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OVERLAPPING SCHWARZ METHODS IN $\boldsymbol{H}(\mathbf{curl})$ ON NONCONVEX DOMAINS

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ABSTRACT. We consider domain decomposition preconditioners for the linear algebraic equations which result from finite element discretization of problems involving the bilinear form $\alpha(\cdot,\cdot) + (\mathbf{curl}\,\cdot, \mathbf{curl}\,\cdot)$ defined on a non-convex domain Ω. Here $(\cdot,\cdot)$ denotes the inner product in $(L^2(\Omega))^3$ and α is a positive number. We use Nedelec's curl-conforming finite elements to discretize the problem. Both additive and multiplicative overlapping Schwarz preconditioners are studied. Our results are uniform with respect to the mesh size and α under standard assumptions concerning the overlapping subdomains.

1. INTRODUCTION

Let Ω be a bounded simply-connected domain in $\mathbb{R}^3$ with a polyhedral boundary Γ. We denote $\boldsymbol{H}(\mathbf{curl};\Omega)$ to be the set of vector functions in $\boldsymbol{L}^2(\Omega) \equiv (L^2(\Omega))^3$ whose curl is also in $\boldsymbol{L}^2(\Omega)$. We consider the bilinear form

$$A(\boldsymbol{u},\boldsymbol{v}) \equiv \alpha(\boldsymbol{u},\boldsymbol{v}) + (\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v}), \quad \text{for all } \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega).$$

When $\alpha = 1$, this bilinear form is the inner product in $\boldsymbol{H}(\mathbf{curl};\Omega)$. Denote by $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ the functions $\boldsymbol{u}$ in $\boldsymbol{H}(\mathbf{curl};\Omega)$ satisfying the homogeneous boundary condition $\boldsymbol{u}\times\boldsymbol{n} = 0$ on Γ.

The bilinear form $A(\cdot,\cdot)$ arises naturally in many problems of practical importance. For example, it appears when time-dependent Maxwell's equations are discretized using an implicit finite difference scheme (cf. [18]). At each time step, we get the variational problem: Given $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, find $\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ satisfying

$$A(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}), \text{ for all } \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega). \tag{1.1}$$

In this case α is related to the time step. The problem (1.1) also arises in elasticity and Stokes' equations with various boundary conditions (cf. [10, 16]).

Date: April 18, 2002.

1991 *Mathematics Subject Classification.* 65F10, 65N55.

Key words and phrases. Schwarz method, domain decomposition, nonconvex domains, Nedelec's finite elements.

This work was supported in part by the National Science Foundation through grant number DMS-9973328.

Schwarz methods provide efficient and easily parallelized preconditioners for the discrete system corresponding to (1.1). Considerable research has been done towards the application of Schwarz methods on these problems. In [21], Toselli analyzed the convergence of overlapping Schwarz methods in the case of convex domains. In [15], Hiptmair and Toselli gave an unified and simplified approach to Schwarz methods for problems in $\boldsymbol{H}(\mathbf{curl};\Omega)$ and $\boldsymbol{H}(\mathrm{div};\Omega)$, again on convex domains. Because of the large kernel of the **curl** operator, the Helmholtz decomposition of an arbitrary vector field into irrotational and solenoidal components plays an important role in the abovementioned work. However, the irrotational component is not in general $\boldsymbol{H}^1$-regular when Ω is nonconvex and many estimates in [15, 21] fail in that case.

In this paper, we extend the scope of the above mentioned theoretical results to the nonconvex case. By introducing a novel decomposition of vector fields in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, we provide a stable decomposition which is critical in the estimate of condition number of the preconditioned system. In addition, our results hold uniformly for $0 < \alpha < \infty$ under standard conditions on the overlapping subdomains.

The outline of the remainder of this paper is as follows. In Section 2, we provide a decomposition for functions in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$. Section 3 describes the finite element spaces and defines the discrete problem. The Schwarz method and results on the conditioning of the preconditioned system are given in Section 4. These results depend on a decomposition lemma which is proved in Section 5. Finally, the results of numerical experiments illustrating the theory are given in Section 6.

2. Decompositions of $\boldsymbol{H}_0(\mathbf{curl};\Omega)$

Throughout this paper, we use boldface type for vector fields, spaces of vector fields, and operators mapping vector fields to vector fields. For any domain $\mathcal{D} \subseteq \mathbb{R}^3$, the norm and seminorm in the Sobolev spaces $H^s(\mathcal{D})$ and $\boldsymbol{H}^s(\mathcal{D})$ are both denoted by $\|\cdot\|_{s,\mathcal{D}}$ and $|\cdot|_{s,\mathcal{D}}$, respectively, with the index s suppressed when $s = 0$. We also drop the subscript $\mathcal{D}$ if $\mathcal{D} = \Omega$.

