of the fundamental group $\pi_1(\Gamma_{\mathcal{U}}, U)$, i.e., a closed path starting and ending in U.

The groupoid $\mathrm{IMG}\,(f,\mathcal{U})$ acts on $\bigsqcup_{U\in\mathcal{U}} \partial T_U$, so the isotropy group of U acts on ∂T_U. We get an action of $\mathrm{IMG}\,(f)$ on the boundary of the tree T_U.

This new definition of the iterated monodromy group coincides with the one given in 4.1.1. Recall that every connected and locally connected space is path connected... Let γ be a path in $\mathcal{X}$ starting in x_1 and ending in x_2. Let $\mathcal{U}$ be a finite cover of $\mathcal{X}$ by open sets of diameter less than δ. We can partition γ into a concatenation $\gamma_1\gamma_2\ldots\gamma_k$ of paths of diameter less than the Lebesgue number of $\mathcal{U}$. Let $U_i \in \mathcal{U}$ be such that the image of γ_i is contained in U_i. Then $\gamma' = (U_1, U_2, \ldots, U_k)$ is a path in $\Gamma_{\mathcal{U}}$. It is easy to see that $S_\gamma : T_{U_1} \longrightarrow T_{U_k}$ is equal to $S_{\gamma'}$. Here T_{U_1} and T_{U_k} are identified with T_{x_1} and T_{x_2}, respectively, by the natural isomorphism, using the fact that $x_1 \in U_1$ and $x_2 \in U_k$. It follows that the group of automorphisms S_γ of a tree of preimages T_t defined by elements γ of the fundamental group $\pi_1(\mathcal{X}, t)$ coincides with the group of automorphisms of T_t defined by the elements of the fundamental group $\pi_1(\Gamma_{\mathcal{U}}, U)$, where $t \in U \in \mathcal{U}$, i.e., that the two definitions of the iterated monodromy groups coincide.

4.4.2. Simplicial models of expanding maps. We will use now the notion of a topological correspondence to approximate arbitrary expanding maps by simplicial complexes.

Let $f \circlearrowright \mathcal{X}$ be an expanding covering, and let δ, as before, be a strong injectivity constant for f. We do not impose any connectivity conditions on $\mathcal{X}$ in this subsection.

Let $\mathcal{U}$ be a finite cover of $\mathcal{X}$ by subsets of diameter less than δ. Denote by $\mathcal{U}_n$ the set of components of $f^{-n}(U)$ for $U \in \mathcal{U}$. We also denote $\mathcal{U}_0 = \mathcal{U}$. Denote by Γ_n the nerve of the cover $\mathcal{U}_n$.

The map f induces simplicial maps $f_n : \Gamma_{n+1} \longrightarrow \Gamma_n$ by the rule $f_n(U) = f(U)$, where $f(U)$ is the image of U as a set under the map $f : \mathcal{X} \longrightarrow \mathcal{X}$, i.e., U is a component of $f^{-1}(f_n(U))$.

Lemma 4.4.3. *The maps $f_n : \Gamma_{n+1} \longrightarrow \Gamma_n$ are coverings.*

Proof. For $U \in \mathcal{U}_n$, denote by N_U the sub-complex of Γ_n equal to the union of simplices containing U.

It is enough to show that $f : N_U \longrightarrow N_{f(U)}$ is an isomorphism for every $U \in \mathcal{U}_{n+1}$. It is obviously a simplicial map.

Let us show that $f : N_U \longrightarrow N_{f(U)}$ is injective on the set of vertices adjacent to U. Suppose that it is not, then there exist elements $A, B, C \in \mathcal{U}_{n+1}$ such that $A \cap C$ and $B \cap C$ are non-empty, and $f(A) = f(B)$. But then there exist $x \in A$ and $y \in B$ such that $d(x, y) < \delta$, which contradicts the conditions of Lemma 1.4.37.

For every simplex $\Delta = \{f(U), A_1, A_2, \ldots, A_k\}$ of Γ_n containing $f(U)$ there exists a unique simplex

$$\Delta' = \{U, B_1, B_2, \ldots, B_k\} = \{U, S_{f(U),A_1}(U), S_{f(U),A_2}(U), \ldots, S_{f(U),A_k}(B_k)\}$$

of Γ_{n+1} containing U such that $f(\Delta') = \Delta$. Consequently, $f : N_U \longrightarrow N_{f(U)}$ is an isomorphism. □

Definition 4.4.4. We say that $\mathcal{U}$ is *semi-Markovian* if for every $U \in \mathcal{U}_1$ there exists $U' \in \mathcal{U}$ such that $U \subset U'$.

The following lemma is a direct corollary of the Lebesgue's covering lemma.

Lemma 4.4.5. *Let $\mathcal{U}$ be an open cover of $\mathcal{X}$ by sets of diameter less than δ. Then there exists $n \geqslant 1$ such that $\mathcal{U}$ is semi-Markovian for $f^n \circlearrowright \mathcal{X}$.*

On the other hand, we do not have to pass to an iterated of f if we are allowed to change the cover.

Lemma 4.4.6. *For every $\delta > 0$ there exists a finite semi-Markovian cover by open sets of diameter less than δ.*

Proof. Let $\mathcal{V}$ be a cover of $\mathcal{X}$ by sets of diameter less than δ_0. As before, we denote by $\mathcal{V}_n$ the set of components of $f^{-n}(A)$ for $A \in \mathcal{V}$. Define, for every $V \in \mathcal{V}$ the sets $V^{(n)}$ inductively by the rule that $V^{(0)} = V$, and $V^{(n+1)}$ is equal to the union of $V^{(n)}$ and all elements $W \in \mathcal{V}_{n+1}$ such that $W \cap V^{(n+1)} \neq \varnothing$. Define $V^{(\infty)} = \bigcup_{n\geqslant 1} V^{(n)}$, and let $\mathcal{V}^{(\infty)} = \{V^{(\infty)} : V \in \mathcal{V}\}$.

Diameter of $V^{(n)}$ is less than

$$2\delta_0(1 + L^{-1} + L^{-2} + \cdots + L^{-n}) < 2\delta_0/(1 - L^{-1}).$$

Consequently, diameter of $V^{(\infty)}$ is not more than $2\delta_0/(1 - L^{-1})$. Assume that $\delta_0 < (1 - L^{-1})\delta/2$. Then all elements of $\mathcal{V}^{(\infty)}$ have diameters less than δ.

It is easy to see that then $(\mathcal{V}_n)^{(\infty)} = (\mathcal{V}^{(\infty)})_n$, and that if $U \in \mathcal{V}_1$ and $V \in \mathcal{V}$ are such that $U \cap V \neq \varnothing$, then $U^{(\infty)}$ (as an element of $\mathcal{V}_1^{(\infty)}$) is contained in $V^{(\infty)}$, which implies that $\mathcal{V}^{(\infty)}$ is semi-Markovian. $\square$

Let $\mathcal{U}$ be a semi-Markovian cover. Choose for every $U \in \mathcal{U}_1$ an element $\iota(U) \in \mathcal{U}_0$ such that $U \subset \iota(U)$. It is easy to see that $\iota : \Gamma_1 \longrightarrow \Gamma_0$ is a simplicial map.

Since U and $\iota(U)$ intersect, the elementary holonomy $S_{U,\iota(U)} : T_U \longrightarrow T_{\iota(U)}$ is defined. For every n it defines a bijection between the set of components of $f^{-n}(U)$ and the set of components of $f^{-n}(\iota(U))$. These sets are subsets of $\mathcal{U}_{n+1}$ and $\mathcal{U}_n$ respectively, and union of the maps $S_{U,\iota(U)}$ for $U \in \mathcal{U}_1$ is a map from $\mathcal{U}_{n+1}$ to $\mathcal{U}_n$, which we will denote ι_n.

Equivalently, $\iota_n(A)$ is the unique component of $f^{-1}(\iota_{n-1}(f(A)))$ containing A.

The map ι_n is uniquely defined by the condition that if A is a component of $f^{-n}(U)$ for $U \in \mathcal{U}_1$, then $\iota_n(A)$ is the unique component of $f^{-n}(\iota(U))$ such that $\iota_n(A) \supset A$. It follows that $\iota_n : \Gamma_{n+1} \longrightarrow \Gamma_n$ is simplicial and that the diagram

$$\begin{array}{ccc} \Gamma_{n+2} & \xrightarrow{\iota_{n+1}} & \Gamma_{n+1} \\ \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} \\ \Gamma_{n+1} & \xrightarrow{\iota_n} & \Gamma_n \end{array} \tag{4.2}$$

is commutative.

Let us show that the pair $f_0, \iota_0 : \Gamma_1 \longrightarrow \Gamma_0$ uniquely determines the sequence $f_n, \iota_n : \Gamma_{n+1} \longrightarrow \Gamma_n$. Namely, we will show that the complexes Γ_n are produced by iteration of the simplicial virtual endomorphism $f_0, \iota_0 : \Gamma_1 \longrightarrow \Gamma_0$.

Proposition 4.4.7. *Let $\widetilde{\Gamma}_n$ and $\tilde{f}_n, \tilde{\iota}_n : \widetilde{\Gamma}_{n+1} \longrightarrow \widetilde{\Gamma}_n$ be the complexes and maps obtained by iterating the simplicial topological correspondence $f_0, \iota_0 : \Gamma_1 \longrightarrow \Gamma_0$. (In particular, $\widetilde{\Gamma}_n = \Gamma_n$ for $n = 0, 1$.)*

Then there exist isomorphisms $\phi_n : \widetilde{\Gamma}_n \longrightarrow \Gamma_n$ such that

$$f_n \circ \phi_{n+1} = \phi_n \circ \tilde{f}_n, \qquad \iota_n \circ \phi_{n+1} = \phi_n \circ \tilde{\iota}_n$$

for all $n \geqslant 1$.

Proof. Let us construct and prove properties of ϕ_n by induction. For $n = 1$ the graph $\widetilde{\Gamma}_1$ coincides with Γ_1, so set ϕ_1 to be equal to the identity map.

Suppose that ϕ_n is defined and satisfies the properties of the proposition. Let $(v_1, v_2, \ldots, v_{n+1})$ be an arbitrary vertex of $\widetilde{\Gamma}_{n+1}$.

If $n = 1$, then we have $v_2 \subset f(v_1)$, since $(v_1, v_2) \in \Gamma_1$. For $n > 1$ we have We have $\phi_n(v_2, v_3, \ldots, v_{n+1}) \subset f(\phi_n(v_1, v_2, \ldots, v_n))$, since $\phi_{n-1}(v_2, v_3, \ldots, v_n) = \iota_n(\phi_n(v_2, v_3, \ldots, v_{n+1}))$ and $f(\phi_n(v_1, v_2, \ldots, v_n)) = \phi_{n-1}(v_2, v_3, \ldots, v_n)$, by the inductive hypothesis.

Consequently, for $n = 1$ there exists a unique component of $f^{-1}(v_2)$ contained in v_1. We set $\phi_2((v_1, v_2))$ to be equal to this component. Similarly, for $n > 1$ there exists a unique component of $f^{-1}(\phi_n(v_2, v_3, \ldots, v_{n+1}))$ contained in $\phi_n(v_1, v_2, \ldots, v_n)$. We set $\phi_{n+1}(v_1, v_2, \ldots, v_{n+1})$ to be equal to it.

Formally, in both cases we defined ϕ_{n+1} by the rule

$$\phi_{n+1}(v_1, v_2, \ldots, v_{n+1}) = S_{f_{n-1}(\phi_n(v_1, v_2, \ldots, v_n)), \phi_n(v_2, v_3, \ldots, v_{n+1})}(\phi_n(v_1, v_2, \ldots, v_n)). \tag{4.3}$$

We get a map $\phi_{n+1} : \widetilde{\Gamma}_{n+1} \longrightarrow \Gamma_{n+1}$ (between sets of vertices). Let us show that it satisfies the conditions of the proposition and that it is an isomorphism of simplicial complexes.

It follows directly from the definition that $f_n(\phi_{n+1}(v_1, v_2, \ldots, v_{n+1})) = \phi_n(v_2, v_3, \ldots, v_{n+1}))$, as we defined $\phi_{n+1}(v_1, v_2, \ldots, v_{n+1})$ as a component of $f^{-1}(\phi_n(v_2, v_3, \ldots, v_{n+1}))$.

The vertex $\iota_n(\phi_{n+1}(v_1, v_2, \ldots, v_{n+1}))$ is, by definition, the component of $f^{-1}(\iota_{n-1} \circ f \circ \phi_{n+1}(v_1, v_2, \ldots, v_{n+1}))$ containing $\phi_{n+1}(v_1, v_2, \ldots, v_{n+1})$. We have $\iota_{n-1} \circ f \circ \phi_{n+1}(v_1, v_2, \ldots, v_{n+1}) = \iota_{n-1}(\phi_n(v_2, v_3, \ldots, v_{n+1})) = \phi_{n-1}(v_2, v_3, \ldots, v_n)$. Consequently, $\iota_n(\phi_{n+1}(v_1, v_2, \ldots, v_{n+1})$ is the component of $f^{-1}(\phi_{n-1}(v_2, v_3, \ldots, v_n))$ containing $\phi_{n+1}(v_1, v_2, \ldots, v_{n+1})$. The set $\phi_n(v_1, v_2, \ldots, v_n)$ satisfies these conditions, since $f(\phi_n(v_1, v_2, \ldots, v_n)) = \phi_{n-1}(v_2, \ldots, v_n)$, by the inductive assumption, and $\phi_n(v_1, v_2, \ldots, v_n) \supset \phi_{n+1}(v_1, v_2, \ldots, v_{n+1})$, by the definition of ϕ_{n+1}. It follows that $\iota_n(\phi_{n+1}(v_1, v_2, \ldots, v_{n+1})) = \phi_n(v_1, v_2, \ldots, v_n)$. The case $n = 1$ is similar.

Let us show (also by induction) that ϕ_{n+1} is simplicial. Suppose that

$$\Delta = \{(v_{1,i}, v_{2,i}, \ldots, v_{n+1,i}) \ : \ i = 1, \ldots, k\}$$

is a simplex of $\widetilde{\Gamma}_{n+1}$. Then $\{\phi_n(v_{2,i}, v_{3,i}, \ldots, v_{n+1,i})\}$ and $\{\phi_n(v_{1,i}, v_{2,i}, \ldots, v_{n,i})\}$ are simplices of $\widetilde{\Gamma}_n$, since ϕ_n is simplicial. It means that $\bigcap_{i=1,\ldots,k} \phi_n(v_{2,i}, v_{3,i}, \ldots, v_{n+1,i})$ and $\bigcap_{i=1,\ldots,k} \phi_n(v_{1,i}, v_{2,i}, \ldots, v_{n,i})$ are non-empty. Then it follows from the definition (4.3) of ϕ_{n+1} and Lemma 4.4.2 that $\{\phi_{n+1}(v_{1,i}, v_{2,i}, \ldots, v_{n+1,i})\}_{i=1,\ldots,k}$ is a simplex of Γ_{n+1}. The case $n = 1$ is similar.

It remains to show that ϕ_{n+1} has an inverse simplicial map. If $n = 1$, then it is checked directly that the inverse map is $\phi_2^{-1}(v) = (\iota(v), f(v))$.

For every $v \in \Gamma_{n+1}$ we have $f_{n-1}(\iota_n(v)) = \iota_{n-1}(f_n(v))$, hence $\phi_n^{-1}(\iota_n(v)) = (v_1, v_2, \ldots, v_n)$ and $\phi_n^{-1}(f_n(v)) = (v_2, v_3, \ldots, v_{n+1})$ for some $v_i \in \Gamma_1$. Define $\phi'_{n+1} = (v_1, v_2, \ldots, v_{n+1})$. It is checked then directly that ϕ'_{n+1} is the inverse of ϕ_{n+1}. It is obvious that ϕ'_{n+1} □

Example 4.4.8. A model of the basilica...

4.4.3. Reconstructing an expanding map from its simplicial model. Let us show that a simplical virtual endomorphism $f_0, \iota_0 : \Gamma_1 \longrightarrow \Gamma_0$ defined in the previous subsection can be used to reconstruct the original expanding map.

Theorem 4.4.9. *Let $f \circlearrowright \mathcal{X}$ be an expanding covering map, and let $\mathcal{U}$ be a semi-Markovian open or closed cover by sufficiently small sets. Let Γ_n be the nerves of the covers $\mathcal{U}_n$, and let $\iota_n, f_n : \Gamma_{n+1} \longrightarrow \Gamma_n$ be the corresponding maps.*

Let $\lim_\iota \Gamma_n$ be the inverse limit of the sequence

$$\Gamma_0 \xleftarrow{\iota} \Gamma_1 \xleftarrow{\iota_1} \Gamma_2 \xleftarrow{\iota_2} \cdots,$$

seen as a topological graph.

Then there exists a homeomorphism of $\mathcal{X}$ with the space of abstract connected components of the graph $\lim_\iota \Gamma_n$ with the topology of the quotient of the space of vertices. Moreover, there exists a homeomorhism conjugating f with the map induced by

$$f_\infty(A_0, A_1, \ldots) = (f(A_1), f(A_2), \ldots).$$

Here the inverse limit $\lim_\iota \Gamma_n$ is considered as a simplicial complex: its set of vertices is the inverse limit of the sets of vertices of Γ_n; and its set of simplices is the inverse limit of the sets of simplices of Γ_n. Note that both sets are compact topologica spaces (homeomorphic to the Cantor sets, if the set of edges is non-empty). As an abstract complex (without topology), the complex $\lim_\iota \Gamma_n$ has uncountably many connected components.

Note that it follows from commutativity of the diagram (4.2) that $f_\infty : \lim_\iota \Gamma_n \longrightarrow \lim_\iota \Gamma_n$ is a continuous simplicial map.

Theorem 4.4.9 is another example of rigidity (structural stability) of hyperbolic dynamical systems. It shows that an expanding covering can be reconstructed from finite amount of (combinatorial) information: a pair of simplicial maps between finite simplicial complexes. We have seen similar statements about hyperbolic dynamical systems in Proposition 1.4.51 and Theorem 1.4.57.

Proof. A vertex of $\lim_\iota \Gamma_n$ is a sequence $(V_0, V_1, V_2, \ldots)$ of vertices $V_n \in \mathcal{U}_n$ of Γ_n such that $\iota_n(V_{n+1}) = V_n$ for all n. Then $V_{n+1} \subset V_n$. Diamenter of V_n is less than $L^{-n}\delta$. It follows that every sequence of points $x_n \in V_n$ is converging and the limit does not depend on the choice of x_n. Let us denote it by $\Phi(V_0, V_1, \ldots)$.

Lemma 4.4.10. *If vertices u, v of $\lim_\iota \Gamma_n$ are adjacent, then $\Phi(u) = \Phi(v)$.*

Proof. Let $u = (A_0, A_1, \ldots)$ and $v = (B_0, B_1, \ldots)$. If u and v are adjacent, then $A_n \cap B_n \neq \varnothing$, and we can choose $x_n \in A_n \cap B_n$. Then $\Phi(u) = \Phi(v) = \lim_{n\to\infty} x_n$. □

Lemma 4.4.11. *The map Φ is onto.*

Proof. Let $x \in \mathcal{X}$ be an arbitrary point. For every n there exists $A_n \in \mathcal{U}_n$ such that $x \in A_n$. Then x belongs to every element of the sequence $\iota_0 \circ \iota_1 \circ \cdots \circ \iota_{n-1}(A_n), \iota_1 \circ \iota_2 \circ \cdots \circ \iota_{n-1}(A_n), \ldots, \iota_{n-1}(A_n), A_n$. Consider the sequence of such sequences as $n \to \infty$. Since every complex Γ_n is finite, we can find a convergent sub-sequence, and its limit will be a vertex $(A_0, A_1, \ldots)$ of $\lim_\iota \Gamma_n$ such that $x \in A_n$ for all n. Then $\Phi(A_0, A_1, \ldots) = x$. □

Proposition 4.4.12. *If elements of $\mathcal{U}$ are closed and u, v are vertices of $\lim_\iota \Gamma_n$ such that $\Phi(u) = \Phi(v)$, then u and v are adjacent.*

If elements of $\mathcal{U}$ are open and $\Phi(u) = \Phi(v)$, then there exists combinatorial distance from u to v in the graph $\lim_\iota \Gamma_n$ is not more than 2.

Proof. If elements of $\mathcal{U}$ are closed (resp., open), then all elements of $\mathcal{U}_n$ are closed (resp., open).

Let $u = (A_0, A_1, \ldots)$ and $v = (B_0, B_1, \ldots)$. Suppose that $x = \Phi(u) = \Phi(v)$. We have $A_0 \supset A_1 \supset A_2 \supset \ldots$, $B_0 \supset B_1 \supset B_2 \supset \ldots$, and x is an accumulation point on both sequences. It follos that x is an accumulation point of each set A_n and B_n for all n. If all A_n, B_n are closed, then this implies that u and v are adjacent.

Suppose that the covers $\mathcal{U}_n$ are open. Then, by the proof of Lemma 4.4.11, there exists a vertex $(C_0, C_1, \ldots)$ such that $x \in C_n$ for all n. Since x belongs to the closure of each set A_n and B_n, we have $C_n \cap A_n \neq \varnothing$ and

$C_n \cap B_n \neq \emptyset$. It follows that $(C_0, C_1, \ldots)$ is adjacent both to $(A_0, A_1, \ldots)$ and to $(B_0, B_1, \ldots)$. □

Lemma 4.4.13. *The map* $\Phi : \lim_\iota \Gamma_n \longrightarrow \mathcal{X}$ *is continuous on the space of vertices of* $\lim_\iota$.

Proof. Define a metric d on the set of vertices of $\lim_\iota \Gamma_n$ by the condition that $d((A_0, A_1, \ldots), (B_0, B_1, \ldots)) = \frac{1}{m+1}$, where m is the minimal index such that $A_m \neq B_m$.

Suppose that $v = (A_0, A_1, \ldots)$ and $u = (B_0, B_1, \ldots)$, and $d(v, u) = \frac{1}{m+1}$. Then $A_m = B_m$, and $\Phi(A_0, A_1, \ldots)$ and $\Phi(B_0, B_1, \ldots)$ both belong to the closure of A_m. The closure of A_m has diameter less than $L^{-m}\delta$, hence

$$d(\Phi(v), \Phi(u)) \leqslant L^{-m}\delta = L^{1-1/d(v,u)}\delta,$$

which implies that Φ is continuous. □

The map Φ induces a continuous bijection between the space of connected components and $\mathcal{X}$. The equivalence relation of belonging to one component is, by Proposition 4.4.12, equal to the relation of adjacency (if the elements of the cover are closed) or to the relation of being on distance less or equal to 2 (if the elements of the cover are open). In both cases the equivalence relation is a closed subset of the direct square of the space of vertices. It follows that the space of connected components is compact Hausdorff. But any continuous bijection between compact Hausdorff spaces is a homeomorphism (since image of a closed, hence compact, set is compact, hence closed). □

Example 4.4.14. Consider the angle doubling map $f : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}$, $f(x) = 2x$. Let $\mathcal{U}$ be the cover of the circle $\mathbb{R}/\mathbb{Z}$ by the arcs $[0, 1/4]$, $[1/4, 1/2]$, $[1/2, 3/4]$, $[3/4, 1]$. Then $\mathcal{U}_n$ consists of arcs of the form $\left[\frac{k}{2^{n+2}}, \frac{k+1}{2^{n+2}}\right]$ for $k = 0, 1, \ldots, 2^{n+2} - 1$. It follows that the graphs Γ_n are cycles of length 2^{n+2}. There is only one choice for the map $\iota_n : \Gamma_{n+1} \longrightarrow \Gamma_n$, since an arc $\left[\frac{k}{2^{n+2}}, \frac{k+1}{2^{n+2}}\right]$ is contained in exactly one arc of the form $\left[\frac{l}{2^{n+1}}, \frac{l+1}{2^{n+1}}\right]$. Namely, $l = k/2$ if k is even and $(k-1)/2$ if k is odd.