Due to the different behavior of $A(\cdot,\cdot)$ on solenoidal and irrotational vector fields, the Helmholtz decomposition is an important tool in the analysis. For any $\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$, we have the continuous Helmholtz decomposition

$$\boldsymbol{u} = \boldsymbol{z} + \nabla\varphi, \tag{2.1}$$

where $\boldsymbol{z} \in \boldsymbol{H}_0(\mathbf{curl};\Omega), \mathrm{div}\,\boldsymbol{z} = 0$, and $\varphi \in H_0^1(\Omega)$. Unfortunately, the vector field $\boldsymbol{z}$ in (2.1) does not in general belong to $\boldsymbol{H}^1(\Omega)$ when the domain Ω is not convex. Our analysis is based on a novel decomposition of $\boldsymbol{z}$. The following two lemmas provide the construction and estimates.

Lemma 2.1. *For any $\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$, there exists $\boldsymbol{w} \in \boldsymbol{H}^1(\Omega)$ such that*

$$\mathbf{curl}\,\boldsymbol{w} = \mathbf{curl}\,\boldsymbol{u}, \quad \textit{and} \quad \mathrm{div}\,\boldsymbol{w} = 0 \textit{ in } \Omega,$$

and the following estimates hold:

$$\|\boldsymbol{w}\| \leq \|\boldsymbol{u}\|, \quad \textit{and} \quad |\boldsymbol{w}|_1 \leq \sqrt{2}\|\mathbf{curl}\,\boldsymbol{u}\|.$$

Proof. The proof follows the argument of Theorem 3.4, chapter I in [11]. Denote by $\widetilde{\boldsymbol{u}}$ the extension by zero of $\boldsymbol{u}$. Then $\widetilde{\boldsymbol{u}}$ is in $\boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3)$. Let $\boldsymbol{v} = \mathbf{curl}\,\widetilde{\boldsymbol{u}}$. Note that $\boldsymbol{v}$ has compact support.

Let $\widehat{\boldsymbol{u}}$ and $\widehat{\boldsymbol{v}}$ be the Fourier transforms of $\widetilde{\boldsymbol{u}}$ and $\boldsymbol{v}$ respectively. Since $\operatorname{div}\boldsymbol{v} = 0$ and $\boldsymbol{v} = \mathbf{curl}\,\widetilde{\boldsymbol{u}}$, we have

$$\xi \cdot \widehat{v} = 0, \text{ and } \widehat{\boldsymbol{v}} = i\xi \times \widehat{\boldsymbol{u}},$$

where $i = \sqrt{-1}$, and $\xi = (\xi_1, \xi_2, \xi_3)^T$ stands for the dual variable of $\boldsymbol{x} = (x_1, x_2, x_3)^T$.

Define $\widehat{\boldsymbol{w}} \equiv (\mathbf{I} - \frac{1}{|\xi|^2}\xi\xi^T)\widehat{\boldsymbol{u}}$ where $\mathbf{I}$ is the identity matrix. It is not hard to see that the matrix $\mathbf{I} - \frac{1}{|\xi|^2}\xi\xi^T$ has the eigenvalue 0 corresponding to the eigenvector ξ, and the eigenvalue 1 of multiplicity two corresponding to two linearly independent eigenvectors orthogonal to ξ. This shows that $\|\widehat{w}\| \leq \|\widehat{\boldsymbol{u}}\|$ and thus the inverse Fourier transform $\boldsymbol{w}$ of $\widehat{\boldsymbol{w}}$ satisfies

$$\|\boldsymbol{w}\|_{0,\mathbb{R}^3} = \|\widehat{\boldsymbol{w}}\|_{0,\mathbb{R}^3} \leq \|\widehat{\boldsymbol{u}}\|_{0,\mathbb{R}^3} = \|\widetilde{\boldsymbol{u}}\|_{0,\mathbb{R}^3} = \|\boldsymbol{u}\|.$$

By the construction of $\widehat{\boldsymbol{w}}$, we also have

$$\xi \cdot \widehat{\boldsymbol{w}} = \xi \cdot \widehat{\boldsymbol{u}} - \frac{1}{|\xi|^2}\xi \cdot \xi(\xi^T\widehat{\boldsymbol{u}}) = 0,$$

and

$$i\xi \times \widehat{\boldsymbol{w}} = i\xi \times \widehat{\boldsymbol{u}} - \frac{i}{|\xi|^2}(\xi^T\widehat{\boldsymbol{u}})\xi \times \xi = i\xi \times \widehat{\boldsymbol{u}}.$$

Thus,

$$\operatorname{div}\boldsymbol{w} = 0 \text{ and } \mathbf{curl}\,\boldsymbol{w} = \mathbf{curl}\,\widetilde{\boldsymbol{u}}.$$

Since $\widehat{\boldsymbol{v}} = i\xi \times \widehat{\boldsymbol{u}}$,

$$\begin{aligned}\xi \times \widehat{\boldsymbol{v}} &= i\xi \times (\xi \times \widehat{\boldsymbol{u}}) = i[(\xi^T\widehat{\boldsymbol{u}})\xi - |\xi|^2\widehat{\boldsymbol{u}}] \\ &= -i|\xi|^2(\widehat{\boldsymbol{u}} - \frac{1}{|\xi|^2}\xi\xi^T\widehat{\boldsymbol{u}}) = -i|\xi|^2\widehat{\boldsymbol{w}}.\end{aligned}$$

It immediately follows that $|\boldsymbol{w}|_{1,\mathbb{R}^3} \leq \sqrt{2}\|\boldsymbol{v}\|_{0,\mathbb{R}^3} \leq \sqrt{2}\|\mathbf{curl}\,\boldsymbol{u}\|$.

The restriction $\boldsymbol{w}$ to Ω is the desired potential. This completes the proof of the lemma. □

The following lemma is a special version of Proposition 5.1 in [7]. However, it provides the additional stability estimate $\|\boldsymbol{w}\| \leq C\|\boldsymbol{z}\|$ for some constant C. The proof follows the argument given there.

Lemma 2.2. *For any* $\boldsymbol{z} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ *with* $\operatorname{div}\boldsymbol{z} = 0$ *in* Ω*, there exist* $\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega)$ *and* $\psi \in H^1(\Omega)$ *such that*

$$\boldsymbol{z} = \boldsymbol{w} + \nabla\psi$$

and the following estimates hold:

$$\|\boldsymbol{w}\| + \|\psi\|_1 \leq C\|\boldsymbol{z}\| \textit{ and } \|\boldsymbol{w}\|_1 \leq C\|\mathbf{curl}\,\boldsymbol{z}\|.$$

Proof. Let Γ_i, $1 \le i \le I$, be the internal connected components of $\partial\Omega$, and Γ_0 the boundary of the only unbounded connected component of $\mathbb{R}^3\backslash\overline{\Omega}$.

Define q_i to be the unique solution in $H^1(\Omega)$ of the problem [1]

$$\begin{cases} -\triangle q_i = 0 \quad \text{in} \quad \Omega, \\ q_i\big|_{\Gamma_0} = 0, \quad q_i\big|_{\Gamma_k} = C_{ik}, 1 \le k \le I, \end{cases}$$

where C_{ik} are constants on Γ_k. These constants are uniquely determined by the following conditions

$$\langle \frac{\partial q_i}{\partial \boldsymbol{n}}, 1\rangle_{\Gamma_0} = -1, \quad \langle \frac{\partial q_i}{\partial \boldsymbol{n}}, 1\rangle_{\Gamma_k} = \delta_{ik},\ 1 \le k \le I.$$

For $\boldsymbol{z}$ given above, we define $\overset{\circ}{\boldsymbol{z}}$ by

$$\overset{\circ}{\boldsymbol{z}} = \boldsymbol{z} - \sum_{i=1}^{I} \langle \boldsymbol{z}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} \nabla q_i.$$