The set of vertices of $\lim_{\iota_n} \Gamma_n$ can be realized as a subset of the circle homeomorphic to the Cantor set, so that edges of $\lim_{\iota_n} \Gamma_n$ connect the endpoints of the components of the complement of the Cantor set (i.e., "filling the gaps" in the Cantor set). It follows that the space of connected components of $\lim_{\iota_n} \Gamma_n$ is homeomorphic to the circle.

Theorem 4.4.9 produces a *finite presentation* of the dynamical system $f \circlearrowleft \mathcal{X}$ in the sense of Definition 1.4.45. The sets of vertices and edges of $\lim_\iota \Gamma_n$ are Markovian subshifts in a natural way. If $\mathcal{U}$ is a cover by closed sets, then we have seen in the proof of the theorem that every connected component of $\lim \iota\Gamma_n$ is a simplex, i.e., edge adjacency is the kernel of the semiconjugacy of f_∞ with f. We get hence a presentation of $f \circlearrowleft \mathcal{X}$ as a quotient of a shift of finite type by a shift of finite type.

4.4.4. Contracting groups and the nucleus.

Definition 4.4.15. Let $(G, \mathfrak{M})$ be a self-similar group, and let X be a basis of $\mathfrak{M}$. The associataed self-similar action is said to be *contracting* if there exists a finite set $\mathcal{N} \subset G$ such that for every $g \in G$ there exists n such that $g|_v \in \mathcal{N}$ for every word $v \in \mathsf{X}^*$ of length at least n. The smallest set $\mathcal{N}$ satisfying this condition is called the *nucleus* of the action.

If the group is contracting, then the nucleus is well defined and is equal to the set

$$\mathcal{N} = \bigcup_{g \in G} \bigcap_{n \geqslant 0} \{g|_v \ : \ v \in \mathsf{X}^*, |v| \geqslant n\}.$$

It is also equal to the set of elements G such that there exist $h \in G, u, v \in \mathsf{X}^*$ such that $|u| \geqslant 1$, $h|_u = h$, and $g = h|_v$. In other words, it is the set of all elements of G that can be reached from the cycles in the Moore diagram of the full automaton of G (see 2.4.7 for the definition of the full automaton). The nucleus is state-closed, i.e., for every $g \in \mathcal{N}$ and $x \in \mathsf{X}$ we have $g|_x \in \mathcal{N}$. We usually consider a nucleus as an automaton. In particular, we will talk sometimes about the Moore diagram of the nucleus.

Example 4.4.16. Consider the binary odometer action generated by $a = \sigma(1, a)$. Since $a^2 = (a, a)$, we have $a^n|_0 = a^{n/2}$ and $a^n|_1 = a^{n/2}$ if n is even, and $a^n|_0 = a^{(n-1)/2}$ and $a^n|_1 = a^{(n+1)/2}$ if n is odd. It follows that the nucleus of this action of $\mathbb{Z}$ is the set $\{1, a, a^{-1}\}$. Its Moore diagram is shown on ...

The following proposition is proved in ...

Proposition 4.4.17. *Let $(G, \mathfrak{M})$ be a covering biset. If some self-similar action associated with it is contracting, then every self-similar action associated with it is contracting.*

In other words, the property of being contracting is a property of the biset. Note that the nucles of the self-similar action depends on the choice of the basis X. In some cases we call the biset *hyperbolic* if the associated self-similar actions are contracting. We say that a biset is *sub-hyperbolic* if its faithful quotient is hyperbolic.

Definition 4.4.18. We say that a self-similar group $(G, \mathfrak{M})$ is *self-replicating* if the left G-action on $\mathfrak{M}$ is transitive. Equivalently, a self-similar action $G \curvearrowright \mathsf{X}^*$ is self-replicating if and only if it is transitive on the first level of the tree X^* and the associated virtual endomorphism is onto.

Proposition 4.4.19. *If $G \curvearrowright \mathsf{X}^*$ is a contracting self-replicating action of a finitely generated group, then G is generated by its nucleus.*

Proposition 4.4.20. *Let $f : \mathcal{X} \longrightarrow \mathcal{X}$ be an expanding self-covering of a compact connected and locally connected metric space. Then $\mathfrak{M}_{t,f}$ is a sub-hyperbolic biset, i.e.,* $\mathrm{IMG}(f)$ *is a contracting self-similar group.*

Proof. ... Use covers $\mathcal{U}$ and consider sums of diameters of elements in a chain.. Show that elements of the nucleus are defined by paths of finite diameter.... □

4.4.5. Contraction coefficient. Let G be a finitely generated group, and let $\phi : G_1 \longrightarrow G$ be a virtual endomorphism. Denote by $|g|$ the length of a group element $g \in G$ with respect to some fixed finite generating set of G. The *contraction coefficient* of ϕ is defined as

$$\rho_\phi = \limsup_{n\to\infty} \limsup_{g\in\mathrm{Dom}\,\phi^n, |g|\to\infty} \sqrt[n]{\frac{|\phi^n(g)|}{|g|}}.$$

The following propositions are proved in...

Proposition 4.4.21. *Let $G \curvearrowright \mathsf{X}^*$ be a level-transitive self-similar action of a finitely generated group. It is contracting if and only if $\rho_\phi < 1$.*

Moreover, if the action is contracting and level-transitive, then the contraction coefficient depends only on the biset.

Self-replicating?...

Proposition 4.4.22. *Let $G \curvearrowright \mathsf{X}^*$ be a self-similar contracting action of a finitely-generated group. Let ρ_ϕ be its contraction coefficient. Then the orbital graphs of the induced action $G \curvearrowright \mathsf{X}^\omega$ on the boundary of the tree X^* have polynomial growth of degree not more than $\frac{\log|\mathsf{X}|}{-\log\rho_\phi}$.*

Proof. .... □

As a corollary of Proposition 4.4.22 we get that the contraction coefficient of an infinite self-similar group is never less than $\frac{1}{|\mathsf{X}|}$. This value is attained by the $|\mathsf{X}|$-adic odometer and by the iterated monodromy group of the Chebyshev polynomial $T_{|\mathsf{X}|}$.

4.4.6. Algebraic properties of contracting groups. Not much is known about algebraic properties of self-similar or even self-similar contracting groups. Many interesting problems remain to be open (we will mention some of them here). We will summarize some of the known facts in this subsection. Throughout this subsection a *contracting group* means a group acting by a faithful self-similar contracting action.

For a proof of the following, see...

Proposition 4.4.23. *The word problem is solvable in finitely generated contracting groups in polynomial time. Namely, if G is a self-similar contracting group acting on X^*, and ρ is its contraction coefficient, then for every $\epsilon > 0$ there exists an algorithm solving the word problem in polynomial time of degree at most $\frac{\log|\mathsf{X}|}{-\log\rho} + \epsilon$.*

Solvability of many algorithmic problems for contracting groups remain to be open: conjugacy, isomorphism, finiteness, etc.. A particularly interesting problem is deciding if two given faithfully acting on X^* contracting self-similar groups are equivalent.

Theorem 4.4.24. *If G is a contracting self-similar group, then it has no free non-abelian subgroups.*

Proof. We will use Theorem 2.4.54. Since the orbital graphs of the action of G on the boundary X^ω of the tree have polynomial growth, G can not contain a free group acting freely on an orbit of a point of X^ω.

The action of $g \in G_w$ on a neighborhood of w is uniquely determined by $g|_v$ for every beginning v of w. It follows that the number of elements of the group of germs $G_w/G_{(w)}$ can not be more than the size of the nucleus (in fact, every group $G_w/G_{(w)}$ is isomorphic to a finite group contained in the nucleus). Consequently, cases (2) and (3) of Theorem 2.4.54 are not possible for contracting groups, hence G has no free subgroups. □

We leave the following theorem as an exercise (see also ...)

Theorem 4.4.25. *Let $(\overline{G}, \overline{\mathfrak{M}})$ be a hyperbolic biset, and let $(G, \mathfrak{M})$ be its faithful quotient. Choose a basis X of $\overline{\mathfrak{M}}$ (identified with the corresponding basis of $\mathfrak{M}$). Suppose that the nucleus of $(\overline{G}, \overline{\mathfrak{M}})$ defined for X does not contain non-trivial elements in the kernel of the epimorphism $\overline{G} \longrightarrow G$. Then an element $g \in \overline{G}$ belongs to the kernel of the epimorphism if and only if there exists $n \geqslant 1$ such that $g|_v = 1$ and $g(v) = v$ for all $v \in \mathsf{X}^n$.*

We call self-similar groups $\overline{G}$ satisfying the conditions of Theorem 4.4.25 *contracting overgroups* of $(G, \mathfrak{M})$. We say that $\overline{G}$ is self-replicating if its left action on the biset $\overline{\mathfrak{M}}$ is transitive. Note that if $\overline{G}$ is self-replicating, then G is too.

It is shown in ... that for any contracting finitely generated self-replicating group $(G, \mathfrak{M})$ there exists a *finitely presented* contracting overgroup $\overline{G}$. Namely, it is enough to take the nucleus of G as a generating set, and define $\overline{G}$ by all relations of length 3 that are valid in G. We will give a geometric proof of this fact later...

Proposition 4.4.26. *Let $\overline{G}$ be a contracting self-replicating overgroup of a contracting finitely generated group G. Then for any finitely presented group F and an epimorphism $F \longrightarrow G$ there exists a subgroup of finite index $F_1 \leqslant F$ and an epimorphism $F_1 \longrightarrow \overline{G}$.*

Proof. By taking images of elements of $\overline{G}$ and F in G and then lifting them to the other group, we can find generating sets $\{g_1, g_2, \ldots, g_m\}$ and $\{f_1, f_2, \ldots, f_m\}$ of $\overline{G}$ and F, respectively, such that the images of g_i and f_i in G are equal for every i. Then F is defined by a presentation with the set of generators $\{f_i\}$ and a finite set of relations R. It follows from Proposition 4.4.25 that there exists n such that if $r(f_1, f_2, \ldots, f_m)$ is a relation from R, then for the corresponding word $r(g_1, g_2, \ldots, g_m)$ in $\overline{G}$ we have $r(g_1, g_2, \ldots, g_m)|_v = 1$ and $r(g_1, g_2, \ldots, g_m)(v) = v$ for every $v \in \mathsf{X}^n$. In other words, every element $r(g_1, g_2, \ldots, g_m)$ for $r \in R$ has trivial image under the wreath recursion $\phi_n : \overline{G} \mapsto \mathsf{S}(\mathsf{X}^n) \ltimes \overline{G}^{\mathsf{X}^n}$ associated with the biset $\overline{\mathfrak{M}}^{\otimes n}$. It follows that the map $f_i \mapsto \phi_n(g_i)$ extends to a homomorphism $F \mapsto \mathsf{S}(\mathsf{X}^n) \ltimes \overline{G}^{\mathsf{X}^n}$. Consider the subgroup F_1 whose image in $\mathsf{S}(\mathsf{X}^n)$ fixes a word $v \in \mathsf{X}^n$. Projecting F_1 onto the coordinate of $\overline{G}^{\mathsf{X}^n}$ corresponding to v, we get a homomorphism $F_1 \longrightarrow \overline{G}$. Since $\overline{G}$ is self-replicating, the homomorphism $F_1 \longrightarrow \overline{G}$ is onto. □

cite Grigorchuk-de la Harpe-Benli...

As a direct corollary of Proposition 4.4.26 and Theorem 4.4.24 we get the following.

Corollary 4.4.27. *Let G be a finitely generated self-replicating contracting group, and let $\overline{G}$ be a contracting overgroup of G. If $\overline{G}$ has a free subgroup, then G is not finitely presented.*

Proof. Suppose that G is finitely presented. Then, by Proposition 4.4.26, there exists an epimorphism from a subgroup of finite index of G to $\overline{G}$. But this implies that G has a free subgroup, which contradicts Theorem 4.4.24. □

Example 4.4.28. It is easy to check that the wreath recursion $a = \sigma(1, b), b = (1, a)$ defining IMG $\left(z^2 - 1\right)$ is contracting on the free group generated by a, b. It follows that IMG $\left(z^2 - 1\right)$ is not finitely presented. We will see later that a general argument shows that the iterated monodromy group of any post-critically finite rational function is not finitely presented unless it is virtually abelian (which happens only for z^d, z^{-d}, Chebyshev polynomials, and Lattés examples).

All known finitely presented contracting groups are virtually nilpotent. It is an open question if there are other finitely presented contracting groups.

Open questions: infinite presentation in general, weak branchness, amenability, growth (say that more will be discussed later)...

4.4.7. The limit dynamical system. Let (G, X) be a contracting self-similar group. Consider the space $\mathsf{X}^{-\omega}$ of left-infinite sequences $\ldots x_2x_1$ of letters $x_i \in \mathsf{X}$ and the space $\mathsf{X}^{-\omega} \times G$, both with the direct product topology (where X and G are discrete). The space $\mathsf{X}^{-\omega}$ is obviously homeomorphic to the space of the right-infinite sequences X^{ω} and is a Cantor set.

Definition 4.4.29. We say that two sequences $\ldots x_2x_1, \ldots y_2y_1$ are *asymptotically equivalent* (with respect to the action of G) if there exists a finite set $N \subset G$ and a sequence $g_n \in N$ such that $g_n(x_n \ldots x_2x_1) = y_n \ldots y_2y_1$ for all n.

We say that $\ldots x_2x_1 \cdot g, \ldots y_2y_1 \cdot h \in \mathsf{X}^{-\omega} \times G$ are *asymptotically equivalent* if there exists a finite set $N \subset G$ and a sequence $g_n \in N$ such that $g_n \cdot x_n \ldots x_2x_1 \cdot g = y_n \ldots y_2y_1h$ for every n. Recall that the last condition means that $g_n(x_n \ldots x_1x_1) = y_n \ldots y_2y_1$ and $g_n|_{x_n \ldots x_2x_1}g = h$.

In particular, if G is finitely generated, the sequences $\ldots x_2x_1, \ldots y_2y_1$ (resp. $\ldots x_2x_1 \cdot g$ and $\ldots y_2y_1 \cdot h$) are equivalent if and only if the distance between $x_n \ldots x_2x_1$ and $y_n \ldots y_2y_1$ (between $x_n \ldots x_2x_1 \cdot g$ and $y_n \ldots y_2y_1 \cdot h$, resp.) in the graphs of the action of G on X^n (on the biset $\mathsf{X}^n \cdot G$, resp.) is uniformly bounded. It is obvious that we get equivalence relations (also in the infinitely generated case). The equivalence relation on $\mathsf{X}^{-\omega}$ is invariant with respect to the shift $\ldots x_2x_1 \mapsto \ldots x_3x_2$, since $g_n(x_n \ldots x_2x_1) = y_n \ldots y_2y_1$ implies $g_n(x_n \ldots x_3x_2) = y_n \ldots x_3x_2$.

Similarly, the equivalence relation on $\mathsf{X}^{-\omega} \times G$ is invariant under the natural right G-action and under tensor products by elements of the biset $\mathsf{X} \cdot G$, i.e., under the maps

$$\ldots x_2x_1 \cdot g \mapsto \ldots x_2x_1 \cdot g \otimes x \cdot h = \ldots x_2x_1g(x) \cdot g|_x h.$$

Definition 4.4.30. The quotient of $\mathsf{X}^{-\omega}$ by the asymptotic equivalence relation is called the *limit space* of the contracting group, and is denoted $\mathcal{J}_{(G,\mathsf{X})}$, or just $\mathcal{J}_G$. The dynamical system $\mathsf{s} \circlearrowleft \mathcal{J}_G$, where s is the map induced by the shift $\mathsf{X}^{-\omega} \longrightarrow \mathsf{X}^{-\omega}$ is called the *limit dynamical system* of the self-similar group.

The quotient of $\mathsf{X}^{-\omega} \times G$ by the asymptotic equivalence relation together with the induced right G-action is the *limit G-space*, and is denoted $\mathcal{X}_{(G,\mathsf{X})}$ or just $\mathcal{X}_G$.

The following is proved in...

Proposition 4.4.31. *Sequences* $\dots x_2x_1, \dots y_2y_1$ *are asymptotically equivalent if and only if there exists a sequence* g_n*,* $n \geqslant 0$*, of elements of the nucleus* $\mathcal{N}$ *such that* $g_n \cdot x_n = y_n \cdot g_{n-1}$ *for every* $n \geqslant 1$*.*

Sequences $\dots x_2x_1 \cdot g$*,* $\dots y_2y_1 \cdot h$ *are asymptotically equivalent if and only if there exists a sequence* $g_n \in \mathcal{N}$ *such that* $g_n \cdot x_n = y_n \cdot g_{n-1}$ *for all* $n \geqslant 1$*, and* $g_0 g = h$*.*

In particular, the stabilizer of a point of $\mathcal{X}_G$ represented by a sequence $\dots x_2x_1 \cdot 1$ is a finite subgroup of G contained in the nucleus. Moreover, it also follows from Proposition 4.4.31 that the action $\mathcal{X}_G \curvearrowleft G$ is proper. Note that $\mathcal{J}_G$ is obviously homeomorphic to the space of orbits of the action $\mathcal{X}_G \curvearrowleft G$. Therefore, it is natural to consider $\mathcal{J}_G$ as an *orbispace* defined by the groupoid of the action $\mathcal{X}_G \curvearrowleft G$.

Lemma 4.4.32. *If the action* $G \curvearrowright \mathsf{X}^*$ *is faithful, then its groupoid of germs coincides with the groupoid of the action.*

Proof. We have to prove that for every non-trivial $g \in G$ and $\xi \in \mathcal{X}_G$ the germ of the action of g on the neighborhoods of ξ is non-trivial. Suppose that it is not true. Let $\dots x_2x_1 \cdot h$ represent ξ. Then there exists n such that for every $\dots y_2y_1 \in \mathsf{X}^{-\omega}$ the sequences $\dots y_2y_1x_nx_{n-1}\dots x_1h$ and $\dots y_2y_1x_nx_{n-1}\dots x_1 \cdot hg$ are equivalent. Let $g_1, g_2, \dots, g_m$ be the list of all non-trivial elements of the nucleus. Then there exists $v_1 \in \mathsf{X}^*$ such that $g_1(v_1) \neq v_1$. Find the smallest index i such that $g_i|_{v_1} \neq 1$, and let $v_2 \in \mathsf{X}^*$ be such that $g_i(v_2) \neq v_2$. Then find the smallest index i such that $g_i|_{v_1v_2} \neq 1$, and let $v_3 \in \mathsf{X}^*$ be such that $g_i(v_3) \neq v_3$. Continue this way until we find a word $w = v_1v_2\dots v_k \in \mathsf{X}^*$ such that for every $g_i \in \mathcal{N}$ either $g_i|_w = 1$ or $g_i(w) \neq w$. Consider then any sequence ending by $wx_nx_{n-1}\dots x_1 \cdot h$. Then it follows from Proposition 4.4.31 that it is asymptotically equivalent only to sequences of the form $w'a_na_{n-1}\dots a_1 \cdot h'$ for some word $w' \neq w$ of the length equal to the length of w, or to a sequence of the form $\dots a_2a_1 \cdot h$. In particular, sequences of the form $\dots y_2y_1wx_nx_{n-1}\dots x_1 \cdot h$ and $\dots y_2y_1wx_nx_{n-1}\dots x_1 \cdot hg$ can not be asymptotically equivalent. □

Let us describe a more natural (without any reference to the basis X of $\mathfrak{M}$) way of defining the space $\mathcal{X}_G$ and the G-action on it. The following proposition is proved in...

Proposition 4.4.33. *Let* $\mathfrak{M}$ *be the biset of a contracting group* G*. Let*

$$\Omega = \bigcup_{A \subset \mathfrak{M}, |A| < \infty} A^{-\omega},$$

where $A^{-\omega}$ *denotes the set of left-infinite sequences* $(\dots, x_2, x_1)$ *of elements of* A *with the direct product topology. Endow* Ω *with the direct limit topology.*

We say that $(\ldots, x_2, x_1)$ *is equivalent to* $(\ldots, y_2, y_1)$ *if there exists a finite set* $N \subset G$ *and a sequence* $g_n \in N$ *such that* $g_n \cdot x_n \otimes x_{n-1} \otimes \cdots \otimes x_1 = y_n \otimes y_{n-1} \otimes \cdots \otimes y_1$ *in* $\mathfrak{M}^{\otimes n}$*. Then the map* $\ldots x_2 x_1 \cdot g \mapsto (\ldots, x_2, x_1 \cdot g)$ *induces a homeomorphism of* $\mathcal{X}_G$ *with the quotient of* Ω *by the defined equivalence relation conjugating the natural action* $\mathcal{X}_G \curvearrowleft G$ *with the action induced by the natural right* G*-action on* Ω *(the right action on the last coordinate).*

Proposition 4.4.33 shows that the right G-space $\mathcal{X}_G$ can be naturally seen as a version of the infinite tensor power $\mathfrak{M}^{\otimes(-\omega)}$.

4.4.8. Limit correspondence on the orbispace $\mathcal{J}_G$. The tensor product $\mathcal{X}_G \otimes_G \mathfrak{M}$ of the right G-space $\mathcal{X}_G$ with the biset $\mathfrak{M}$ is naturally identified with $\mathcal{X}_G$, see Proposition 4.4.33. After choosing a basis X of $\mathfrak{M}$, this identification is given by the rule

$$\ldots x_2 x_1 \cdot g \otimes y \cdot h = \ldots x_2 x_1 g(y) \cdot g|_y h.$$

This gives us a natural orbispace version of the limit dynamical system $\mathsf{s} \circlearrowright \mathcal{J}_G$. Recall that the orbispace $\mathcal{J}_G$ is defined by the groupoid $\mathfrak{G} = \mathcal{X}_G \rtimes G$ of the action $\mathcal{X}_G \rtimes G$. (Recall that since the G-action is from the right, we have $\mathbf{s}(\xi, g) = \xi \cdot g$ and $\mathbf{r}(\xi, g) = \xi$.)

Namely, consider the space $\mathcal{M} = \mathcal{X}_G \times \mathfrak{M}$, the anchors $P_l, P_r : \mathcal{M} \longrightarrow \mathcal{X}_G$ for the left and the right actions, respectively, given by

$$P_l(\xi, x) = \xi, \qquad P_r(\xi, x) = \xi \otimes x$$

and the $\mathcal{X}_G \rtimes G$-actions

$$(\zeta_1, g) \cdot (\xi, x) = (\xi \cdot g^{-1}, g \cdot x), \qquad (\xi, x) \cdot (g, \zeta_2) = (\xi, x \cdot g)$$

for $\xi, \zeta_1, \zeta_2 \in \mathcal{X}_G, x \in \mathfrak{M}, g \in G$, and we have $\zeta_1 \cdot g = \xi$, $\zeta_2 = \xi \otimes x$.

Note that the right action is free and proper, since $(\xi, x) \cdot (g, \zeta_2) = (\xi, x)$ implies that $x = x \cdot g$, hence $g = 1$ (as the right action $\mathfrak{M} \curvearrowleft G$ is free). The quotient $\mathcal{M}/\mathfrak{G}$ by the right action is naturally $\mathcal{X}_G \times \mathfrak{M}/G$, and the map $P_l/\mathfrak{G} : \mathcal{M}/\mathfrak{G} \longrightarrow \mathfrak{G}^{(0)}$ is the projection $\mathcal{X}_G \times \mathfrak{M}/G \longrightarrow \mathcal{X}_G$ on the first coordinate. Since $\mathfrak{M}/G$ is finite and discrete, $P_l/\mathfrak{G}$ is a $|\mathfrak{M}/G|$-to-one covering map. Consequently, the constructed biaction $\mathfrak{G} \curvearrowright \mathcal{M} \curvearrowleft \mathfrak{G}$ is a virtual endomorphism of $\mathfrak{G}$.

Equivalently, we may choose a basis X of $\mathfrak{M}$, and define the virtual endomorphism by a groupoid automaton given by the map

$$((\xi, g), x) \mapsto (g(x), (\xi \otimes x, g|_x)) : \mathfrak{G} \times \mathsf{X} \longrightarrow \mathsf{X} \times \mathfrak{G}.$$

The map $\iota : \mathfrak{G}^{(0)} \times \mathsf{X} \longrightarrow \mathfrak{G}^{(0)}$ is then the map $(\xi, x) \mapsto \xi \otimes x$. According to Definition 4.3.18, the associated biaction will be on the space

$$\{(\xi, x, (\zeta, g)) \in \mathcal{X}_G \times \mathsf{X} \times \mathfrak{G} \ : \ \zeta \cdot g = \xi \otimes x\},$$

which is naturally identified with $\mathcal{M}$ using the map $(\xi, x \cdot g) \mapsto (\xi, x, (\xi \otimes x \cdot g^{-1}, g))$. We leave it as an exercise to check that this identification agrees with the left and right actions given in Definition 4.3.18.

We denote the orbispace defined by the biaction groupoid $\mathfrak{G} \ltimes \mathcal{M} \rtimes \mathfrak{G}$ by $\mathcal{J}_1$, see Exercise 3.9. Denote the morphisms $\mathcal{J}_1 \longrightarrow \mathcal{J}_G$ given by the projections $\mathfrak{G} \ltimes \mathcal{M} \rtimes \mathfrak{G} \longrightarrow \mathfrak{G}$ onto the left and the right copies of $\mathfrak{G}$ by f and ι, respectively.

Since the right action $\mathcal{M} \curvearrowleft \mathfrak{G}$ is free and proper, an equivalent atlas for $\mathcal{J}_1$ is the groupoid $\mathfrak{G} \ltimes (\mathcal{M}/\mathfrak{G})$. If we choose a basis X of $\mathfrak{M}$, then $\mathcal{M}/\mathfrak{G}$ is naturally identified with $\mathcal{X}_G \times \mathsf{X}$, and the groupoid $\mathfrak{G} \ltimes (\mathcal{M}/\mathfrak{G})$ as the restriction of $\mathfrak{G} \ltimes \mathcal{M} \rtimes \mathfrak{G}$ to the transversal $\mathcal{X}_G \times \mathsf{X} \subset \mathcal{M}$. The corresponding action of $\mathfrak{G}$ on $\mathcal{X}_G \times \mathsf{X}$ is given by

$$(\zeta_1, g) \cdot (\xi, x) = (\xi \cdot g^{-1}, g(x)),$$

where the action is defined if and only if $\zeta_1 \cdot g = \xi$ (compare this with the definition of the left action $\mathfrak{G} \curvearrowright \mathcal{M}$).

Every element $((\zeta_1, g), (\xi, x)) \in \mathfrak{G} \ltimes (\mathcal{X}_G \times \mathsf{X})$ satisfies $\zeta \cdot g = \xi$, hence is uniquely determined by ξ and $g \in G$. Therefore, we can describe the elements of $\mathfrak{G} \ltimes (\mathcal{M}/\mathfrak{G}) = \mathfrak{G} \ltimes (\mathcal{X}_G \times (\mathfrak{M}/G)$ as triples $(g, \xi, x) \in G \times \mathcal{X}_G \times \mathsf{X}$ with the source and range maps

$$\mathbf{s}(g, \xi, x) = (\xi, x), \quad \mathbf{r}(\xi, g) = (\xi \cdot g^{-1}, g(x)),$$

and multiplication

$$(g_1, \xi_1, x_1)(g_2, \xi_2, x_2) = (g_1 g_2, \xi_2, x_2).$$

We see that the groupoid $\mathfrak{G} \ltimes (\mathcal{M}/\mathfrak{G})$ is isomorphic to the skew-product groupoid $\mathfrak{G} \ltimes \sigma$ defined by the natural cocycle $\sigma : \mathfrak{G} \longrightarrow \mathsf{S}(\mathsf{X})$:

$$\sigma(g, \xi)(x) = g(x).$$

Therefore the morphism $f : \mathcal{J}_1 \longrightarrow \mathcal{J}_G$ is identified with the covering of orbifolds $\mathcal{J}_1 \longrightarrow \mathcal{J}_G$ defined by the cocycle σ, i.e., with the projection

$$F : (g, \xi, x) \mapsto (\xi, g^{-1}). \tag{4.4}$$

(See ... for the definition of coverings of orbifolds...).

Let us describe the projection $\iota : \mathcal{J}_1 \longrightarrow \mathcal{J}_G$ onto the right action in terms of the groupoid $\mathfrak{G} \ltimes (\mathcal{X}_G \times \mathsf{X}) = \mathfrak{G} \ltimes \sigma$. An element $((\zeta_1, g), (\xi, x), (\zeta_2, h))$ of $\mathfrak{G} \ltimes \mathcal{M} \rtimes \mathfrak{G}$ has source $(\xi, x) \cdot (\zeta_2, h) = (\xi, x \cdot h)$ and range $(\zeta_1, g) \cdot (\xi, x) = (\xi \cdot g^{-1}, g \cdot x)$, and is projected by ι to the element (ζ_2, h), where $\zeta_2 = \xi \otimes x$. Let us restrict this to the transversal $\mathcal{X}_G \times \mathsf{X}$, and consider the projection of the element $(g, \xi, x) \in G \times \mathcal{X}_G \times \mathsf{X} = \mathfrak{G} \ltimes (\mathcal{X}_G \times \mathsf{X})$. Consider the corresponding element $((\xi \cdot g^{-1}, g), (\xi, x), (\xi \otimes x, 1))$ of $\mathfrak{G} \ltimes \mathcal{M} \rtimes$. Its source belongs to $\mathcal{X}_G \times \mathsf{X}$, but its range is $(\xi \cdot g^{-1}, g \cdot x)$ is equal to $(\xi \cdot g^{-1}, g(x)) \cdot (\xi \cdot g^{-1} \otimes g(x), g|_x)$, where the dot on the right-hand side of the equality is the right $\mathfrak{G}$-action. It follows that the projection $\iota : \mathcal{J}_1 \longrightarrow \mathcal{J}_G$ can be defined by the functor

$$I : (g, \xi, x) \mapsto (\xi \otimes x, g|_x) \tag{4.5}$$

from $\mathfrak{G} \ltimes \sigma$ to $\mathfrak{G}$. More precisely, this functor as a morphism of groupoids is isomorphic to the composition of the projection $\iota : \mathfrak{G} \ltimes \mathcal{M} \rtimes \mathfrak{G} \longrightarrow \mathfrak{G}$ with the equivalence of $\mathfrak{G} \ltimes \sigma$ and $\mathfrak{G} \ltimes \mathcal{M} \rtimes \mathfrak{G}$.

The limit dynamical system is a self-covering of a topological space if and only if the groupoid of germs of IMG (f) is principal... The limit dynamical system is a self-covering of an orbispace if and only if the groupoid is Hausdorff...

4.4.9. Contracting correspondences. Let us generalize the notion construction of the limit space $\mathcal{X}_G$ to the case of virtual endomorphisms of groupoids.

Let $\mathfrak{G} \curvearrowright \mathcal{M} \curvearrowleft \mathfrak{G}$ be a virtual endomorphism. Let A be a compact subset of $\mathcal{M}$, and denote by Ω_A the set of all sequences $(\ldots, x_2, x_1) \in A^{-\omega}$ such that $P_r(x_n) = P_l(x_{n-1})$ for all $n \geqslant 2$. We take Ω_A with the topology of a subspace of the direct product $A^{-\omega}$. Let $\Omega = \Omega_{\mathcal{M}}$ be the inductive limit of the spaces Ω_A with respect to the natural embeddings $\Omega_{A_1} \hookrightarrow \Omega_{A_2}$ for $A_1 \subset A_2$.

We say that $(\ldots x_2, x_1), (\ldots, y_2, y_1) \in \Omega$ are *asymptotically equivalent* if there exists a compact subset $N \subset \mathfrak{G}$ and a sequence $g_n \in \mathfrak{G}$ such that $g_n \cdot x_n \otimes x_{n-1} \otimes \cdots \otimes x_1 = y_n \otimes y_{n-1} \otimes \cdots \otimes y_1$ in $\mathcal{M}^{\otimes n}$.

Denote by $\mathcal{M}^{\otimes(-\omega)}$ the quotient of Ω by the asymptotic equivalence. We denote an element of $\mathcal{M}^{\otimes(-\omega)}$ represented by a sequence $(\ldots, x_2, x_1)$ either by the sequence itself or by $\ldots \otimes x_2 \otimes x_1$. We have a natural right $\mathfrak{G}$-action induced by $(\ldots \otimes x_2 \otimes x_1) \cdot g = \ldots \otimes x_2 \otimes (x_1 \cdot g)$.

Example 4.4.34. If $\mathfrak{G}$ has one unit, i.e., if it is a group, then $\mathcal{M}$ is a covering biset in the sense of Definition 4.2.2. If $\mathcal{M}$ is hyperbolic, then $\mathcal{M}^{\otimes(-\omega)}$ is the limit G-space $\mathcal{X}_G$, see Proposition 4.4.33.

Example 4.4.35. If $\mathfrak{G}$ is trivial, then $\mathcal{M}$ is a covering correspondence, as in Definition 4.3.1, defined by the anchors $P_l = f$, and $P_r = \iota$. The map f is a finite degree covering. Then $\mathcal{M}^{\otimes n}$ is the space $\mathcal{M}_n$ of orbits of length n, see Proposition 4.3.20, and the space $\mathcal{M}^{\otimes(-\omega)}$ is the inverse limit of the spaces $\mathcal{M}_n$ with respect to the maps $\iota_n(x_1, x_2, \ldots, x_{n+1}) = (x_2, x_3, \ldots, x_{n+1})$, see the comments after Proposition 4.3.20. If ι is a homeomorphic embedding, so that the correspondence is interpreted as a partial self-covering f of a topological space $\mathcal{X}$, then $\mathcal{M}_n$ is naturally identified with the domain of the nth iterate of f, by the homeomorphism $(x_1, x_2, \ldots, x_n) \mapsto x_1$. Then the maps ι_n are identical embeddings, and therefore $\mathcal{M}^{\otimes(-\omega)}$ is the intersection of the domains $\mathcal{M}_n$, i.e., the set of points $x \in \mathcal{X}$ such that $f^n(x)$ is defined for every $n \geqslant 1$.

The endomorphism is contracting if the action of $\mathfrak{G}$ on $\mathcal{M}^{\otimes(-\omega)}$ is proper and co-compact and the maps $\xi \mapsto \xi \otimes x : \mathcal{M}^{\otimes(-\omega)}$ are uniformly contracting (in the sense that uniformly bounded distances can be made arbitrarily small)...

Shadowing property for virtual endomorphisms...

Definition 4.4.36. Let $\mathfrak{G} \curvearrowright \mathcal{M} \curvearrowleft \mathfrak{G}$ be a virtual endomorphism of an étale co-compact groupoid. We say that it is *contracting* if for every compact subset $X \subset \mathcal{M}$ there exists a neighborhood of the diagonal $U \subset P_r(X) \times P_r(X)$ and a compact set $N \subset \mathfrak{G}$ such that for every pair $(\ldots x_2, x_1), (\ldots, y_2, y_1) \in \Omega \cap X^{-\omega}$ if there exists a bounded sequence $g_n \in \mathfrak{G}$ such that $(P_r(g_n \cdot x_n), P_r(y_n)) \in U$ for all n, then there exists a sequence $h_n \in N$ such that $g_n \cdot x_n \otimes x_{n-1} \otimes \ldots \otimes x_1 = y_n \otimes y_{n-1} \otimes \ldots \otimes y_1 \cdot h_n$ for all n big enough.

Example 4.4.37. If the virtual endormorphism is associated with a self-covering of a topological space (i.e., if $\mathfrak{G}$ is trivial, and P_r is a homeomorphism), then every element of Ω is interpreted as a forward P_l-orbit, and we get the usual definition of an expansive (equivalently, expanding) self-covering, see Definition 1.4.1.

Example 4.4.38. If $\mathfrak{G}$ is a group, then we can take U to be equal to the diagonal, and we get a version of the definition of a hyperbolic covering biset.

Definition of a contracting virtual endomorphism of a groupoid... Prove structural stability of virtual enodomorphisms with shadowing property and that contracting implies shadowing property and homotopical structural stability...

Theorem 4.4.39. *Let $\mathfrak{G} \curvearrowright \mathcal{M} \curvearrowleft \mathfrak{G}$ be a contracting virtual endomorphism of a path-connected étale groupoid $\mathfrak{G}$. Let $\mathfrak{M}$ be the induced biset on the iterated monodromy group* $\mathrm{IMG}(\mathcal{M})$*. Then the action groupoid $\mathcal{M}^{\otimes(-\omega)} \rtimes \mathfrak{G}$ is equivalent to the action groupoid $\mathcal{X}_{\mathrm{IMG}(\mathcal{M})} \rtimes \mathrm{IMG}(\mathcal{M})$.*

Proof. .... □

4.4.10. Simplicial contracting models. Prove that for every cover by small open sets there is an iteration f^n such that the corresponding simplical correspondence is homotopic to a contracting correspondence...

Explain how finite covers lead to complexes of groups...

The case of expanding Thurston maps.. [BM] monograph...

Definition of the topological nucleus... prove that it is homotopic to a contracting map...

Basilica, z^2+i, Gupta-Sidki group... Hubbard tree and Hubbard cactus for polynomials...

Proposition 4.4.40. *Let $f \circlearrowleft \mathcal{X}$ be an expanding covering map, and let δ be as in Lemma 1.4.37. Then the covering dimension of $\mathcal{X}$ is equal to the minimal value of n such that there exists a cover of $\mathcal{X}$ by open sets of diameter less than δ of multiplicity $n+1$.*

Proof. For every finite open cover $\mathcal{U}$ of $\mathcal{X}$ by sets of diameter less than δ the maximal diameter of the covers $\mathcal{U}_n$ exponentially decrease. By Lebegues covering lemma, this implies that for every open cover $\mathcal{V}$ of $\mathcal{X}$ there exists n such that $\mathcal{U}_n$ is subordinate to $\mathcal{V}$. The multiplicity of $\mathcal{U}_n$ is equal to the multiplicity of $\mathcal{U}$ by Lemma 4.4.3. □

4.4.11. Topological properties of expanding maps. Local properties and their formulation in algebraic terms: connectedness, local connectedness, when the limit dynamical system is a covering of spaces, when it is a covering of orbi-spaces, principal groupoid of germs, Hausdorff groupoid of germs of the group action vs the properties of the limit dynamical system...

Proposition 4.4.41. *Let $f \circlearrowleft \mathcal{X}$ be an expanding self-covering of a compact metric space, and let $\delta > 0$ be its strong injectivity constant. Then the topological dimension of $\mathcal{X}$ is equal to the smallest d such that there exists an open cover of $\mathcal{X}$ of multiplicity $d+1$ by sets of diameter less than δ.*

Proof. One inequality is obvious, the other follows by lifting covers by f^n... □

Corollary 4.4.42. *Let $f \circlearrowleft \mathcal{X}$ be an expanding self-covering. The topological dimension of $\mathcal{X}$ is equal to the smallest dimension of a contracting simplicial model of $f \circlearrowleft \mathcal{X}$.*

4.4.12. Expanding endomorphisms of orbifolds. Gromov-Shub Theorem, and its extension to orbifolds and locally simply connected spaces.... Numeration systems on $\mathbb{R}^n$, digit tiles, literature on this, also in the nilpotent case...

4.4.13. Topological dimension one. Remind what it is... All rational functions whose Julia set is not the whole sphere... Show that they are contracting on a virtually free group, and so can not be finitely presented... Prove also that iterated monodromy groups of rational functions are not finitely presented except for the obvious exceptions ... Examples with Sierpinski carpet... Mention D. Thurston's work...

4.5. Thurston maps and related structures

4.5.1. Basic definitions. See the definition of Thurston maps, i.e., post-critically finite orientation preserving branched self-coverings of the sphere S^2 in 4.3.3.

It is natural to consider Thurston maps up to the following equivalence relation.

Definition 4.5.1. Thurston maps f_1, f_2 with post-critical sets P_{f_1}, P_{f_2} are *combinatorially equivalent* if there exist homeomorphisms $\phi_1, \phi_2 : S^2 \longrightarrow S^2$ such that $\phi_i(P_{f_1}) = P_{f_2}$, ϕ_1 is homotopic to ϕ_2 relative to P_{f_1}, and the diagram

$$\begin{array}{ccc} S^2 & \xrightarrow{f_1} & S^2 \\ \big\downarrow{\scriptstyle \phi_1} & & \big\downarrow{\scriptstyle \phi_2} \\ S^2 & \xrightarrow{f_2} & S^2 \end{array}$$

is commutative.

We consider and sometimes define Thurston maps up to combinatorial equivalence. For example, we can compose f with an element of the *pure mapping class group* G_{P_f} of (S^2, P_f), i.e., with a homeomorphism $h : S^2 \longrightarrow S^2$ acting identically on P_f and defined up to a homotopy relative to P_f. The composition does not depend, up to combinatorial equivalence, on the choice of a particular representative h in the homotopy class.

One of standard ways of describing a Thurston map is using *subdivision rules*. A subdivision rule is a topological correspondence $f, \iota : \Delta_1 \longrightarrow \Delta_0$, where Δ_i are finite CW-complexes homeomorphic to S^2, f is an orientation preserving branched covering that maps cells of Δ_1 homeomorphically to cells of Δ_0, and ι is a homeomorphism such that $\iota(\Delta_1)$ is a subdivision of Δ_0. Note that it follows from the definitions that the post-critical set of $f \circ \iota^{-1}$ is a subset of the set of vertices of Δ_0, hence $f \circ \iota^{-1}$ is a Thurston map.

Example 4.5.2. Consider the CW complex Δ_0 obtained by gluing two copies of a right isosceles triangle along the boundary (by the identity map). Let Δ_1 be obtained from Δ_0 by subdividing both faces in two congruent right triangles, as it is shown on the top half of Figure 4.15, and let $\iota : \Delta_1 \longrightarrow \Delta_0$ be the identity homeomorphism. We color the faces of Δ_1 black and white so that no two faces sharing an edge have the same color, see Figure 4.15, where only the "front" part is shown. Let $f : \Delta_1 \longrightarrow \Delta_0$ be the branched covering that maps white triangles homeomorphically to the front triangle of Δ_0, black triangles the back of Δ_0, and maps the vertices of Δ_1 to the vertices of Δ_0 as it is shown on the figure. Note that it is not important how exactly the cells are mapped by f (we can choose the affine maps, for example), since we consider the Thurston map up to a combinatorial equivalence. Note that the critical points of f are the vertex of the right angle and the midpoint of the hypotenuse. It follows that the post-critical set of the described Thurston map is the set of vertices of Δ_0.

Example 4.5.3. One can also consider a similar "square" example, where a square pillow made of two squares (black and white) is covered by itself via a branched covering of degree four. The covering pillow is the same square pillow in which both squares are subdivided into four squares colored black and white in a checkerboard manner, so that no two small squares sharing a side have the same color. We may choose the covering preserving the colors and the vertical and horizontal directions. This example can be also modified in the way shown on the bottom half of Figure **??**, where a "flap" is added to the covering surface. Namely, we make a slit along a side of a square of the covering surface and attach to it a "pocket" obtained by gluing two squares along three sides. We also color the squares of the subdivision in two colors so that no two squares of the same color share an edge, and map black (resp., white) squares of the covering surface to the black (resp., white) square of the pillow.

Combinatorial equivalence of Thurston maps can be formulated in terms of the iterated monodromy groups (i.e., of the associated bisets) in the following way.

Theorem 4.5.4. *Let f_1, f_2 be Thurston maps with post-critical sets P_{f_1}, P_{f_2}. Let $\mathfrak{M}_1, \mathfrak{M}_2$, be the associated $\pi_1(S^2\backslash P_{f_i})$-bisets. The maps f_1 and f_2 are combinatorially equivalent if and only if there exists a bijection $\phi : \mathfrak{M}_1 \longrightarrow \mathfrak{M}_2$ and an isomorphism $h_* : \pi_1(S^2\backslash P_{f_1}) \longrightarrow \pi_1(S^2\backslash P_{f_2})$ induced by an orientation-preserving homeomorphism $h : S^2\backslash P_{f_1} \longrightarrow S^2\backslash P_{f_2}$, such that $\phi(g_1 \cdot x \cdot g_2) = h_*(g_1) \cdot \phi(x) \cdot h_*(g_2)$ for all $g_1, g_2 \in \pi_1(S^2\backslash P_{f_1})$ and $x \in \mathfrak{M}_1$.*

A proof of this theorem can be found in ... and [**Nek05**, Theorem 6.5.2], see also...

Iterated monodromy groups (i.e., the associated bisets) of Thurston maps are easy to compute in the case when the maps are given by subdivision rules. One can replace the topological correspondence $f, \iota : \Delta_1 \longrightarrow \Delta_0$ by the induced correspondence on the dual graphs in the following way. Let Γ_0 be the dual graph of Δ_0, seen as a subset of S^2. Then $\Gamma_1 = f^{-1}(\Gamma_0)$ is the dual graph of Δ_1. Suppose that e is an edge of Γ_1 corresponding to a common side of two adjacent cells A and B of Δ_0. If $\iota(A)$ and $\iota(B)$ are contained in different cells of Δ_0, then the image of the common side either belongs to an edge of Δ_0, which we will denote $\iota'(e)$. If $\iota(A)$ and $\iota(B)$ are contained in the same cell, then we denote this cell by $\iota'(e)$. We get

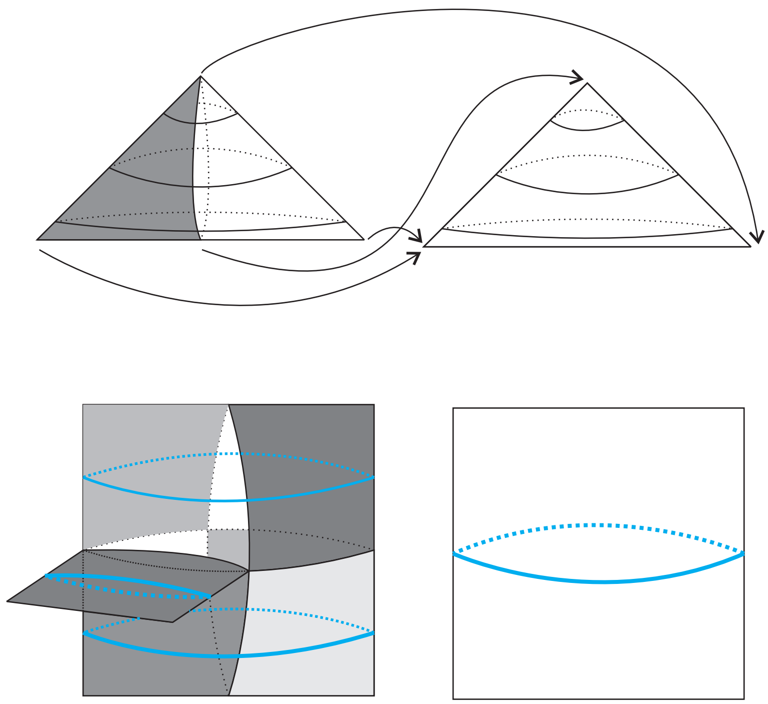

Figure 4.15. Subdivision rules

then a continuous map $\iota' : \Gamma_1 \longrightarrow \Gamma_0$. It is easy to see that the iterated monodromy group of the correspondence $f, \iota' : \Gamma_1 \longrightarrow \Gamma_0$ is equivalent as a self-similar group to the iterated monodromy group of $f, \iota : \Delta_1 \longrightarrow \Delta_0$, i.e., to the iterated monodromy group of the Thurston map $f \circ \iota^{-1}$ defined by the subdivision rule.

Example 4.5.5. Consider Example 4.5.2. The dual graphs Γ_1, Γ_0 are shown of Figure 4.16. Take the front vertex t as the basepoing in Γ_0. Let e_1, e_2, e_3 be the edges of Γ_0 oriented from t to the other vertex of Γ_0. The preimages of t are t_0 and t_1 as shown on the left-hand side of Figure 4.16.

The map $\iota' : \Gamma_1 \longrightarrow \Gamma_0$ collapses the preimages of e_1 to the vertices of Γ_0, maps the two preimages of e_2 to e_3 (once preserving and once reverting the orientation) and maps the preimages of e_3 to e_2 and e_1 (preserving the orientation in the case of e_1 and reverting it for e_2), see the figure.

Denote $a = e_3^{-1}e_2, b = e_1^{-1}e_3, c = e_2^{-1}e_1$. Note that $acb = 1$. Let t_0, t_1 be the f-preimages of t in the front and the back sells of Δ_0, respectively. Then $\iota'(t_0) = t$ and $\iota'(t_1)$ is the other vertex of Γ_0. Let us choose the trivial connecting path from t to $t = \iota'(t_0)$ and the path e_1 from t to $\iota'(t_1)$. Then taking lifts of the generators by f and mapping them back by ι', we get the

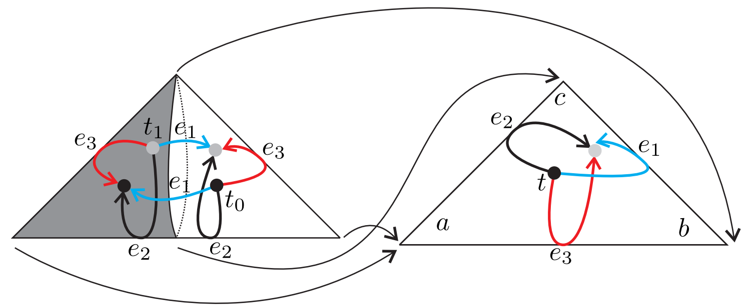

Figure 4.16.

wreath recursion

$$\begin{aligned}
a &= (e_1^{-1}e_3, e_1^{-1}e_2e_3^{-1}e_1) = (b, c^{-1}b^{-1}) = (b, c^{-1}a^{-1}c),\\
b &= \sigma(e_1^{-1}e_1, e_2^{-1}e_1) = \sigma(1, c),\\
c &= \sigma(e_1^{-1}e_3, e_3^{-1}e_1) = \sigma(b, b^{-1}).
\end{aligned}$$

4.5.2. Thurston theorem. It is natural to ask when a Thurston map is combinatorially equivalent to a rational function on $\widehat{\mathbb{C}}$. This question was answered by W. Thurston, by showing that it happens if and only if the map has no topological obstruction defined as follows.

Let $f : S^2 \longrightarrow S^2$ be a Thurston map with post-critical set P_f. A simple closed curve γ in $S^2\backslash P_f$ is said to be *periferal* if one of the regions bounded by γ contains less than two points of P_f. An *f-invariant multicurve* is a collection $\mathcal{C}$ of simple closed curves in $S^2\backslash P_f$ that are disjoint, non-periferal, pairwise non-homotopic and such that for every $\gamma \in \mathcal{C}$ each connected component of $f^{-1}(\gamma)$ is either peripheral or homotopic to an element of $\mathcal{C}$. By slightly abusing notation, we will denote sometimes by $f^{-1}(\gamma)$ the set of connected components of $f^{-1}(\gamma)$.

If $\mathcal{C}$ is an f-invariant multicurve, then we consider the following linear map $T_{\mathcal{C}} : \mathbb{R}^{\mathcal{C}} \longrightarrow \mathbb{R}^{\mathcal{C}}$:

$$A_{\mathcal{C}}(e_\gamma) = \sum_{\alpha \in f^{-1}(\gamma)} \frac{[\alpha]}{\deg(f : \alpha \longrightarrow \gamma)},$$

where e_γ is the basic vector of $\mathbb{R}^{\mathcal{C}}$ corresponding to $\gamma \in \mathcal{C}$; $[\alpha] = e_{\alpha'}$, where $\alpha' \in \mathcal{C}$ homotopic to α, if α is non-peripheral, and 0 otherwise; and $deg(f : \alpha \longrightarrow \gamma)$ is the degree of the corresponding covering.

Let S^2_ν be the Thurston orbifold associated with f, see 4.3.3. The *Euler characteristic* of S^2_ν is the number $2 - \sum_{x \in P_f} \left(1 - \frac{1}{\nu(x)}\right)$. If the Euler characteristic is positive, then the fundamental group of the orbifold is finite (which never happens for orbifolds of Thurston maps). If it is equal to zero, then the fundamental group (and hence also the iterated monodromy group) is virtually abelian. If the Euler characteristic is negative, then the fundamental group is hyperbolic, see....

Theorem 4.5.6. *A Thurston map $f : S^2 \longrightarrow S^2$ with negative Euler characteristic of the associated orbifold is combinatorially equivalent to a rational function if and only if for any f-invariant multicurve $\mathcal{C}$ the spectral radius of $A_{\mathcal{C}}$ is less than one. If it is the case, the rational function is unique up to a conjugation by a Möbius transformation.*

In the Euclidean case (Euler characteristic zero) the Thurston map f is equivalent to an endormophism of the orbifold of an action of an affine group on $\mathbb{R}^2$. The affine group (the fundamental group of the orbifold) contains a subgroup of finite index isomorphic to $\mathbb{Z}^2$. The virtual endomorphism of the fundamental group associated with the self-covering will induce a virtual endomorphism ϕ of $\mathbb{Z}^2$. If the eigenvalues of ϕ are complex, then f is equivalent to a unique (up to conjugation) rational function. If they are real and different, then f is not equivalent to a rational function. If they are real and equal, then it is equivalent to a rational function, but the function is not unique.

An f-invariant multicurve with spectral radius of $A_{\mathcal{C}}$ greater or equal to one is called an *obstruction*. The simplest class of obstructions are *Levy cycles*: a sequence of simple, disjoint, pairwise non-homotopic, non-periferal curves $\gamma_1, \gamma_2, \ldots, \gamma_n$ such that $f : \gamma_i \longrightarrow f(\gamma_i)$ is of degree 1 and $f(\gamma_i)$ is homotopic to γ_{i+1}, where the indices are taken modulo n. Note that the fact that a Levy cycle is an obstruction for a Thurston map to be equivalent to a rational function follows just from the fact that the $\pi_1(S_\nu)$-biset associated with a post-critically finite rational function is hyperbolic, see... Existence of a Levy cycle contradicts hyperbolicity of the biset, since we get then an element of infinite order $g \in \pi_1(S_\nu)$ represented by γ_1 such that $\phi^n(g) = g$ for a virtual endomorphism associated with the biset. Then all the elements of the cyclic group generated by g must belong to the nucleus. In fact, the converse statement is also true, and was proved by L. Bartholid and D. Dudko, see...

Theorem 4.5.7. *A Thurston map f has a Levy cycle if and only if the $\pi_1(S_\nu)$-biset associated with it (where S_ν is the Thurston orbifold of f) is not hyperbolic.*

We will see examples of Levy cycles later ... Here is a simple example of an obstruction which is not a Levy cycle.

Example 4.5.8. Consider Example 4.5.3. It is easy to check that the Euler characteristic of the associated orbifold is negative. Let γ be a simple curve formed by two horizontal medians of the squares, see Figure 4.15, where it is drawn blue. Then $f^{-1}(\gamma)$ is the union of horizontal medians of the small squares of the covering surface. One of the connected components is a closed curve on the flap and is periferal. Two other connected components are homotopic to γ and are mapped to γ by a degree two covering. It follows that $\mathcal{C}$ is an f-invariant multicurve and $A_{\mathcal{C}}$ is the identity map.

4.5.3. Teichmüller space dynamics. Let $f : S^2 \longrightarrow S^2$ be a Thurston map with the post-critical set P_f. The *Teichmüller space* $\mathcal{T}_{P_f}$ of (S^2, P_f) is the space of homeomorphisms $\tau : S^2 \longrightarrow \widehat{\mathbb{C}}$ modulo the equivalence relation identifying two homeomorphisms τ_1, τ_2 if there exists a Möbius transformation $\phi : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ such that $\phi \circ \tau_1$ is isotopic to τ_2 relative to P_f. We interpret elements $\tau : S^2 \longrightarrow \widehat{\mathbb{C}}$ as complex structures on S^2. For every complex structure $\tau \in \mathcal{T}_{P_f}$ there exists a unique complex structure τ' such that the map $f_\tau = \tau \circ f \circ (\tau')^{-1}$ closing the diagram

$$\begin{array}{ccc} S^2 & \xrightarrow{f} & S^2 \\ \downarrow{\scriptstyle \tau'} & & \downarrow{\scriptstyle \tau} \\ \widehat{\mathbb{C}} & \xrightarrow{f_\tau} & \widehat{\mathbb{C}} \end{array} \tag{4.6}$$

is a rational function. Essentially, τ' is obtained from τ by pulling it back by the branched covering f. Let us denote $\tau' = \sigma_f(\tau)$. It follows from the definitions that if g is a homeomorphism of S^2 acting identically on P_f, then we have

$$\sigma_{f\circ g}(\tau) = \sigma_f(\tau) \circ g, \qquad \sigma_{g\circ f}(\tau) = \sigma_f(\tau \circ g). \tag{4.7}$$

Thurston's theorem 4.5.7 is proved by studying dynamics of σ_f. Namely, it follows from the definitions that f is combinatorially equivalent to a rational function if and only if σ_f has a fixed point. One can show that σ_f is non-uniformly contracting, and hence either iterations of σ_f converge to a unique fixed point, or there is no fixed point and the iterations of σ_f converge to infinity. The latter implies degeneration of the associated complex structures and existence of an obstruction. One can read about the details of the proof in... and ...

The *Moduli space* $\mathcal{M}_{P_f}$ is the space of injective maps $\tau : P_f \longrightarrow \widehat{\mathbb{C}}$ modulo post-compositions with Möbius transformations. It is a classical fact that $\mathcal{T}_{P_f}$ is naturally identified with the universal covering of $\mathcal{M}_{P_f}$, and that the fundamental group of $\mathcal{M}_{P_f}$ is the (pure) mapping class group of (S^2, P_f), i.e., the group of homeomorphisms of S^2 fixing pointwise P_f modulo isotopies relative to P_f. We will denote this group G_{P_f}. The covering $\mathcal{T}_{P_f} \longrightarrow \mathcal{M}_{P_f}$ is the map $\tau \mapsto \tau|_{P_f}$. The action of the fundamental group on the universal covering by deck transformations is identified with the natural right action of the mapping class group on $\mathcal{T}_{P_f}$ by pre-compositions.

If we choose three points of P_f, and specify the values of τ on them (e.g., $\infty, 0$, and 1) then a point of $\mathcal{M}_{P_f}$ is uniquely determined by the values of $\tau : P_f \longrightarrow \widehat{\mathbb{C}}$ on the remaining points of P_f. This gives us an identification of $\mathcal{M}_{P_f}$ with the subset of $\mathbb{C}^{|P_f|-3}$ consisting of vectors $(z_1, z_2, \ldots, z_{|P_f|-3})$ such that z_i are pairwise different and not equal to 0 or 1.

The map $\sigma_f : \mathcal{T}_{P_f} \longrightarrow \mathcal{T}_{P_f}$ naturally induces a correspondence on $\mathcal{M}_{P_f}$ in the following way. Let G_1 be the subgroup of elements of the mapping class group G_{P_f} consisting of all *liftable* homeomorphisms, i.e., homeomorphisms $g : (S^2, P_f) \longrightarrow (S_2, P_f)$ for which there exists a homeomorphism g' of S^2 fixing pointwise P_f and such that the diagram

$$\begin{array}{ccc} S^2 & \xrightarrow{g'} & S^2 \\ \big\downarrow f & & \big\downarrow f \\ S^2 & \xrightarrow{g} & S^2 \end{array}$$

is commutative.

It is known that G_1 is a subgroup of finite index in G_{P_f} and that g' is unique, so that $\phi_f : g' \mapsto g$ is a virtual endomorphism of the mapping class group, see [**KPS16**, Proposition 3.1].

Let $\mathcal{W}$ be the quotient of $\mathcal{T}_{P_f}$ by G_1. Since G_1 is a finite index subgroup of the mapping class group, the identity map on $\mathcal{T}_{P_f}$ induces a finite degree covering map $F : \mathcal{W} \longrightarrow \mathcal{M}_{P_f}$. For every $g \in G_1$ and $\tau \in \mathcal{T}_{P_f}$, we have $\sigma_f(\tau \circ g) = \sigma_{g \circ f}(\tau) = \sigma_{f \circ g'} = \sigma_f(\tau) \circ g'$, by (4.7). If we use right actions for both G_{P_f} and σ_f, then we get the relation $g \cdot \sigma_f = \sigma_f \cdot g'$.

It follows that $\tau \mapsto \sigma_f(\tau)$ induces a continuous map $\iota : \mathcal{W} \longrightarrow \mathcal{M}_{P_f}$. We get a correspondence $F, \iota : \mathcal{W} \longrightarrow \mathcal{M}_{P_f}$. We call it the *moduli space correspondence associated with* f. If we interpret ι as a model of the identity map, then the correspondence is the projection of the correspondence $\sigma_f(\tau) \mapsto \tau$ to the moduli space. More on this correspondence and a more direct description of it see in [**Koc13**].

Denote by $\mathfrak{T}$ be associated biset over the fundamental group G_{P_f} of $\mathcal{M}_{P_f}$. According to Propsition 4.3.2, it is naturally identified with the set of maps $g_1 \cdot \sigma_f \cdot g_2 = \sigma_{g_1 \circ f \circ g_2}$ for $g_1, g_2 \in G_{P_f}$ with the natural action of G_{P_f}.

4.5.4. Maps on the moduli spaces and skew products. The moduli space correspondence $F, \iota : \mathcal{W} \longrightarrow \mathcal{M}_{P_f}$ associated with a Thurston map f is sometimes (but not always) a partial self-covering, i.e., ι is a homeomorphic embedding, and we get a commutative diagram

$$\begin{tikzcd} \mathcal{T}_{P_f} \arrow[d] & \mathcal{T}_{P_f} \arrow[l, "\sigma_f"'] \arrow[d] \\ \mathcal{M}_{P_f} \arrow[r, "F"] & \mathcal{M}_{P_f}. \end{tikzcd}$$

Example 4.5.9. Let f be a degree two branched covering, with one fixed critical point x, and one critical point y belonging to a cycle $y_0 = y, y_1, \ldots, y_{n-1}$ with $f(y_i) = y_{i+1}$, where indices are taken modulo n. Let τ be a point of $\mathcal{T}_{P_f}$. The corresponding point of the moduli space is the restriction of τ to the set $P_f = \{x, y_0, y_1, \ldots, y_{n-1}\}$ to $\widehat{\mathbb{C}}$ modulo Möbius transformations. By choosing the appropriate Möbius transformation, we may assume that $\tau(x) = \infty$, $\tau(y_0) = 0$, and $\tau(y_1) = 1$. Then the point of the moduli space is identified with the vector $(\tau(y_2), \tau(y_3), \ldots, \tau(y_{n-1}))$.

Consider the diagram (4.6) for this situation. If τ' is also normalized the same way as τ (i.e., if $\tau'(x) = \infty, \tau'(y_0) = 0, \tau'(y_1) = 1$), then f_τ is a rational function such that ∞ is a totally invariant critical point, 0 is a critical point, mapped by f_τ to 1, and $f_\tau(\tau'(y_i)) = \tau(y_{i+1})$. If $(p_2, p_3, \ldots, p_{n-1}) = (\tau'(y_2), \tau'(y_3), \ldots, \tau'(y_{n-1}))$ is the tuple representing τ', then we have $f_\tau(0) = 1, f_\tau(1) = z_2$, $f_\tau(p_i) = z_{i+1}$ for $i = 2, \ldots, n-2$, and $f_\tau(p_{n-1}) = 0$.

We conclude that f_τ is a quadratic polynomial with critical point 0 such that $f_\tau(0) = 1$, hence $f_\tau(z) = az^2 + 1$ for some non-zero coefficient a, and we have $f_\tau(1) = z_2, f_\tau(p_2) = z_3, \ldots, f_\tau(p_{n-2}) = z_{n-1}, f_\tau(p_{n-1}) = 0$. Note that the last equality implies $a = -\frac{1}{p_{n-1}^2}$, so that

$$f_\tau(z) = 1 - \frac{z^2}{p_{n-1}^2},$$

and

$$z_2 = 1 - \frac{1}{p_{n-1}^2}, z_3 = 1 - \frac{p_2^2}{p_{n-1}^2}, \ldots, z_{n-1} = 1 - \frac{p_{n-2}^2}{p_{n-1}^2}.$$

It follows that if we identify the moduli space with a subset of $\mathbb{C}^{n-2}$ as above, then the map F equal to the projection of the correspondence $\sigma_f(\tau) \mapsto \tau$ to the moduli space is given by the formula

$$F(p_2, p_3, \ldots, p_{n-1}) = \left(1 - \frac{1}{p_{n-1}^2}, 1 - \frac{p_2^2}{p_{n-1}^2}, \ldots, 1 - \frac{p_{n-2}^2}{p_{n-1}^2}\right).$$

Note that in this case the map F can be extended to an endomorphism of the projective space $\mathbb{PC}^{n-2}$ given in the homogeneous coordinates by

$$[p_1 : p_2 : \ldots : p_{n-1}] \mapsto [p_{n-1}^2 : p_{n-1}^2 - p_1^2 : p_{n-1}^2 - p_2^2 : \ldots : p_{n-1}^2 - p_{n-2}^2].$$

S. Koch showed that F extends to an endomorphism of $\mathbb{PC}^{n-2}$ for any unicritical (i.e., having a unique finite critical point) post-critically finite polynomial...

We can also put the moduli space correspondence F and the rational function f_τ together into one *skew product map*:

$$(z, p_2, p_3, \ldots, p_{n-1}) \mapsto \left(1 - \frac{z^2}{p_{n-1}^2}, 1 - \frac{1}{p_{n-1}^2}, 1 - \frac{p_2^2}{p_{n-1}^2}, \ldots, 1 - \frac{p_{n-2}^2}{p_{n-1}^2}\right).$$

It also extends to an edomorphism of the projective space.

Such and similar constructions are sources of (otherwise hard to find) examples of post-critically finite endomorphisms of $\mathbb{PC}^n$, i.e., endomorphisms such that the set of post-critical points is a union of a finite number of varieties. See more on this in ...

Example 4.5.10. Consider a degree 2 Thurston map with two critical points x_0, x_1 such that $f^2(x_i) = x_i$. Then the post-critical set consists of four points $x_0, f(x_0), x_1, f(x_1)$. Let as identify x_0 with 0, x_1 with ∞, and $f(x_1)$ with 1. Then a point of the moduli space is represented by the position p of $f(x_0)$. Then f_τ is a rational function with critical points $0, \infty$, such that $f_\tau(\infty) = 1$, $f_\tau(1) = \infty$, $f_\tau(p) = 0$, $f_\tau(0) = F(p)$, where F is the moduli space correspondence.

Any quadratic rational function $f_\tau(z)$ with critical points 0 and ∞ is of the form $\frac{az^2+b}{cz^2+d}$. If $f_\tau(\infty) = 1$ and $f_\tau(1) = \infty$, then it is of the form $\frac{z^2+b}{z^2-1}$. It follows from the condition $f_\tau(p) = 0$ that $f_\tau(z) = \frac{z^2-p^2}{z^2-1}$. Consequently, $F(p) = f_\tau(0) = p^2$. Note that the fixed points of $F(p)$ are $0, 1, \infty$, which do not belong to $\mathcal{M}$. This implies that there are no rational functions with the given dynamics on the post-critical set. We will see later that there exist obstructed Thurston maps realizing this dynamics (e.g., the mating of two copies of $z^2 - 1$, see...). Any such Thurston map f has a Levy cycle consisting of a single closed curve separating $\{x_0, x_1\}$ from $\{f(x_0), f(x_1)\}$.

The corresponding skew product map is given by $(z, p) \mapsto \left(\frac{z^2-p^2}{z^2-1}, p^2\right)$. It does not extend to an endomorphism of $\mathbb{PC}^2$.

Example 4.5.11. This is an example from [buff-coch... page 571]... Consider a Thurston map f of degree 3 with two fixed simple (i.e., of local degree 2) critical points and two simple critical points that are interchanged by f. An example of such a map is $f(z) = \frac{3z^2}{2z^3+1}$ with critical points $0, 1, -1/2 \pm i\sqrt{3}/2$. The first two critical points are fixed, the other two are interchanged. Let us assume that one of the fixed critical points of the Thurston map f is 1, and that the critical points that are swapped are $\omega = -1/2 + i\sqrt{3}/2$ and $\overline{\omega} = -1/2 - \sqrt{3}/2$. Then a point $\tau|_{P_f}$ of $\mathcal{M}_{P_f}$ is represented by the position p of the other critical point. Then the corresponding rational function f_τ is represented by a rational of degree 3 that has a fixed critical point 1, swaps two critical points ω and $\overline{\omega}$, and has the fourth critical point p.

For a point $\alpha = [a : b]$ of $\widehat{\mathbb{C}} = \mathbb{P}^1$, consider the function $g_\alpha = \frac{az^3+3bz^2+2a}{2bz^3+3az+b}$. Its derivative is $\frac{6(b^2z-a^2)(1-z^3)}{(2bz^3+3az+b)^2}$, hence the critical points of g_α are $1, \omega, \overline{\omega}$, and α^2. Note also that $g_\alpha(1) = \frac{a+3b+2a}{2b+3a+b} = 1$, $g_\alpha(\omega) = \frac{a+3b\overline{\omega}+2a}{2b+3a\omega+b} = \frac{\overline{\omega}(3b+3a\omega}{3b+3a\omega} = \overline{\omega}$, and similarly $g_\alpha(\overline{\omega}) = \omega$. It follows that g_α satisfies the same conditions on the critical points as f_τ. Then $g_\alpha - f_\tau$ has critical points at $1, \omega, \overline{\omega}$ and equal to zero at these critical points. It follows that the numerator of $g_\alpha - f_\tau$ is divisible by $(z^3-1)^2$, and since it is of degree not more than 6, we get that it is equal to $a(z^3-1)^2$ for some $a \in \mathbb{C}$. In particular, it is true for the numerators of $g_0 - f_\tau$ and $g_\infty - f_\tau$. Let $P(z)$ and $Q(z)$ be the numerator and the denominator of f_τ, respectively. Then $(2z^3+1)P(z) - 3z^2Q(z)$ and $3zP(z) - (z^3+2)Q(z)$ are both equal to $(z^3-1)^2$ times a complex number. It follows that there exist $a, b \in \mathbb{C}$ such that

$$a(3zP(z) - (z^3+2)Q(z)) + b((2z^3+1)P(z) - 3z^2Q(z)) = 0,$$

which implies $f_\tau(z) = \frac{P(z)}{Q(z)} = \frac{a(z^3+2)+3bz^2}{3az+b(2z^3+1)} = g_{[a:b]}(z)$. Consequently, for every point of $\mathcal{M}_{P_f}$ there exists an $\alpha \in \widehat{\mathbb{C}}$ such that $f_\tau = g_\alpha$. The point of the moduli space is represented by α^2. The image of the critical point α^2 is $g_\alpha(\alpha^2) = \frac{\alpha^7+3\alpha^4+2\alpha}{2\alpha^6+3\alpha^3+1} = \frac{(\alpha^3+1)(\alpha^4+2\alpha)}{(\alpha^3+1)(2\alpha^3+1)} = \frac{\alpha^4+2\alpha}{2\alpha^3+1}$.

We see that in this case the moduli space correspondence $F, \iota : \mathcal{M}' \longrightarrow \mathcal{M}$ is given by two maps $\iota : \alpha \mapsto \alpha^2$ and $F : \alpha \mapsto \frac{\alpha(\alpha^3+2)}{2\alpha^3+1}$. Here $\mathcal{M} = \widehat{\mathbb{C}} \backslash \{1, \omega, \overline{\omega}\}$ and $\mathcal{M}' = \widehat{\mathbb{C}} \backslash \{\pm 1, \pm\omega, \pm\overline{\omega}\}$ (check that if ξ is a cubic root of 1, then the only solutions of $\frac{\alpha(\alpha^3+2)}{2\alpha^3+1} = \xi$ are ξ (three times) and $-\xi$).

Here is another interesting example from... [BEKP]...

Example 4.5.12. Consider the polynomial $f(z) = ((i-1)z^2+1)^2$. Its finite critical points are $z = 0$ and $z = \pm\sqrt{\frac{1}{1-i}}$. The orbits of the critical points are

$$\pm\sqrt{\frac{1}{1-i}} \mapsto 0 \mapsto 1 \mapsto -1 \mapsto -1,$$

hence $\{\infty, 0, 1, -1\}$ is the post-critical set of f.

We can write $f(z)$ as the composition $h \circ g$ of $g : z \mapsto z^2$ and $h : z \mapsto ((i-1)z+1)^2$. The sets of critical values of both g and h are $\{0, \infty\}$. Let $A = \{0, 1, \infty\}$ and $B = \{0, \infty, 1, -1\}$. Then $B = g^{-1}(A)$ and $A \subset h^{-1}(B)$. It follows that g and h induce pull-back maps $\sigma_g : \mathcal{T}_A \longrightarrow \mathcal{T}_B$ and $\sigma_h : \mathcal{T}_B \longrightarrow \mathcal{T}_A$, and that $\sigma_f = \sigma_g \circ \sigma_h$. But $\mathcal{T}_A$ is a single point, since the dimension of $\mathcal{T}_P$ is equal to $|P| - 2$. It follows that σ_f is constant. In particular, the map ι in the associated moduli space correspondence $F, \iota : \mathcal{M}' \longrightarrow \mathcal{M}$ is constant.

4.5.5. The iterated monodromy group of the moduli space correspondence. Let f be a Thurston map with post-critical set P_f. Recall that the fundamental group of the moduli space $\mathcal{M}_{P_f}$ is naturally identified with the pure mapping class group of (S^2, P_f), i.e., with the group of homeomorphisms $g : S^2 \longrightarrow S^2$ acting identically on P_f modulo isotopies relative to P_f. Every such a homeomorphism induces an automorphism g_* of the fundamental group $S^2 \backslash P_f$, defined up to inner automorphisms. We get a homomorphism from $\pi_1(\mathcal{M}_{P_f})$ to the group of outer automorphisms of $\pi_1(S^2 \backslash P_f)$. It is known that this homomorphism is an embedding...

The associated biset over the mapping class group also has a natural interpretation. Consider the set $\mathcal{F}$ of all homotopy classes relative to P_f of the maps of the form $g_1 \circ f \circ g_2$, where g_1, g_2 are elements of the mapping class group G. Then $\mathcal{F}$ is naturally a G-biset. It follows from the description of the virtual endomorphism ϕ_f associated with the correspondence $F, \iota : \mathcal{M}' \longrightarrow \mathcal{M}$ that $\mathcal{F}$ is isomorphic to the biset associated with the correspondence....

Let ϕ be the virtual endomorphism of $\pi_1(S^2 \backslash P_f)$ induced by f, i.e., by lifting loops by f. Let μ be the virtual endomrophism of $\pi_1(\mathcal{M}_{P_f})$ induced by the moduli space correspondence. Its domain is the subgroup G_1 of homeomorphisms liftable by f, and if $g \in G_1$, then $\mu(g)$ is the lift of g by f acting identically on $f^{-1}(P_f)$, see.... It follows directly from the definition that the action of $\pi_1(\mathcal{M}_{P_f})$ on $\pi_1(S^2 \backslash P_f)$ agrees with ϕ and μ in the sense that for every $g \in G_1$ there exists $\delta \in \pi_1(S^2 \backslash P_f)$ such that

$$\phi(\gamma^g) = \phi(\gamma)^{\mu(g)\delta} \tag{4.8}$$

for all γ in the domain of ϕ. (relation between the domains of ϕ and μ...) Since ϕ is surjective, equation 4.8 determines $\mu(g)$ uniquely as the only automorphism of G mapping $\phi(\gamma)$ to $\phi(\gamma^g)$. This makes it possible to compute μ.

The procedure is interpreted in terms of bisets in the following way. If ϕ is an endomorphism of a group G, then the associated biset is G as a set with the following left and right actions:

$$h_1 \cdot g \cdot h_2 = h_1^{\phi} g h_2,$$

where the dots on the left hand side represent the actions on the biset, h_1^{ϕ} is the action of the automorphism on h_1, and multiplication on the right-hand side is the usual multiplication in G. In order to avoid confusion, we will denote the element of the biset corresponding to $g \in G$ by $\phi \cdot g$. Then the formula for the actions looks more natural:

$$h_1 \cdot \phi \cdot g \cdot h_2 = \phi \cdot h_1^{\phi} g h_2.$$

We will denote this biset by $[\phi]$. Note that $[\phi]$ is a covering biset, the singleton $\{\phi \cdot 1\}$ is its basis, and the homomorphism ϕ is the virtual endomorphism (this time everywhere defined) associated with $[\phi]$ and $\phi \cdot 1$.

It is also easy to see that two bisets $[\phi_1]$, $[\phi_2]$ are isomorphic if and only if there exists an element $g \in G$ such that $x^{\phi_1} = (x^{\phi_1})^g$ for all $x \in G$. In other words, if and only if ϕ_1 and ϕ_2 differ by an inner automorphism of G.

It follows directly from the definitions that $[\phi_1 \cdot \phi_2]$ is isomorphic to $[\phi_1] \otimes [\phi_2]$, where $\phi_1 \cdot \phi_2$ is the composition (in terms of a right action) of the endomorphisms. In particular, the set of isomorphism classes of bisets $[\phi]$ defined by automorphisms ϕ of G is a group in terms of tensor products naturally isomorphic to the outer automorphism group of G.

Suppose now that $\mathfrak{M}$ is a G-biset, and let H be a subgroup of the outer automorphism group of G. Then the *induced H-biset* is the set of isomorphism classes of bisets of the form $[h_1] \otimes \mathfrak{M} \otimes [h_2]$ where $h_1, h_2 \in H$ with the natural left and right H-actions.

A particular case of this situation is the moduli space correspondence. If f is a Thurston map, then the biset over the pure mapping class group associated with the moduli space correspondence is naturally isomorphic to the biset induced on the mapping class group (seen as a subgroup of the outer automorphism group of $\pi_1(S^2\backslash P_f)$) induced by the $\pi_1(S^2\backslash P_f)$-biset $\mathfrak{M}_f$ associated with the Thurston map. A biset $[h_1] \otimes \mathfrak{M}_f \otimes [h_2]$ is isomorphic to the biset $\mathfrak{M}_{h_1 \circ f \circ h_2}$ (check the sides...) associated with the Thurston map $h_1 \circ f \circ h_2$.

The biset ... is essential in the study of combinatorial equivalence of Thurston maps (see...) and obstructions. Recall that if γ is a closed simple curve on a surface, then the *Dehn twist* about γ is the following homeomorphism (defined up to an isotopy). Consider a narrow annulus along γ, and let γ_1 and γ_2 be the inner and the outer curves bounding it (both of them are homotopic and close to γ). The twist acts identically outside the annulus, and rotates γ_1 by a full turn, see Figure... write better...

Let $\mathcal{C}$ be a multicurve. The Dehn twists about the elements of $\mathcal{C}$ pairwise commute and freely generate the abelian group $\mathbb{Z}^{\mathcal{C}} \leqslant G_{P_f}$. It is easy to check that the map $A_{\mathcal{C}}$ from Theorem 4.5.7 is precisely the restriction of the virtual endomorphism associated with ... to this group. In particular, Theorem 4.5.7 implies that if f has an obstruction, then ... is not hyperbolic. In fact, it also follows that if any of the Thurston maps $h_1 \circ f \circ h_2 \in$ ... is obstructed, then the biset ... is not hyperbolic.

The bisets of the form $[\phi] \otimes \mathfrak{M}$ and $\mathfrak{M} \otimes [\phi]$ are easy to compute using the following description of the corresponding wreath recursions.

Proposition 4.5.13. *Let $\mathfrak{M}$ be a covering G-biset, and let X be a basis. Let $\Psi_{\mathfrak{M}} : G \longrightarrow \mathsf{S}_{\mathsf{X}} \ltimes G^{\mathsf{X}}$ be the corresponding wreath recursion. Let ϕ be an endomorphism of G. Then $\{\phi\} \otimes \mathsf{X}$ and $\mathsf{X} \otimes \{\phi\}$ are bases of the bisets $[\phi] \otimes \mathfrak{M}$ and $\mathfrak{M} \otimes [\phi]$, respectively. Let $\Psi_{[\phi]\otimes\mathfrak{M}}, \Psi_{\mathfrak{M}\otimes[\phi]} : G \longrightarrow \mathsf{S}_{\mathsf{X}} \ltimes G^{\mathsf{X}}$ be the associated wreath recursions, where we identify $\{\phi\} \otimes \mathsf{X}$ and $\mathsf{X} \otimes \{\phi\}$ with X by the bijections $\phi \otimes x \mapsto x$ and $x \otimes \phi \mapsto x$. Then*

$$\Psi_{[\phi]\otimes\mathfrak{M}}(g) = \Psi(g^{\phi}), \qquad \Psi_{\mathfrak{M}\otimes[\phi]}(g) = (\Psi(g))^{\phi},$$

where ϕ acts in the second equality on $\mathsf{S}_{\mathsf{X}} \ltimes G^{\mathsf{X}}$ diagonally: $(\sigma(g_x)_{x\in\mathsf{X}})^{\phi} = \sigma(g_x^{\phi})_{x\in\mathsf{X}}$.

The proposition follows directly from the definitions and we leave its proof to the reader as an exercise.

Example 4.5.14. Consider the case when f is the *rabbit polynomial* $z^2 + c$, where for $c \approx -0.1226 + 0.7449i$ is such that $f^3(0) = 0$. The biset $\mathfrak{M}_f$ is given by the wreath recursion

$$a = \sigma(1, c), \quad b = (1, a), \quad c = (1, b),$$

see ... Denote this biset over the free group $\langle a, b, c \mid \varnothing\rangle$ (which is the fundamental group of the corresponding punctured sphere $\widehat{\mathbb{C}} \backslash P_f$) by $\mathfrak{M}_0$. Consider also the biset $\mathfrak{M}_1$ given by

$$a = \sigma(1, c), \quad b = (a, 1), \quad c = (1, b).$$

The pure mapping class group is generated by two Dehn twists S and T acting on the fundamental group (from the right) by the automorphisms

$$a^S = a, \quad b^S = b^{cb}, \quad c^S = c^{cb},$$

and

$$a^T = a^{ba}, \quad b^T = b^{ba}, \quad c^T = c.$$

Using Proposition 4.5.13, we see that the biset $[S] \otimes \mathfrak{M}_0$ is given then by the wreath recursion

$$\begin{aligned} a &= \sigma(1, c), \\ b &= (1, a)^{(1,ba)} = (1, a^{ba}), \\ c &= (1, b)^{(1,ba)} = (1, b^{ba}), \end{aligned}$$

which implies that $[S] \otimes \mathfrak{M}_0$ is isomorphic to $\mathfrak{M}_0 \otimes [T]$.

Similarly, $[S] \otimes \mathfrak{M}_1$

$$\begin{aligned} a &= \sigma(1, c), \\ b &= (a, 1)^{(a,b)} = (a, 1), \\ c &= (1, b)^{(a,b)} = (1, b), \end{aligned}$$

which gives $[S] \otimes \mathfrak{M}_1 \cong \mathfrak{M}_1$.

The biset $[T] \otimes \mathfrak{M}_0$ is given by

$$\begin{aligned} a &= (\sigma(1, c))^{\sigma(a,c)} = \sigma(a, a^{-1}c), \\ b &= (1, a)^{\sigma(a,c)} = (a, 1), \\ c &= (1, b). \end{aligned}$$

Conjugating the right-hand side by $(a^{-1}, 1)$, we get the wreath recursion defining $\mathfrak{M}_1$.

The biset $[T] \otimes \mathfrak{M}_1$ is given by

$$\begin{aligned} a &= (\sigma(1, c))^{\sigma(1,ac)} = \sigma(c^{-1}a^{-1}c, ac), \\ b &= (a, 1)^{\sigma(1,ac)} = (1, c^{-1}ac), \\ c &= (1, b). \end{aligned}$$

Conjugating the right hand side by $(c^{-1}ac, 1)$, we get an isomorphism of $[T] \otimes \mathfrak{M}_1$ with $\mathfrak{M}_0 \otimes [R]$, where R is the automorphism

$$a^R = a^{ac}, \quad b^R = b, \quad c^R = c^{ac}.$$

Note that $a^{S^{-1}} = a, b^{S^{-1}} = b^{c^{-1}}, c^{S^{-1}} = c^{b^{-1}c^{-1}}$ and $a^{T^{-1}} = a^{b^{-1}}, b^{T^{-1}} = b^{a^{-1}b^{-1}}, c^{T^{-1}} = c$. It follows that

$$\begin{aligned} a^{T^{-1}S^{-1}} &= (bab^{-1})^{S^{-1}} = cbc^{-1}acb^{-1}c^{-1} = a^{cb^{-1}c^{-1}}, \\ b^{T^{-1}S^{-1}} &= (baba^{-1}b^{-1})^{S^{-1}} = b^{c^{-1}a^{-1}cb^{-1}c^{-1}}, \\ c^{T^{-1}S^{-1}} &= c^{S^{-1}} = c^{b^{-1}c^{-1}}. \end{aligned}$$

Conjugating the right-hand side by $cbc^{-1}ac$, we conclude that $R = T^{-1}S^{-1}$, so that $[T] \otimes \mathfrak{M}_1$ is isomorphic to $\mathfrak{M}_0 \otimes [T^{-1}S^{-1}]$.

Consequently, the biset associated with the corresponding moduli space correspondence is given by the wreath recursion

$$S = (T, 1), \qquad T = \sigma(1, T^{-1}S^{-1}).$$

It is easy to check that it coincides with the iterated monodromy group of the rational function $p \mapsto 1 - \frac{1}{p^2}$, which is a realization of the moduli space correspondence, as explained in Example 4.5.9. (make an exercise...)

Example 4.5.15. $z^2 + i$... obstructed maps in the corresponding family... then move to exercises...

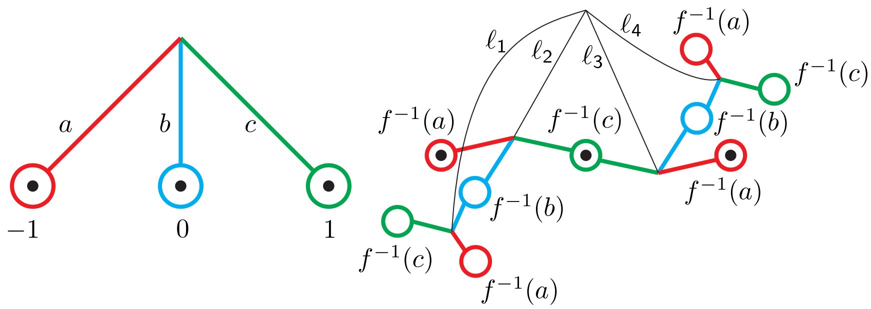

Figure 4.17. The iterated monodromy group of $((i-1)z^2+1)^2$

Example 4.5.16. Consider the polynomial $f(z) = ((i-1)z^2+1)^2$ from Example 4.5.12. Take the generators a, b, c shown on the left-hand side part of Figure.., going around the post-critical points in the positive direction. The right-hand side part shows their preimages and the connecting paths we are using to compute the wreath recursion.. We get

$$a = (1, a, c, 1), \quad b = (12)(34), \quad c = (23)(1, b, 1, 1).$$

Denote by $\mathfrak{M}_0$ the corresponding biset. Let $\mathfrak{M}_1$ be the biset given by

$$a = (1, a, c, 1), \quad b = (13)(24), \quad c = (14)(1, 1, 1, b).$$

Consider the same Dehn twists S and T as in the previous example. Let $\mathfrak{M}$ be the biset associated with f. We have $cb = (1342)(b, 1, 1, 1)$, $ba = (12)(34)(a, 1, 1, c)$ for $\mathfrak{M}_0$, and $cb = (1342)(1, b, 1, 1)$, $ba = (13)(24)(c, 1, 1, a)$ for $\mathfrak{M}_1$.

The biset $[S] \otimes \mathfrak{M}_0$ is then given by

$$\begin{aligned} a &= (1, a, c, 1), \\ b &= (13)(24)(b, 1, b^{-1}, 1), \\ c &= (14)(b, 1, 1, 1). \end{aligned}$$

Conjugating the right-hand side by $(b^{-1}, 1, 1, 1)$, we get $\mathfrak{M}_1$.

The biset $[S] \otimes \mathfrak{M}_1$ is given by

$$\begin{aligned} a &= (1, a, c, 1), \\ b &= (12)(34)(b^{-1}, b, 1, 1), \\ c &= (23)(1, b, 1, 1). \end{aligned}$$

Conjugating by $(b, 1, 1, 1)$, we get that $[S] \otimes \mathfrak{M}_1$ is isomorphic to $\mathfrak{M}_0$.

The biset $[T] \otimes \mathfrak{M}_0$ is given by

$$\begin{aligned} a &= (a, 1, 1, c), \\ b &= (12)(34)(a, a^{-1}, c, c^{-1}), \\ c &= (23)(1, b, 1, 1). \end{aligned}$$

Conjugating by $(a^{-1}, 1, 1, c)(1342)$ we get $\mathfrak{M}_1$.

Similarly, $[T] \otimes \mathfrak{M}_1$ is given by

$$\begin{aligned} a &= (c, 1, 1, a), \\ b &= (13)(24)(c, a^{-1}, c^{-1}, a), \\ c &= (14)(1, 1, 1, b). \end{aligned}$$

Conjugation of the right-hand side by $(1, a, c, 1)(1243)$ produces the wreath recursion associated with $\mathfrak{M}_0$.

We see that the subgroup G_1 of the liftable elements of the mapping class group has index 2 and is mapped to the trivial element by the virtual endomorphism. This agrees with the fact that the lifting map σ_f is constant.

Example 4.5.17. Computations become much more complicated for higher degree and for post-critical sets of larger size. The iterated monodromy group of the map $F(p_1, p_2) = \left(1 - \frac{p_2^2}{p_1^2}, 1 - \frac{1}{p_1^2}\right)$, which equal to the moduli space map associated with a quadratic polynomial whose critical point belongs to a cycle of length 4, was computed by J. Belk and S. Koch in [BK08...]. It is generated by the wreath recursion

$$\begin{aligned} a &= (b, 1, 1, b), & d &= (12)(34)(1, a, 1, a), \\ b &= (c, c, 1, 1), & e &= (f, 1, f, 1), \\ c &= (14)(23)(d, d_y, d_x, 1), \\ f &= (13)(24)(b^{-1}, 1, eb, e), \end{aligned}$$

where $d_x = (fa)^{-1}$ and $d_y = (ceb)^{-1}$.

Even though the computations often become too complicated to do them by hand, they can be efficiently implemented on computer, see the papers ... and the computer packages...

Solution of the "twisted rabbit" problem... The problem is to decide for a given Thurston map f and a homeomorphisms h_1, h_2 of $S^2 \setminus P_f$ when the Thurston maps $h_1 \circ f$ and $h_2 \circ f$ are combinatorially equivalent.

It follows from Theorem 4.5.4 that $h_1 \circ f$ and $h_2 \circ f$ are combinatorially equivalent if and only if there exists an element h of the pure mapping class group such that the bisets $[h^{-1}h_1] \otimes \mathfrak{M}_f \otimes [h]$ and $[h_1] \otimes \mathfrak{M}_f$ are isomorphic. Thus, the question of combinatorial equivalence of compositions

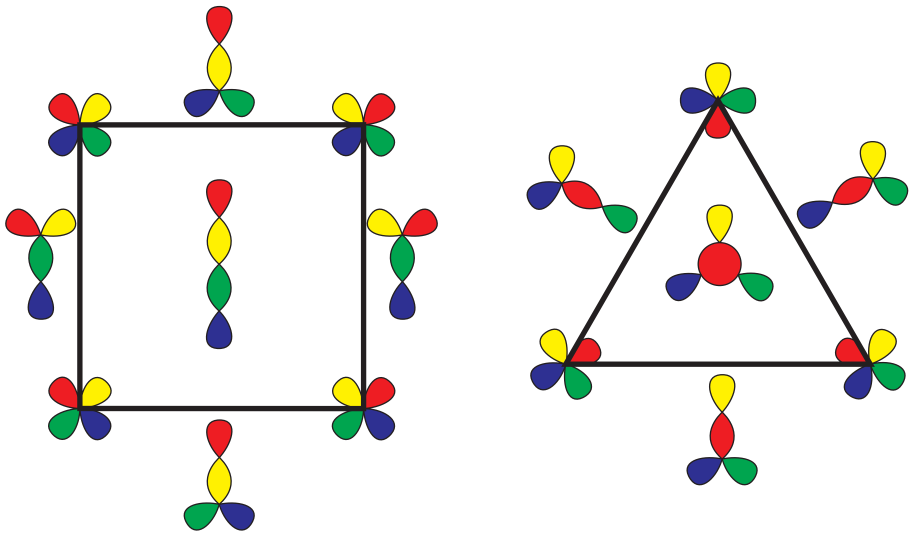

Figure 4.18.

of f with homeomorphisms is equivalent to the conjugacy in the biset over the mapping class group..., i.e., to the question when for given two elements x_1, x_2 of the biset ... there exists h such that $x_2 = h^{-1} \cdot x_1 \cdot h$.

Note that if $h_1 \cdot x = x \cdot h_2$, then $h_1 \cdot x$ and $h_2 \cdot x$ are conjugate (since we have $h_2 \cdot x = h_2 \cdot (x \cdot h_2) \cdot h_2^{-1}$). Suppose that $\{x = x_1, x_2, \ldots, x_m\}$ is a basis of the biset ..., and suppose that $f_1, f_2, \ldots, f_m$ are such that $x = f_i \cdot x_i$ for every i (we may assume that $f_1 = 1$). We have $g \cdot x = x_i \cdot g|_x$ for some x_i. Then we have $g \cdot x = f_i^{-1} \cdot x \cdot g|_x$, hence $g \cdot x$ is combinatorially equivalent to $g|_x f_i^{-1}$. We have a map $\Delta : g \mapsto g|_x f_i^{-1}$, where i is such that $g \cdot x = x_i \cdot g|_x$. If the biset ... is hyperbolic, then Δ is contracting the length of g, and there exists a finite subset $A \subset G$ such that for every $g \in G$ there exists $n \geqslant 0$ such that $\Delta^n(g) \in A$. Then the problem of classifying the elements of ... up to combinatorial equivalence is reduced to classification of the elements of $A \otimes \mathfrak{M}_f$. Even if ... is not hyperbolic, it is often possible to understand the dynamics of the map Δ on G and reduce the problem to a manageable part of ...

Example 4.5.18. The twisted rabbit problem...

Example 4.5.19. $z^2 + i$...

Combinatorial models for hyperbolic polynomials...

4.5.6. Iterated monodromy groups of skew product maps. Recall the definition of the skew product map...

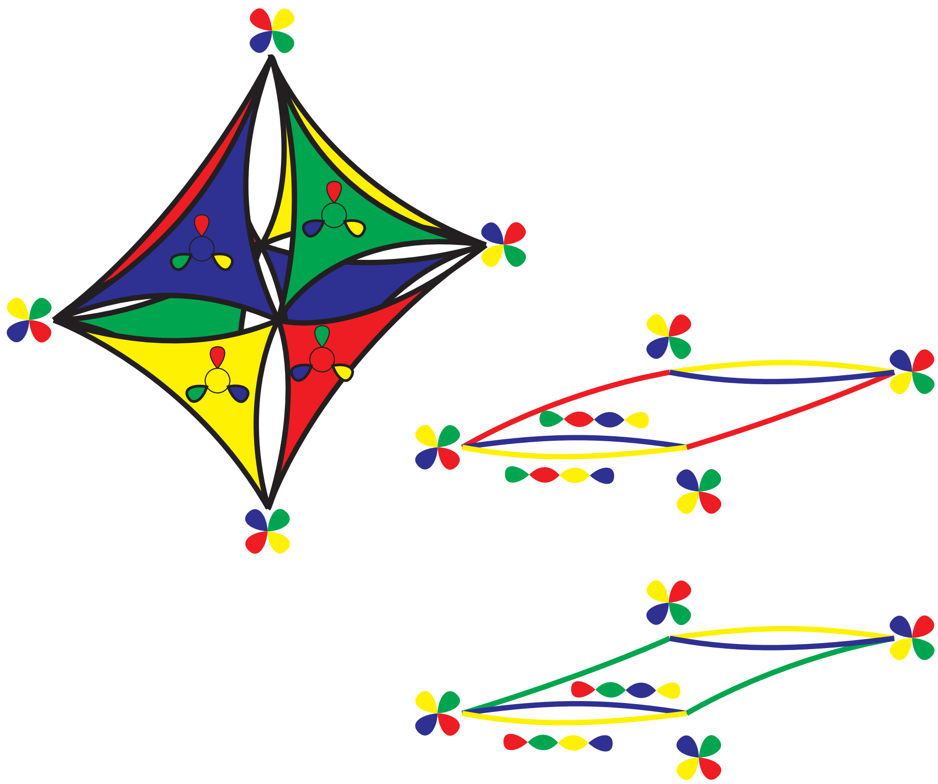

Figure 4.19.

The skew product structure of the map makes it possible to draw the z-slices... They are the Julia sets of *forward iterations* ...

For example, see Figure 4.20, where the z-slices of the Julia set of $F(z, p) = \left(1 - \frac{z^2}{p^2}, 1 - \frac{1}{p^2}\right)$ are shown. The figure shows the Julia set of the second coordinate $1 - \frac{1}{p^2}$ (of the moduli space map) and the slices of the Julia set for the corresponding values of p.

The iterated monodromy groups of the skew product correspondences can be computed in a way similar to the computation of the iterated monodromy groups of moduli space correspondences....

Let $\mathfrak{M}$ be the $\pi_1(S^2 \backslash P_f)$-biset associated with the Thurston map f. The fundamental group of the space ... of the skew product correspondence is the semidirect product $\pi_1(S^2 \backslash P_f) \ltimes G_{P_f}$. It acts faithfully on $\pi_1(S^2 \backslash P_f)$...

Let $\{\mathfrak{M}_0, \mathfrak{M}_1, \dots, \mathfrak{M}_{m-1}\}$ be a basis of the biset $\mathfrak{T}$ associated with the moduli space correspondence. Each of $\mathfrak{M}_i$ is an (isomorphism class) of a $\pi_1(S^2 \backslash P_f)$-biset of the form $[h_1] \otimes \mathfrak{M} \otimes [h_2]$ for $h_1, h_2 \in G_{P_f}$. Consider the $\pi_1(S^2 \backslash P_f)$-biset $\mathfrak{M}_0 \otimes \mathfrak{M}_1 \otimes \cdots \otimes \mathfrak{M}_{m-1}$ (where, as always, $\oplus$ denotes the

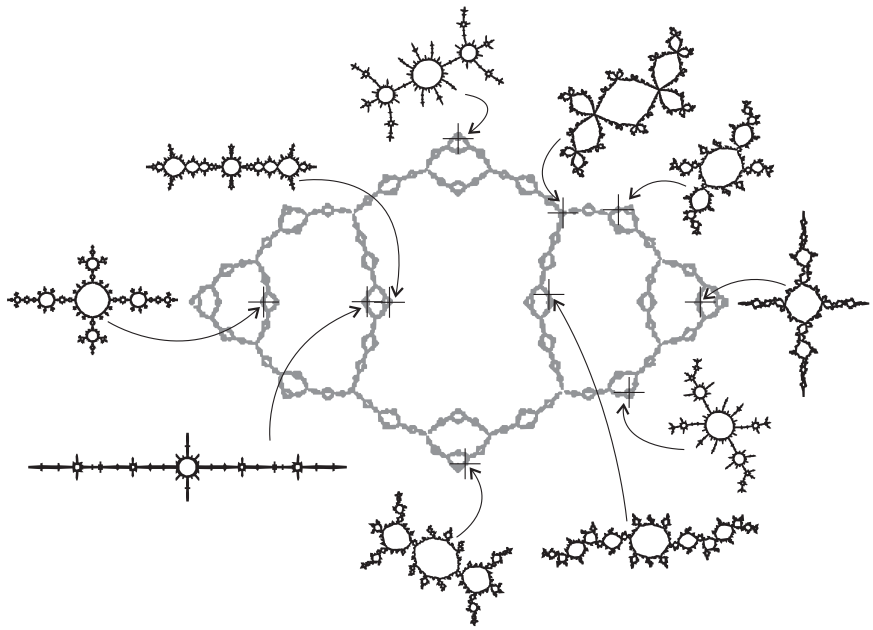

Figure 4.20.

disjoint union of bisets), and choose bases X_i of $\mathfrak{M}_i$. Let h be an element of the mapping class group G_{P_f}. Then for every $i = 0, 1, \ldots, m-1$, the biset $[h]\otimes\mathfrak{M}_i$ is isomorphic to a biset of the form $\mathfrak{M}_j\otimes[h_i]$ for some $h_i \in G_{P_f}$. The set $h\otimes\mathsf{X}_i = \{h\otimes x \ : x \in \mathsf{X}_i\}$ is a basis of $[h]\otimes\mathfrak{M}_i$, while $\{x\otimes h_i \ : \ x \in \mathsf{X}_j\}$ is a basis of $\mathfrak{M}_j \otimes [h_i]$. The wreath recursion associated with the basis $h \otimes \mathsf{X}_i$ is $g \mapsto \Psi_i(g^h)$, where Ψ_i is the wreath recursion associated with $\mathfrak{M}_i$ and X_i. The wreath recursion associated with $\mathsf{X}_j \otimes h_i$ is $g \mapsto (\Psi_j(g))^{h_i}$. Since two bisets are isomorphic, there exists an element $t_{h,i} \in \mathsf{S}_{\mathsf{X}_j} \rtimes \pi_1(S^2\backslash P_f)$ such that

$$\Psi_i(g^h) = (\Psi_j(g))^{h_i t_{h,i}}$$

for all $g \in \pi_1(S^2\backslash P_f)$. ...

Example 4.5.20. Consider again the case of the rabbit polynomial from Example 4.5.14. The biset $\mathfrak{M}_0 \oplus \mathfrak{M}_1$ is given by the recursion

$$a = (12)(34)(1, c, 1, c), \quad b = (1, a, a, 1), \quad c = (1, b, 1, b).$$

The computation in Example 4.5.14 shows the following relations:

$$[S] \otimes \mathfrak{M}_1 = \mathfrak{M}_1 \otimes [T], \quad [S] \otimes \mathfrak{M}_2 = \mathfrak{M}_2,$$

which implies

$$S = (T, T, 1, 1).$$

We also have

$$[T]\otimes\mathfrak{M}_0=\mathfrak{M}_1\cdot(a,1),\quad [T]\otimes\mathfrak{M}_1=\mathfrak{M}_0\otimes[R]\cdot(c^{-1}a^{-1}c,1),$$

where $R=T^{-1}S^{-1}cbc^{-1}ac$. It follows that

$$T=(13)(24)(a,1,T^{-1}S^{-1}cb,T^{-1}S^{-1}cbc^{-1}ac).$$

Consequently, the iterated monodromy group of the skew product map $(z,p)\mapsto\left(1-\frac{z^2}{p^2},1-\frac{1}{p^2}\right)$ is given by the wreath recursion

$$\begin{aligned}
a&=\sigma(1,c,1,c),\\
b&=(1,a,a,1),\\
c&=(1,b,1,b),\\
S&=(T,T,1,1),\\
T&=\pi(a,1,T^{-1}S^{-1}cb,T^{-1}S^{-1}cbc^{-1}ac),
\end{aligned}$$

where $\sigma=(12)(34)$ and $\pi=(13)(24)$.

Example 4.5.21. Consider the mating of z^2-1 with itself. It is generating by two copies of the iterated monodromy group $a=\sigma(a^{-1},ba), b=(1,a)$ of z^2-1. We get the wreath recursion

$$\begin{aligned}
a_1&=\sigma(a_1^{-1},b_1a_1), & a_2&=\sigma(a_2^{-1},b_2a_2),\\
b_1&=(1,a_1), & b_2&=(1,a_2).
\end{aligned}$$

We impose the relation $b_1a_1=b_2a_2$, so that the above wreath recursion is considered to be on a free group of rank 3 generated by a_1,b_1,b_2. We have $a_2=b_2^{-1}b_1a_1$.

We get then the recursion $\mathfrak{M}_0$ over the free group generated by a_1,b_1,b_2:

$$\begin{aligned}
a_1&=\sigma(a_1^{-1},b_1a_1),\\
b_1&=(1,a_1),\\
b_2&=(1,b_2^{-1}b_1a_1).
\end{aligned}$$

Consider also another recursion, denoted $\mathfrak{M}_1$:

$$\begin{aligned}
a_1&=\sigma(1,a_1^{-1}b_1a_1),\\
b_1&=(a_1,1),\\
b_2&=(1,b_2^{-1}b_1a_1).
\end{aligned}$$

Consider the following Dehn twists:

$$a_1^T=a_1^{b_1a_1},\quad b_1^T=b_1^{a_1},\quad b_2^T=b_2,$$

and

$$a_1^D=a_1^{b_1^{-1}b_2},\quad b_1^D=b_1,\quad b_2^D=b_2.$$

Direct computations show then the following relations:

$$[T] \otimes \mathfrak{M}_0 = \mathfrak{M}_1, \qquad [T] \otimes \mathfrak{M}_1 = \mathfrak{M}_0 \otimes [T]$$

and

$$[D] \otimes \mathfrak{M}_1 = \mathfrak{M}_1 \otimes [D] \cdot (1, b_2^{-1}b_1), \qquad [D] \otimes \mathfrak{M}_2 = \mathfrak{M}_2 \cdot (a_1^{-1}, b_2^{-1}b_1a_1).$$

It follows that the iterated monodromy group of the corresponding skew product $(z,p) \mapsto \left(\frac{z^2-p^2}{z^2-1}, p^2\right)$ is given by the recursion

$$\begin{aligned} a_1 &= \sigma(a_1^{-1}, b_1a_1, 1, a_1^{-1}b_1a_1), \\ b_1 &= (1, a_1, a_1, 1), \\ a_2 &= \sigma(a_2^{-1}, b_2a_2, a_2^{-1}, b_2a_2), \\ b_2 &= (1, a_2, 1, a_2), \\ T &= \pi(1, 1, T, T), \\ D &= (D, Db_2^{-1}b_1, a_1^{-1}, a_2), \end{aligned}$$

where $\sigma = (12)(34)$ and $\pi = (13)(24)$.

The skew product structure of the map $(z,\tau) \mapsto (f_\tau(z), F(\tau))$ is reflected in the structure of the iterated monodromy group. Namely, as the map does not depend on z in the second coordinate, we get a semiconjugacy from the skew product map to the moduli space map F. This semiconjugacy induces a natural semiconjugacy from the iterated monodromy group of the skew product to the iterated monodromy group of the moduli space map acting on the corresponding trees.

For instance, if we take the group $G = \langle a, b, c, S, T\rangle$ from Example 4.5.20, then the map $\{1,2,3,4\}^* \longrightarrow \{0,1\}^*$ is generated by $1 \mapsto 0, 2 \mapsto 0, 3 \mapsto 1, 4 \mapsto 1$. This maps agrees with the wreath recursion so that it induces the natural epimorphism from the group G to the group $\mathrm{IMG}\left(1 - 1/p^2\right)$ generated by

$$S = (T, 1), \qquad T = (01)(1, T^{-1}S^{-1}).$$

The group $G_0 = \langle a, b, c\rangle$ generated by the loops in the z-plane belongs to the kernel of the epimorphism. (In fact, one can prove that it is equal to the kernel.)

In general we have the map .... and the epimorphism...

The group $G_0 = \langle a, b, c\rangle$ is self-similar, but not level-transitive. The quotient of the tree $\{1,2,3,4\}$ is the tree $\{0,1\}^*$ on which $\mathrm{IMG}\left(1 - 1/p^2\right)$ (i.e., the quotient G/G_0) acts. The group G_0 is the faithful quotient of the free group for the biset $\mathfrak{M}_0 \oplus \mathfrak{M}_1$.

The preimage of a path in the quotient tree $\{1,2,3,4\}^*/G_0$ is a binary G_0-invariant subtree on which G_0 acts level-transitively. The action is naturally identified with the left action on the space of right orbits of the biset $\oplus_{n=0}^{\infty}\mathfrak{M}_{i_1}\otimes\mathfrak{M}_{i_2}\otimes\cdots\otimes\mathfrak{M}_{i_n}$ for the sequence $w=(i_1,i_2,\ldots)$ describing the path in the quotient tree. (make the indexing nice...) The action is not faithful. Let us denote by ... Description of the family...

Example 4.5.22. Let $f(z) = z^2+i$. The corresponding skew product map is conjugate to $F(z,p) = \left(\left(1-\frac{2z}{p}\right)^2, \left(1-\frac{2}{p}\right)^2\right)$, see ... Its iterated monodromy group is generated by

$$\begin{aligned} a &= (12)(34), & R &= (13)(24)(1,b,1,b),\\ b &= (a,c,a,c^b), & S &= (T,T,S,S),\\ c &= (b,1,1,b), \end{aligned}$$

where $T = babcbS^{-1}R^{-1}$, see Exercise 4.17.

Consider a bigger group ...

Example 4.5.23. Let $m = \sum_{i=0}^{\infty} k_i 2^i$ for an infinite sequence $k_0k_1\ldots \in \{0,1\}^\omega$ of zeros and ones be a dyadic integer. Then τ^m is well defined and is given by the wreath recursion $\tau^m = (\tau^{m/2}, \tau^{m/2})$ if m is even (i.e., if $k_0=0$) and $\tau^m = \sigma(\tau^{(m-1)/2}, \tau^{(m+1)/2})$ if m_w is odd (if $k_0=1$).

Let

$$a = \sigma(a^{-1}, ba), \quad b = (1,a)$$

be the generators of basilica, and consider the family of groups

$$G_m = \langle a, b, a^{\tau^m}, b^{\tau^m}\rangle.$$

Informally, the group G_m is obtained by taking two copies of the basilica group IMG $\left(z^2-1\right)$ and "rotating" one of them by a power of the adding machine (i.e., the loop around infinity).

If m is even, then we have

$$a^{\tau^m} = \sigma(a^{-\tau^{m/2}}, (ba)^{\tau^{m/2}}), \quad b^{\tau^m} = (1, a^{\tau^{m/2}}).$$

If m is odd, then

$$a^{\tau^m} = \sigma(1, (a^{-1}ba)^{\tau^{(m-1)/2}}), \quad b^{\tau^m} = (a^{\tau^{(m-1)/2}}, 1).$$

Note that $m/2$ if m is even and $(m-1)/2$ if m is odd is the dyadic integer $\sum_{i=0}^{\infty} k_{i+1}2^i$, i.e., the integer corresponding to the shift $k_1k_2\ldots$ of the sequence $k_0k_1\ldots$. We see that the groups G_m form a family with the universal group equal to the subgroup $\langle a_1, b_1, a_2, b_2\rangle$ of the group from Example 4.5.21.

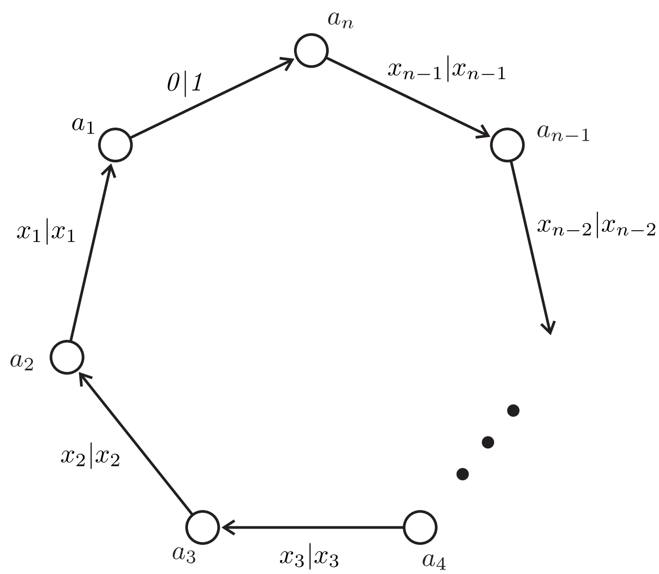

Figure 4.21. Automata generating IMG (f) for hyperbolic quadratic polynomials

4.6. Iterations of polynomials

4.6.1. Iterated monodromy groups of sequences of polynomials. Let $a_1, a_2, \dots, a_k$ be a sequence of elements of the symmetric group $\mathsf{S}(\mathsf{X})$ for a finite set X. Consider the following oriented CW-complex. Its set of vertices is X; for every permutation a_i and for every cycle $(x_1, x_2, \dots, x_n)$ of a_i we have a cell with vertices $x_1, x_2, \dots, x_n$ going around the cell in the given order, according to the orientation. Different cells do not have common edges. We will call this complex the *cycle diagram* of the sequence a_i.

Definition 4.6.1. A sequence $a_1, a_2, \dots, a_k \in \mathsf{S}(\mathsf{X})$ is *dendroid* if its cycle diagram is contractible.

For example (12)....

Polynomial iterations, relation with dendroid automorphisms of a rooted tree... Examples of self-similar families... random compositions of z^2 and $1 - z^2$... coming from the skew product maps...

4.6.2. External rays and iterated monodromy groups of polynomials. The case of one polynomial, kneading automata,

4.6.3. Quadratic polynomials. Quadratic polynomials, their symbolic dynamics...

For example, for $\mathfrak{K}_0$ is the Basilica group from Example... The groups $\mathfrak{K}_{00}$ and $\mathfrak{K}_{11}$ are the iterated monodromy groups of the "Rabbit" and "Airplane" quadratic polynomials, respectively...

4.7. Functoriality

4.7.1. General discussion. Maps between bisets and the induced semi-conjugacies of the limit dynamical systems... When it is onto, when it is one-to-one... ....

Let $G_i \curvearrowright X_i$ be actions of groups. We say that a bijection $f : X_1 \longrightarrow X_2$ is a *bounded orbit equivalence* if for every $g \in G_1$ there exists a finite set $A_g \subset G_2$ such that for every $x \in X_1$ and $g \in G_1$ there exists $h \in A_g$ such that $f(g(x)) = h(f(x))$, and if also for every $h \in G_2$ there exists a finite set $B_h \subset G_1$ such that for every $x \in X_2$ and $h \in G_2$ there exists $g \in B_h$ such that $f^{-1}(h(x)) = g(f^{-1}(x))$. See ...

Proposition 4.7.1. *Let $G_i \curvearrowright \mathsf{X}^\omega$ be contracting self-similar group actions. If the identity map $\mathsf{X}^\omega \longrightarrow \mathsf{X}^\omega$ is a bounded orbit equivalence (equivalently, if the identity map $\mathsf{X}^* \longrightarrow \mathsf{X}^*$ is a bounded orbit equivalence) then the identity map $\mathsf{X}^{-\omega} \longrightarrow \mathsf{X}^{-\omega}$ induces a conjugacy of the limit dynamical systems $\mathsf{s} \circlearrowleft \mathcal{J}_{G_1}$ and $\mathsf{s} \circlearrowleft \mathcal{J}_{G_2}$.*

Proof. .... □

4.7.2. Plane filling curves. Let $\mathsf{X} = \{\mathsf{0}, \mathsf{1}, \mathsf{2}\}$. The group generated by the recursion

$$k = (\mathsf{02})(k, k, k), b = (\mathsf{012})(1, 1, b).$$

It is checked directly that if we naturally identify X^ω with the set of 3-adic integers (identifying $\mathsf{0}, \mathsf{1}, \mathsf{2}$ with the digits $0, 1, 2$), then k acts on $\mathbb{Z}_3$ by $x \mapsto -x$, and b acts by $x \mapsto x+1$. It follows that the limit dynamical system of the group $\langle k, b\rangle$ is the map induced by $x \mapsto 3x$ on the orbifold of the action of $\mathbb{R}$ of the infinite dihedral group generated by the transformations $x \mapsto -x$ and $x \mapsto x + 1$ of $\mathbb{R}$. (It topologically conjugate to the action of the Chebyshev polynomial T_3 on $[-1, 1]$.)

Consider the direct square of the action, i.e., the action of the direct square of $\langle k, b\rangle$ on $\mathsf{X}^\omega \times \mathsf{X}^\omega$. Let us identify $\mathsf{X}^\omega \times \mathsf{X}^\omega$ with X^ω by the map

$$(x_0x_1 \ldots, y_0y_1 \ldots) \mapsto x_0y_0x_1y_1 \ldots.$$

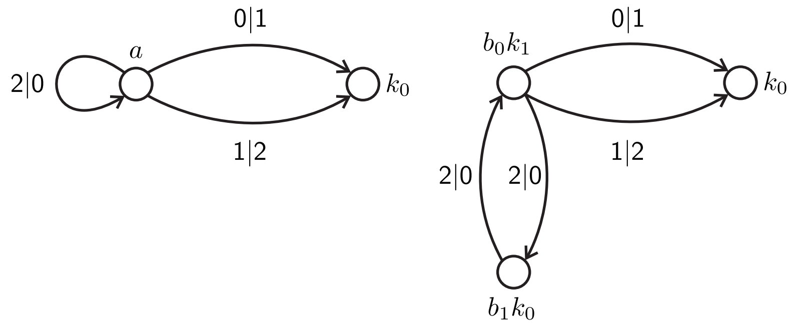

Figure 4.22. Automaton for the Peano curve

Then the action of $\langle k, b\rangle^2$ is also self-similar and generated by the recursion

$$\begin{aligned} b_0 &= (012)(1, 1, b_1), \\ b_1 &= (b_0, b_0, b_0), \\ k_0 &= (02)(k_1, k_1, k_1), \\ k_1 &= (k_0, k_0, k_0). \end{aligned}$$

The limit dynamical system of $H = \langle k_0, k_1, b_0, b_1\rangle$ is then the map induced by $(x, y) \mapsto (y, 3x)$ on the quotient of $\mathbb{R}^2$ by the group of the affine transformations of the form $(x, y) \mapsto ((-1)^{k_0}x + n_0, (-1)^{k_1}y + n_1)$ for $k_0, k_1, n_0, n_1 \in \mathbb{Z}$.

Let us through in a new element

$$a = (012)(k_0, k_0, a),$$

and let $G = \langle H, a\rangle$. Note that we have

$$\begin{aligned} b_0k_1 &= (012)(k_0, k_0, b_1k_0), \\ b_1k_0 &= (02)(b_0k_1, b_0k_1, b_0k_1), \\ a &= (012)(k_0, k_0, a). \end{aligned}$$

If $k \geqslant 0$ is the number of leading digits 2 of a word v, then it follows from the above recursions that $a(v) = b_0k_1(v)$, if k is even, and $a(v) = b_1k_0(v)$, if k is odd, see Figure 4.22.

It follows that the limit dynamical systems of G and H are topologically conjugate (moreover, the identity map on $\mathsf{X}^{-\omega}$ induces the topological conjugacy).

If we conjugate the right-hand side of the wreath recursion by $(1, k_0, 1)$, then we get

$$a = (012)(1, 1, a),$$

hence $\langle a\rangle$ is the usual 3-adic odometer. Its limit space is the circle $\mathbb{R}/\mathbb{Z}$. Note that it follows from ... that the corresponding map between the bisets induces a surjective semiconjugacy from the limit dynamical system of the

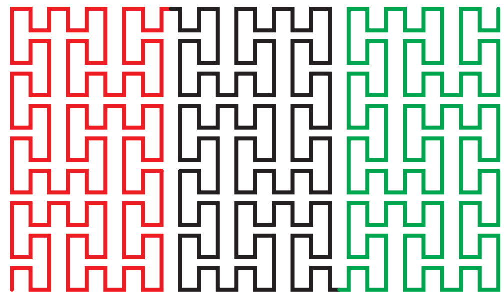

Figure 4.23. Peano curve

odometer onto the limit dynamical system of G (equivalently, of G). This map is precisely the classical Peano curve (cite...). Its approximation using orbital graphs of the action on X^n for $n = ...$ is shown on Figure 4.23.

We have "unwrapped" the limit space and the orbital graph, i.e., have drawn a tile diagram....

It is interesting that our description is very close to the original description of the Peano curve given in ... It is described there using 3-adic reals and using symbolic transformations. It remained only to translate it into the language of self-similar groups in a very straightforward fashion.

Some other classical plane filling curves can be naturally described using self-similar groups. For example, the Sierpinski curve is associated with the following group acting on $\{0, 1\}^*$.

$$\begin{aligned} a &= \sigma, \\ b &= (a, c), \\ c &= (b, b), \\ x &= (a, x). \end{aligned}$$

We have seen in ... that $\langle a, b, c\rangle$ is virtually abelian, and that its limit dynamical system is folding of a right isosceles triangle. We also know ... that the limit dynamical system of the group $\langle a, x\rangle$ is the tent map acting on the segment. Check that if v starts with an even number of 1s then $x(v) = b(v)$, otherwise $x(v) = c(v)$. It follows that the limit dynamical system of $\langle a, b, c, x\rangle$ is the same as of $\langle a, b, c\rangle$. The embedding $\langle a, x\rangle < \langle a, b, c, x\rangle$ induces a surjective map from the segment to the isosceles right triangle. Its approximation using the graphs of actions is shown on Figure 4.24.

4.7.2.1. *A surjective map from the Julia set of $z^2 + i$ to a triangle.* An example somewhat analogous to the Sierpiński curve is a surjective map from the Julia set of $z^2 + i$ to the triangle, defined in the following way.

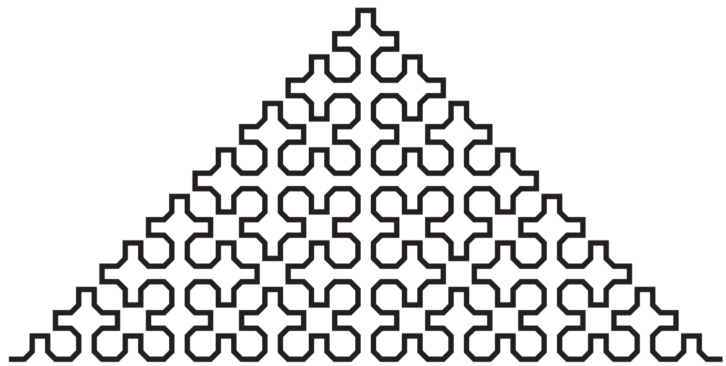

Figure 4.24. Sierpiński curve

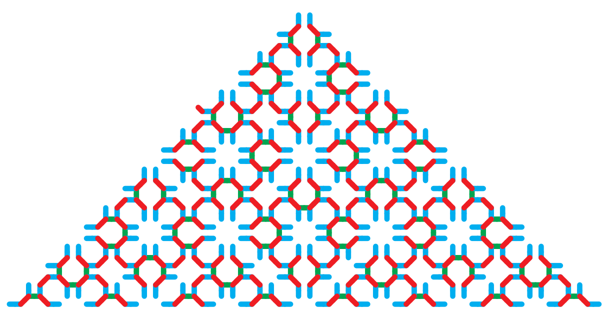

Figure 4.25. A surjection from the Julia set of $z^2 + i$ to the triangle

Consider the iterated monodromy group of $z^2 + i$:

$$a = \sigma, \quad b = (a, c), \quad c = (b, 1).$$

It is easy to see that the graphs of the action of this group on the levels of the tree are subgraphs of the graphs of the action of

$$a = \sigma, \quad b = (a, c), \quad c = (b, b),$$

which, as we have seen ..., are “triangles”, see... The inclusion of the graphs defines in the limit a semiconjugacy of the action of $z^2 + i$ on its Julia set with the triangle folding map. See Figure 4.25.

4.7.3. Mating and tuning. Definition of mating...

As an example, consider the mating of $z^2 + i$ with itself. The iterated monodromy IMG $\left(z^2 + i\right)$ is, in terms of 4.6.3, the group $G_{1/6}$, so it is given by the wreath recursion

$$\begin{aligned}
\tau_{1/6} &= \sigma(\tau_{1/3}\tau_{1/6}, \tau_{1/6}\tau_{1/3}),\\
\tau_{1/3} &= (\tau_{1/6}, \tau_{2/3}),\\
\tau_{2/3} &= (\tau_{1/3}, 1),
\end{aligned}$$

where $\tau = \tau_{1/6}\tau_{1/3}\tau_{2/3} = \sigma(1, \tau)$ is the adding machine.

The other copy of IMG $\left(z^2 + i\right)$ can be considered as the iterated monodromy group of the complex conjugate polynomial IMG $\left(z^2 - i\right)$, i.e., the group $G_{5/6}$. Its recursion is

$$\begin{aligned}
\delta_{5/6} &= \sigma(\delta_{5/6}\delta_{2/3}, \delta_{2/3}\delta_{5/6}),\\
\delta_{2/3} &= (\delta_{1/3}, \delta_{5/6}),\\
\delta_{1/3} &= (1, \delta_{2/3}).
\end{aligned}$$

Note that $\delta_{1/3}\delta_{2/3}\delta_{5/6} = \tau$.

The group $G = \langle \tau_{1/6}, \tau_{1/3}, \tau_{2/3}, \delta_{1/3}, \delta_{2/3}\rangle$ is the iterated monodromy group of the formal mating of $z^2 + i$ with itself. We have removed $\delta_{5/6}$ from the generating set, since $\delta_{5/6} = \delta_{2/3}\delta_{1/3}\tau_{1/6}\tau_{1/3}\tau_{2/3}$.

Conjugation of the right-hand side by $(1, \tau_{1/3}\tau_{1/6})$ produces the following recursion for G

$$\begin{aligned}
\tau_{1/6} &= \sigma,\\
\tau_{1/3} &= (\tau_{1/6}, \tau_{2/3}^{\tau_{1/3}\tau_{1/6}}),\\
\tau_{2/3} &= (\tau_{1/3}, 1),\\
\delta_{2/3} &= (\delta_{1/3}, \delta_{5/6}^{\tau_{1/3}\tau_{1/6}}),\\
\delta_{1/3} &= (1, \delta_{2/3}^{\tau_{1/3}\tau_{1/6}}).
\end{aligned}$$

We have $\delta_{2/3}^{\tau_{1/3}\tau_{1/6}} = (\delta_{5/6}^{\tau_{2/3}\tau_{1/3}\tau_{1/6}}, \delta_{1/3}^{\tau_{1/6}})$ and $\delta_{1/3}^{\tau_{1/6}} = (\delta_{2/3}^{\tau_{1/3}\tau_{1/6}}, 1)$.

Let us rename $a_1 = \tau_{1/6}$, $b_1 = \tau_{1/3}^{\tau_{1/6}}$, $c_1 = \tau_{2/3}^{\tau_{1/3}\tau_{1/6}}$, $a_2 = \delta_{5/6}^{\tau_{2/3}\tau_{1/3}\tau_{1/6}} = \tau_{1/6}\tau_{1/3}\tau_{2/3}\delta_{2/3}\delta_{1/3}$, $b_2 = \delta_{2/3}^{\tau_{1/3}\tau_{1/6}}$, and $c_2 = \delta_{1/3}^{\tau_{1/6}}$. Then we have $a_2 = c_1b_2b_1c_2a_1$, and the new recursion for G is

$$\begin{aligned}
a_1 &= \sigma,\\
b_1 &= (c_1, a_1),\\
c_1 &= (1, b_1),\\
b_2 &= (a_2, c_2),\\
c_2 &= (b_2, 1).
\end{aligned}$$

Figure 4.26. Mating of $z^2 + i$ with itself

Consider the subgroup $H = \langle a_1, b_1c_2, c_1b_2\rangle$ of G. Denote $b_1c_2 = B$ and $c_1b_2 = C$. Then it satisfies the recursion

$$a_1 = \sigma, \qquad B = (C, a_1), \qquad C = (CBa_1, B).$$

This is a virtually abelian group of affine transformations...

A similar example is given by the mating of the polynomial $f_{1/4}$ with itself....

Paper-folding family, rotated matings...

Other examples of Lattes matings: 5/12+1/12, 1/6+5/14, 1/4+1/4 (Milnor's example)

4.7.4. More examples. Show how to understand the topology of a Julia set...

The limit spaces of families of groups: rabbit and airplane family, rotated basilicas family...

Skew product examples, their topology from the iterated monodromy group...

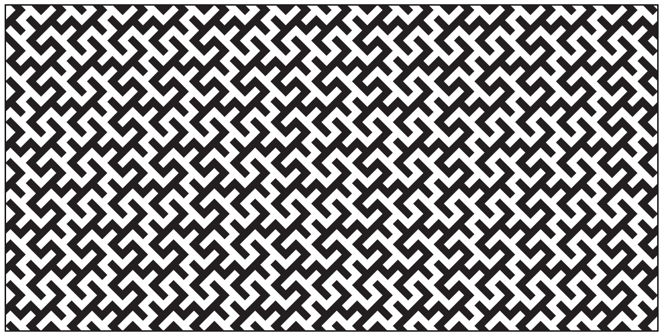

Figure 4.27. Paper folding curve

The 2D moduli space map, the tori in the Julia set...

Exercises

4.1. Compute the iterated monodromy groups (as self-similar groups) of the following rational functions:
(a) $z^2 - 2$;
(b) $z^2 + i$;
(c) $z^2 + c$ for every c such that 0 belongs to a cycle of length 3: $0 \mapsto c \mapsto c^2 + c \mapsto 0$.
(d) $z^2 - \frac{16}{27z}$;
(e) $\left(\frac{2-z}{z}\right)^2$.
(f) Chebyshev polynomials.

4.2. Let $c \approx 0.2956$ be the real root of the polynomial x^3+x^2+3x-1. Prove that the iterated monodromy group of $\frac{z^2-c}{z^2+c}$ is generated by the wreath recursion

$$a = \sigma(1, b), \quad b = (c, 1),$$
$$c = (d, a), \quad d = \sigma(a, a^{-1}),$$

where $d = a^{-1}b^{-1}c^{-1}$. The Julia set of this rational function is shown on Figure 1.31.

4.3. Let c be one of the complex roots $\approx -0.6478 \pm 1.7214i$ of the polynomial $x^3 + x^2 + 3x - 1$. Prove that the iterated monodromy group of $\frac{z^2-c}{z^2+c}$ is generated by the wreath recursion

$$a = \sigma(c^{-1}a^{-1}, d), \quad b = (1, c),$$
$$c = (a, d), \quad d = \sigma,$$

where $d = bac = c^{-1}a^{-1}b^{-1}$.

4.4. Consider the standard action $a = \sigma(1, a), b = (b, 1)$ of the iterated monodromy group of the Chebyshev polynomial T_2. Prove that if we start from $v = 00\ldots0 \in \mathsf{X}^n$ and apply the generators a, b, we get a Hamiltonian path $v, a(v), ba(v), aba(v), \ldots$ passing through every vertex of the n-dimensional cube $\{0, 1\}^n$. It is called the *Gray code*, see...

4.5. Show that the iterated monodromy group of any uniformization of the tent map is equivalent to one of the groups described in 4.3.4. In other words, by considering only graphs of two groups connected by an edge instead of considering all possible orbispaces with the underlying space a segment we did not change the set of the iterated monodromy groups.

4.6. Prove that the map $f_n : \mathcal{X}_{n+1} \longrightarrow \mathcal{X}_n$ defined in ... is a covering.

4.7. Prove that the group generated by

$$a = \sigma(b, b), \quad b = (c, a), \quad c = (a, a)$$

contains a finite index subgroup isomorphic to $\mathbb{Z}^3$.

4.8. Prove that the group generated by

$$a = \sigma(b, b), \quad b = (c, c), \quad c = (c, a)$$

contains a finite index subgroup isomorphic to $\mathbb{Z}^5$.

4.9. Show (using the wreath recursion) that the group generated by

$$\alpha = \sigma, \quad \beta = (\alpha, \alpha), \quad \gamma = (\delta, \beta), \quad \delta = (\delta, \gamma)$$

contains an index two subgroup equivalent to the iterated monodromy group from **??**.

4.10. Let $\phi : \mathbb{Z}^n \dashrightarrow \mathbb{Z}^n$ be a virtual endomorphism, and let A be its matrix as a linear operator on $\mathbb{Q}^n$. Show that the associated self-similar action is faithful if and only if no eigenvalue of A is an algebraic integer.

4.11. Describe, up to equivalence of self-similar groups, all self-replicating contracting actions of $\mathbb{Z}^2$ on the binary rooted tree.

4.12. Prove that if $|c| > 4$, then the restriction of $f_c(z) = 1 + \frac{c}{z^2}$ to its Julia set is topologically conjugate to the action of z^{-2} on the unit circle. (Hint: Show that f_c is hyperbolic, compute its iterated monodromy group, and then use...)

4.13. Prove that the iterated monodromy group of $1 + \frac{c_n}{z^2}$, where c_n is defined in Problem..., is equivalent as a self-similar group to G_{n+3} from ...

4.14. Consider a degree 4 Thurston map f with one totally invariant point x, and three simple critical points a_1, a_2, a_3 such that $f(a_1) = f(a_2) = a_3$, and $f^2(a_3)$ is a fixed point, so that we have four post-critical points

$a_3, f(a_3), f^2(a_3)$, and x. Let us, as before, choose three values of post-critical points: $x = \infty$, $a_3 = 0$, and $f(a_3) = 1$. Then a point of the moduli space is uniquely determined by the position p of $f^2(a_3)$. Show that for $p \neq -1$ the associated moduli space correspondence is given by $F(p) = \left(\frac{p^2-1}{p^2+1}\right)^2$. (We have seen in Example 4.5.12 that in the case $p = -1$ the pull-back map σ_f and hence the correspondence F are constant.)

4.15. The mapping class group biset for $z^2 + i$ and for external ray 1/4...

4.16. Solution of the twisted z^2+i problem (without classifying the obstructed cases)...

4.17. Show that the iterated monodromy group of the skew product $F(z,p) = \left(\left(1 - \frac{2z}{p}\right)^2, \left(1 - \frac{2}{p}\right)^2\right)$ is generated by the wreath recursion

$$\begin{aligned} a &= (12)(34), & R &= (13)(24)(1, b, 1, b), \\ b &= (a, c, a, c^b), & S &= (T, T, S, S), \\ c &= (b, 1, 1, b), \end{aligned}$$

where $T = babcbS^{-1}R^{-1}$.

4.18. Show that the iterated monodromy group of the skew product map $F(z,p) = \left(\left(\frac{2z}{p+1} - 1\right)^2, \left(\frac{p-1}{p+1}\right)^2\right)$ is generated by

$$\begin{aligned} a &= \sigma(b, b, ba, ab), & P &= \pi, \\ b &= (1, bab, a, 1), & S &= \sigma\pi(P\tau^{-1}, P, S^{-1}\tau^{-1}, S^{-1}), \\ c &= (c, b, c, b), \end{aligned}$$

where $\tau = cab$, $\sigma = (12)(34)$, $\pi = (13)(24)$.

4.19. Let $f(z)$ be a rational function with real coefficients and real critical values. Suppose that its post-critical set has n points. Consider the quotient $\overline{f} \circlearrowleft D$ of the dynamical system $f \circlearrowleft \widehat{\mathbb{C}}$ by the complex conjugation, see 4.3.2.3. Show that $\mathrm{IMG}\left(\overline{f}\right)$ is generated by a self-similar set S consisting of $n+1$ elements such that for every $s \in S$ and $x \in \mathsf{X}$ if $s(x) \neq x$, then $s|_x = 1$. In other words, only the arrows ending in the trivial state are labeled by pairs of different letters in the Moore diagram of S.

4.20. Consider a self-similar action of $\mathbb{Z}^n$ transitive on the first level. Let ϕ be the associated virtual endomorphism of $\mathbb{Z}^n$, seen as a linear transformation. Prove that the action is contracting if and only if the spectral radius of ϕ is less than one, and that then the contraction coefficient ρ_ϕ is equal to the spectral radius of ϕ.

4.21. Let us mate the quadratic polynomials corresponding to the external angles 1/6 and 5/14. The iterated monodromy group of the formal mating is generated by $G_{1/6}$ and $G_{9/14}$. Let us denote the standard generators of $G_{1/6}$ by $\tau_{1/6}, \tau_{1/3}, \tau_{2/3}$, and the standard generators of $G_{9/14}$ by $\delta_{9/14}, \delta_{2/7}, \delta_{4/7}, \delta_{1/7}$. Let $\tau = \tau_{1/6}\tau_{1/3}\tau_{2/3} = \delta_{1/7}\delta_{2/7}\delta_{4/7}\delta_{9/14}$ be the odometer.

Denote $a = \tau_{1/6}^{\delta_{1/7}}$, $b = \tau_{1/3}^{\delta_{2/7}\delta_{1/7}}$, $c = \tau_{2/3}^{\tau^{-1}}$, and $x = \delta_{1/7}, y = \delta_{2/7}^{\tau_{1/6}}, z = \delta_{4/7}^{\tau_{1/3}\tau_{1/6}}$. Show that $\langle b, c\rangle \cong (\mathbb{Z}/2\mathbb{Z})^2$, $\langle x, y, z\rangle \cong (\mathbb{Z}/2\mathbb{Z})^3$, and that the iterated monodromy group of the mating is generated by the recursion

$$\begin{aligned} a &= \sigma, & x &= (1, z),\\ b &= (a, c^{aQ}), & y &= (PaQ, x),\\ c &= (1, b^{aQ}), & z &= (1, y),\\ PaQ &= \sigma(Qa, aQ), \end{aligned}$$

where $P = bc$, $Q = xyz$.

4.22. Show that the subgroup $\langle a, P, Q\rangle$ of the iterated monodromy group from the previous problem has the same limit dynamical system as the iterated monodromy group and is equivalent as a self-similar group to the group of affine transformations of $\mathbb{C}$ generated by

$$z \cdot a = -z, \quad z \cdot P = -z + 2\lambda, \quad z \cdot Q = -z + 1,$$

and the biset generated by

$$z \otimes 0 = \lambda z + \frac{1+\lambda}{2}, \qquad z \otimes 1 = -\lambda z + \frac{1+\lambda}{2},$$

where $\lambda = \frac{-1+\sqrt{7}i}{4}$ is the root of $2\lambda^2 + \lambda + 1$.

4.23. Consider $f(z) = 1 + \frac{c}{z^2}$, where $c \approx -2.02949$ is such that the critical points of f belong to a cycle of length 6. Prove that its iterated monodromy group is generated by

$$\begin{aligned} \alpha_1 &= \sigma(\alpha_2, 1),\\ \beta_1 &= (\beta_2, 1),\\ \gamma_1 &= (\gamma_2, 1),\\ \alpha_2 &= \sigma(\alpha_1\beta_1, \alpha_1^{-1}),\\ \beta_2 &= (\gamma_1, 1),\\ \gamma_2 &= (1, \alpha_1^{-1}). \end{aligned}$$

(Probably the easiest way is to use the method of 4.3.2.3.)

4.24. Use the last recursion to prove that $f(z)$ is equivalent to the following Thurston map $\widetilde{f}$. Take two complex planes C_1, C_2 compactified by the circle at infinity. Let $f_1 : C_1 \longrightarrow C_2$ and $f_2 : C_2 \longrightarrow C_1$ be given

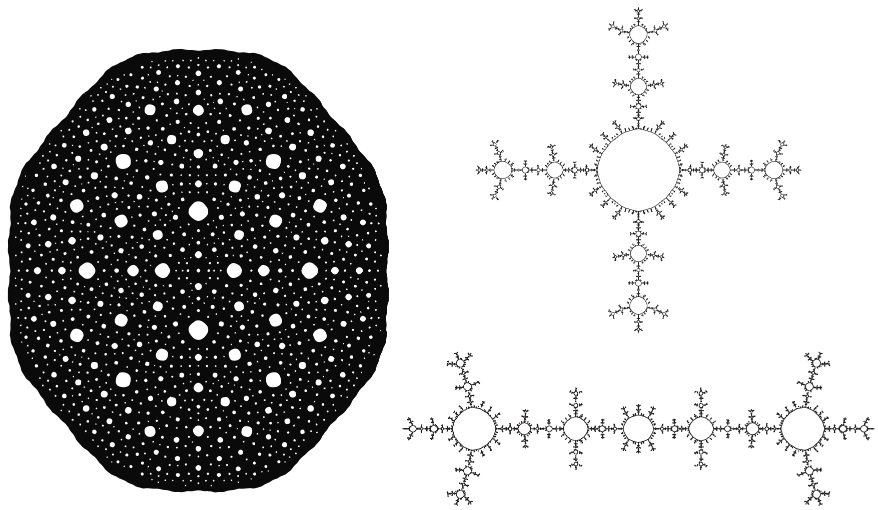

Figure 4.28.

by $f_1(z) = z^2$, $f_2(z) = z^2 + c$, where $c \approx -1.229$ is the real root of $x^3(x^3+1)^4 + 1$. Paste C_1 and C_2 along the circle at infinity (in the same way as it is done for matings). Define $\widetilde{f} : C_1 \cup C_2 \longrightarrow C_1 \cup C_2$ by $\widetilde{f}|_{C_1} = f_1$ and $\widetilde{f}|_{C_2} = f_2$.

In particular, we get that the second iteration of f is the mating of the polynomials $z^4 + c$ and $(z^2+c)^2$.

4.25. Let $f(z) = z^4 + c$ be a polynomial such that 0 belongs to a cycle. Consider, generalizing the previous example, two complex planes C_1, C_2 and the maps $f_1 : C_1 \longrightarrow C_2 : z \mapsto z^2$ and $f_2 : C_2 \longrightarrow C_1 : z \mapsto z^2 + c$. Show that if we paste C_1 and C_2 along the circle at infinity, then the obtained Thurston map is combinatorially equivalent to a rational function of the form $1 + \frac{c'}{z^2}$.

Bibliography

[Abé05] Miklós Abért, *Group laws and free subgroups in topological groups*, Bull. London Math. Soc. **37** (2005), no. 4, 525–534.

[AH03] Valentin Afraimovich and Sze-Bi Hsu, *Lectures on chaotic dynamical systems*, AMS/IP Studies in Advanced Mathematics, vol. 28, American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2003.

[Bau93] Gilbert Baumslag, *Topics in combinatorial group theory*, Lectures in Mathematics, ETH Zürich, Birkhäuser Verlag, Basel, 1993.

[BB17] James Belk and Collin Bleak, *Some undecidability results for asynchronous transducers and the Brin-Thompson group* $2V$, Trans. Amer. Math. Soc. **369** (2017), no. 5, 3157–3172.

[BBM17] J. Belk, C. Bleak, and F. Matucci, *Rational embeddings of hyperbolic groups*, (preprint, arXiv:1711.08369), 2017.

[BCM+16] Collin Bleak, Peter Cameron, Yonah Maissel, Andrés Navas, and Feyishayo Olukoya, *The further chameleon groups of richard thompson and graham higman: Automorphisms via dynamics for the higman groups* $g_{n,r}$, (preprint, arXiv:1605.09302), 2016.

[Bea91] Alan F. Beardon, *Iteration of rational functions. Complex analytic dynamical systems*, Graduate Texts in Mathematics, vol. 132, Springer-Verlag. New York etc., 1991.

[BGŠ03] Laurent Bartholdi, Rostislav I. Grigorchuk, and Zoran Šunić, *Branch groups*, Handbook of Algebra, Vol. 3, North-Holland, Amsterdam, 2003, pp. 989–1112.

[BH99] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften, vol. 319, Springer, Berlin, 1999.

[BMH17] J. Belk, F. Matucci, and James Hyde, *On the asynchronous rational group*, (preprint, arXiv:1711.01668), 2017.

[BN03] Evgen Bondarenko and Volodymyr Nekrashevych, *Post-critically finite self-similar groups*, Algebra and Discrete Mathematics **2** (2003), no. 4, 21–32.

[Bow70] Rufus Bowen, *Markov partitions for Axiom* A *diffeomorphisms*, Amer. J. Math. **92** (1970), 725–747.

[Bra72] Ola Bratteli, *Inductive limits of finite-dimensional C^*-algebras*, Transactions of the American Mathematical Society **171** (1972), 195–234.

[BŠ01] Laurent Bartholdi and Zoran Šunić, *On the word and period growth of some groups of tree automorphisms*, Comm. Algebra **29** (2001), no. 11, 4923–4964.

[BS02] Michael Brin and Garrett Stuck, *Introduction to dynamical systems*, Cambridge University Press, Cambridge, 2002.

[CDTW12] Douglas Cenzer, Ali Dashti, Ferit Toska, and Sebastian Wyman, *Computability of countable subshifts in one dimension*, Theory Comput. Syst. **51** (2012), no. 3, 352–371.

[CFP96] John W. Cannon, William I. Floyd, and Walter R. Parry, *Introductory notes on Richard Thompson groups*, L'Enseignement Mathematique **42** (1996), no. 2, 215–256.

[CN10] Julien Cassaigne and François Nicolas, *Factor complexity*, Combinatorics, automata and number theory, Encyclopedia Math. Appl., vol. 135, Cambridge Univ. Press, Cambridge, 2010, pp. 163–247.

[Dev89] Robert L. Devaney, *An introduction to chaotic dynamical systems*, second ed., Addison-Wesley Studies in Nonlinearity, Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1989.

[DHS99] F. Durand, B. Host, and C. Skau, *Substitutional dynamical systems, Bratteli diagrams and dimension groups*, Ergod. Th. and Dynam. Sys. **19** (1999), 953–993.

[DL06] David Damanik and Daniel Lenz, *Substitution dynamical systems: characterization of linear repetitivity and applications*, J. Math. Anal. Appl. **321** (2006), no. 2, 766–780.

[DM02] Stefaan Delcroix and Ulrich Meierfrankenfeld, *Locally finite simple groups of* 1-*type*, J. Algebra **247** (2002), no. 2, 728–746.

[Dur03] Fabien Durand, *Corrigendum and addendum to: "Linearly recurrent subshifts have a finite number of non-periodic subshift factors" [Ergodic Theory Dynam. Systems* **20** *(2000), no. 4, 1061–1078; MR1779393 (2001m:37022)]*, Ergodic Theory Dynam. Systems **23** (2003), no. 2, 663–669.

[Dye59] Henry A. Dye, *On groups of measure preserving transformations I*, Amer. J. Math. **81** (1959), 119–159.

[Eil74] Samuel Eilenberg, *Automata, languages and machines*, vol. A, Academic Press, New York, London, 1974.

[Ele18] Gábor Elek, *Uniformly recurrent subgroups and simple C^*-algebras*, J. Funct. Anal. **274** (2018), no. 6, 1657–1689.

[Fek23] M. Fekete, *über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten*, Math. Z. **17** (1923), no. 1, 228–249.

[Fer99] Sébastien Ferenczi, *Complexity of sequences and dynamical systems*, Discrete Math. **206** (1999), no. 1-3, 145–154, Combinatorics and number theory (Tiruchirappalli, 1996).

[Fer02] S. Ferenczi, *Substitutions and symbolic dynamical systems*, Substitutions in dynamics, arithmetics and combinatorics, Lecture Notes in Math., vol. 1794, Springer, Berlin, 2002, pp. 101–142.

[FG91] Jacek Fabrykowski and Narain D. Gupta, *On groups with sub-exponential growth functions. II*, J. Indian Math. Soc. (N.S.) **56** (1991), no. 1-4, 217–228.

[Fre04] D. H. Fremlin, *Measure theory. Vol. 3*, Torres Fremlin, Colchester, 2004, Measure algebras, Corrected second printing of the 2002 original.

[Fri83] David Fried, *Métriques naturelles sur les espaces de Smale*, C. R. Acad. Sci. Paris Sér. I Math. **297** (1983), no. 1, 77–79.

[Fri87] ———, *Finitely presented dynamical systems*, Ergod. Th. Dynam. Sys. **7** (1987), 489–507.

[GN05] R. I. Grigorchuk and V. V. Nekrashevych, *Amenable actions of nonamenable groups*, Zapiski Nauchnyh Seminarov POMI **326** (2005), 85–95.

[GNS00] Rostislav I. Grigorchuk, Volodymyr V. Nekrashevich, and Vitaliĭ I. Sushchanskii, *Automata, dynamical systems and groups*, Proceedings of the Steklov Institute of Mathematics **231** (2000), 128–203.

[GPS95] Thierry Giordano, Ian F. Putnam, and Christian F. Skau, *Topological orbit equivalence and C^*-crossed products*, Journal für die reine und angewandte Mathematik **469** (1995), 51–111.

[GPS99] ———, *Full groups of Cantor minimal systems*, Israel J. Math. **111** (1999), 285–320.

[Gri80] Rostislav I. Grigorchuk, *On Burnside's problem on periodic groups*, Functional Anal. Appl. **14** (1980), no. 1, 41–43.

[Gri83] Rostislav I. Grigorchuk, *Milnor's problem on the growth of groups*, Sov. Math., Dokl. **28** (1983), 23–26.

[Gri85] Rostislav I. Grigorchuk, *Degrees of growth of finitely generated groups and the theory of invariant means*, Math. USSR Izv. **25** (1985), no. 2, 259–300.

[Gri73] Christian Grillenberger, *Constructions of strictly ergodic systems. I. Given entropy*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **25** (1972/73), 323–334.

[Gro81] Mikhael Gromov, *Groups of polynomial growth and expanding maps*, Publ. Math. I. H. E. S. **53** (1981), 53–73.

[GS83] Narain D. Gupta and Said N. Sidki, *On the Burnside problem for periodic groups*, Math. Z. **182** (1983), 385–388.

[GW15] Eli Glasner and Benjamin Weiss, *Uniformly recurrent subgroups*, Recent trends in ergodic theory and dynamical systems, Contemp. Math., vol. 631, Amer. Math. Soc., Providence, RI, 2015, pp. 63–75.

[Hal76] P.R. Halmos, *Measure theory*, Graduate Texts in Mathematics, 18, Springer New York, 1976.

[Hir70] Morris W. Hirsch, *Expanding maps and transformation groups*, Global Analysis, Proc. Sympos. Pure Math., vol. 14, American Math. Soc., Providence, Rhode Island, 1970, pp. 125–131.

[Hou79] C. H. Houghton, *The first cohomology of a group with permutation module coefficients*, Arch. Math. (Basel) **31** (1978/79), no. 3, 254–258.

[HP09] Peter Haïssinsky and Kevin M. Pilgrim, *Coarse expanding conformal dynamics*, Astérisque (2009), no. 325, viii+139 pp.

[HPS92] Richard H. Herman, Ian F. Putnam, and Christian F. Skau, *Ordered Bratteli diagrams, dimension groups, and topological dynamics*, Intern.J̃. Math. **3** (1992), 827–864.

[Kai01] Vadim A. Kaimanovich, *Equivalence relations with amenable leaves need not be amenable*, Topology, Ergodic Theory, Real Algebraic Geometry. Rokhlin's Memorial, Amer. Math. Soc. Transl. (2), vol. 202, 2001, pp. 151–166.

[Koc13] Sarah Koch, *Teichmüller theory and critically finite endomorphisms*, Adv. Math. **248** (2013), 573–617.

[KPS16] Sarah Koch, Kevin M. Pilgrim, and Nikita Selinger, *Pullback invariants of Thurston maps*, Trans. Amer. Math. Soc. **368** (2016), no. 7, 4621–4655.

[Kro84] L. Kronecker, *Näherunsgsweise ganzzahlige auflösung linearer gleichungen*, Monatsberichte Königlich Preussischen Akademie der Wissenschaften zu Berlin (1884), 1179–1193, 1271–1299.

[Ku03] Petr K ˚ urka, *Topological and symbolic dynamics*, Cours Spécialisés [Specialized Courses], vol. 11, Société Mathématique de France, Paris, 2003.

[LN02] Yaroslav V. Lavreniuk and Volodymyr V. Nekrashevych, *Rigidity of branch groups acting on rooted trees*, Geom. Dedicata **89** (2002), no. 1, 155–175.

[LN07] Y. Lavrenyuk and V. Nekrashevych, *On classification of inductive limits of direct products of alternating groups*, Journal of the London Mathematical Society **75** (2007), no. 1, 146–162.

[LP03] Felix Leinen and Orazio Puglisi, *Diagonal limits of of finite alternating groups: confined subgroups, ideals, and positive defined functions*, Illinois J. of Math. **47** (2003), no. 1/2, 345–360.

[LP05] ———, *Some results concerning simple locally finite groups of 1-type*, Journal of Algebra **287** (2005), 32–51.

[LY75] T. Y. Li and James A. Yorke, *Period three implies chaos*, Amer. Math. Monthly **82** (1975), no. 10, 985–992.

[Lys85] Igor G. Lysionok, *A system of defining relations for the Grigorchuk group*, Mat. Zametki **38** (1985), 503–511.

[Mat12] Hiroki Matui, *Homology and topological full groups of étale groupoids on totally disconnected spaces*, Proc. Lond. Math. Soc. (3) **104** (2012), no. 1, 27–56.

[Mat15] ———, *Topological full groups of one-sided shifts of finite type*, J. Reine Angew. Math. **705** (2015), 35–84.

[Mat16] ———, *Étale groupoids arising from products of shifts of finite type*, Adv. Math. **303** (2016), 502–548.

[MBT17] Nicolás Matte Bon and Todor Tsankov, *Realizing uniformly recurrent subgroups*, (preprint arxiv:1702.07101, 2017.

[Med11] K. Medynets, *Reconstruction of orbits of Cantor systems from full groups*, Bull. Lond. Math. Soc. **43** (2011), no. 6, 1104–1110.

[MH38] M. Morse and G. A. Hedlund, *Symbolic dynamics*, American Journal of Mathematics **60** (1938), no. 4, 815–866.

[Mil06] John Milnor, *Dynamics in one complex variable*, third ed., Annals of Mathematics Studies, vol. 160, Princeton University Press, Princeton, NJ, 2006.

[Mor21] Harold Marston Morse, *Recurrent geodesics on a surface of negative curvature*, Trans. Amer. Math. Soc. **22** (1921), no. 1, 84–100.

[MRW87] Paul S. Muhly, Jean N. Renault, and Dana P. Williams, *Equivalence and isomorphism for groupoid C^*-algebras*, J. Oper. Theory **17** (1987), 3–22.

[MS20] S. Mazurkiewicz and W. Sierpiński, *Contribution à la topologie des ensembles dénombrables*, Fundamenta Mathematicae **1** (1920), 17–27.

[Nek05] Volodymyr Nekrashevych, *Self-similar groups*, Mathematical Surveys and Monographs, vol. 117, Amer. Math. Soc., Providence, RI, 2005.

[Nek14] ———, *Combinatorial models of expanding dynamical systems*, Ergodic Theory and Dynamical Systems **34** (2014), 938–985.

[Nek18] ———, *Palindromic subshifts and simple periodic groups of intermediate growth*, Annals of Math. **187** (2018), no. 3, 667–719.

[Per54] Oskar Perron, *Die Lehre von den Kettenbrüchen. Bd I. Elementare Kettenbrüche*, B. G. Teubner Verlagsgesellschaft, Stuttgart, 1954, 3te Aufl.

[Ply74] R. V. Plykin, *Sources and sinks of* A-*diffeomorphisms of surfaces*, Mat. Sb. (N.S.) **94(136)** (1974), 243–264, 336.

[Pro51] E. Prouhet, *Mémoire sur quelques relations entre les puissances des nombres*, C. R. Acad. Sci. Paris **33** (1851), 225.

[PS72] G. Polya and G. Szego, *Problems and theorems in analysis, volume I*, Springer, 1972.

[Que87] Martine Queffélec, *Substitution dynamical systems - spectral analysis*, Lecture Notes in Mathematics, vol. 1294, Berlin etc.: Springer-Verlag, 1987.

[Röv99] Claas E. Röver, *Constructing finitely presented simple groups that contain Grigorchuk groups*, J. Algebra **220** (1999), 284–313.

[Roz86] A. V. Rozhkov, *On the theory of groups of Aleshin type*, Mat. Zametki **40** (1986), no. 5, 572–589, 697. MR 886178

[Rub89] Matatyahu Rubin, *On the reconstruction of topological spaces from their groups of homeomorphisms*, Trans. Amer. Math. Soc. **312** (1989), no. 2, 487–538.

[Rue78] D. Ruelle, *Thermodynamic formalism*, Addison Wesley, Reading, 1978.

[Sav15] Dmytro Savchuk, *Schreier graphs of actions of Thompson's group F on the unit interval and on the Cantor set*, Geom. Dedicata **175** (2015), 355–372.

[Shu69] Michael Shub, *Endomorphisms of compact differentiable manifolds*, Am. J. Math. **91** (1969), 175–199.

[Shu70] ———, *Expanding maps*, Global Analysis, Proc. Sympos. Pure Math., vol. 14, American Math. Soc., Providence, Rhode Island, 1970, pp. 273–276.

[Sid00] Said N. Sidki, *Automorphisms of one-rooted trees: growth, circuit structure and acyclicity*, J. of Mathematical Sciences (New York) **100** (2000), no. 1, 1925–1943.

[Sma67] S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–817.

[Šun07] Zoran Šunić, *Hausdorff dimension in a family of self-similar groups*, Geometriae Dedicata **124** (2007), 213–236.

[Tho80] Richard J. Thompson, *Embeddings into finitely generated simple groups which preserve the word problem*, Word Problems II (S. I. Adian, W. W. Boone, and G. Higman, eds.), Studies in Logic and Foundations of Math., 95, North-Holand Publishing Company, 1980, pp. 401–441.

[Thu12] A. Thue, *über die gegenseitige lage gleicher teile gewisser zeichenreihen*, Norske vid. Selsk. Skr. Mat. Nat. Kl. **1** (1912), 1–67.

[Vor12] Yaroslav Vorobets, *Notes on the Schreier graphs of the Grigorchuk group*, Dynamical systems and group actions (L. Bowen et al., ed.), Contemp. Math., vol. 567, Amer. Math. Soc., Providence, RI, 2012, pp. 221–248.

[Wie14] Susana Wieler, *Smale spaces via inverse limits*, Ergodic Theory Dynam. Systems **34** (2014), no. 6, 2066–2092.

[Wil67] R. F. Williams, *One-dimensional non-wandering sets*, Topology **6** (1967), 473–487.

[Wil74] ———, *Expanding attractors*, Inst. Hautes Études Sci. Publ. Math. (1974), no. 43, 169–203.

[Yi01] Inhyeop Yi, *Canonical symbolic dynamics for one-dimensional generalized solenoids*, Trans. Amer. Math. Soc. **353** (2001), no. 9, 3741–3767.