Then $\overset{\circ}{\boldsymbol{z}} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ satisfies that

$$\mathbf{curl}\,\overset{\circ}{\boldsymbol{z}} = \mathbf{curl}\,\boldsymbol{z}, \quad \operatorname{div}\overset{\circ}{\boldsymbol{z}} = 0,$$

$$\begin{aligned} \langle \overset{\circ}{\boldsymbol{z}}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_k} &= \langle \boldsymbol{z}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_k} - \sum_{i=1}^{I} \langle \boldsymbol{z}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} \langle \frac{\partial q_i}{\partial \boldsymbol{n}}, 1\rangle_{\Gamma_k} \\ &= 0, \quad 1 \le k \le I, \end{aligned}$$

and

$$\begin{aligned} \|\overset{\circ}{\boldsymbol{z}}\| &\le \|\boldsymbol{z}\| + \sum_{i=1}^{I} |\langle \boldsymbol{z}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i}| \cdot \|\nabla q_i\| \\ &\le \|\boldsymbol{z}\| + C\|\boldsymbol{z}\|_{\boldsymbol{H}(\mathrm{div};\Omega)} \le C\|\boldsymbol{z}\|. \end{aligned}$$

It follows from Corollary 3.19 of [1] that

$$\|\overset{\circ}{\boldsymbol{z}}\| \le C\|\mathbf{curl}\,\overset{\circ}{\boldsymbol{z}}\|. \tag{2.2}$$

Denote by $\widetilde{\boldsymbol{z}}$ the extension by zero of $\overset{\circ}{\boldsymbol{z}}$ to an open ball $B(0;r)$ which contains $\overline{\Omega}$. Let $\Omega^c \equiv B(0;r)\backslash\overline{\Omega}$.

By Lemma 2.1, there is a $\widetilde{\boldsymbol{w}} \in \boldsymbol{H}^1(B(0;r))$ such that

$$\mathbf{curl}\,\widetilde{\boldsymbol{w}} = \mathbf{curl}\,\widetilde{\boldsymbol{z}} \text{ and } \operatorname{div}\widetilde{\boldsymbol{w}} = 0.$$

Moreover, $\|\widetilde{\boldsymbol{w}}\|_{0,B(0;r)} \le \|\overset{\circ}{\boldsymbol{z}}\|$ and $\|\widetilde{\boldsymbol{w}}\|_{1,B(0;r)} \le \sqrt{2}\|\overset{\circ}{\boldsymbol{z}}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \le C\|\mathbf{curl}\,\overset{\circ}{\boldsymbol{z}}\|$. In the last inequality, we used 2.2.

Since $\mathbf{curl}\,(\widetilde{\boldsymbol{w}} - \widetilde{\boldsymbol{z}}) = 0$, there is a $\widetilde{\varphi} \in H^1(B(0;r))/\mathbb{R}$ such that $\widetilde{\boldsymbol{w}} - \widetilde{\boldsymbol{z}} = \nabla\widetilde{\varphi}$ and $\|\widetilde{\varphi}\|_{1,B(0;r)} \le C\|\widetilde{\boldsymbol{w}} - \widetilde{\boldsymbol{z}}\|_{0,B(0;r)}$ (*cf*. Theorem 2.9, Chapter I in [11]). Note that in Ω^c, $\nabla\widetilde{\varphi} = \widetilde{\boldsymbol{w}} \in \boldsymbol{H}^1(\Omega^c)$ since $\widetilde{\boldsymbol{z}} = 0$, and thus $\widetilde{\varphi} \in H^2(\Omega^c)$. Using Theorem 5 in [20], we can extend this $\widetilde{\varphi}$ in $H^2(\Omega^c)$ to φ defined on $B(0;r)$ satisfying

$$\|\varphi\|_{1,B(0;r)} \le C\|\widetilde{\varphi}\|_{1,\Omega^c} \le C\|\widetilde{\boldsymbol{w}} - \widetilde{\boldsymbol{z}}\|_{0,B(0;r)}, \tag{2.3}$$

and

$$\|\varphi\|_{2,B(0;r)} \le C\|\widetilde{\varphi}\|_{2,\Omega^c} = C(\|\widetilde{\boldsymbol{w}}\|_{1,\Omega^c} + \|\widetilde{\boldsymbol{z}}\|). \tag{2.4}$$

Now, we have

$$\begin{aligned}\widetilde{\boldsymbol{z}} &= \widetilde{\boldsymbol{w}} - \nabla\widetilde{\varphi}\\ &= (\widetilde{\boldsymbol{w}} - \nabla\varphi) + \nabla(\varphi - \widetilde{\varphi}).\end{aligned}$$

Note that $\widetilde{\boldsymbol{w}} - \nabla\varphi$ is in $\boldsymbol{H}^1(B(0;r))$ and its trace to $\partial\Omega$ from Ω^c vanishes. Thus, $\widetilde{\boldsymbol{w}} - \nabla\varphi$ is in $\boldsymbol{H}_0^1(\Omega)$ and satisfies

$$\|\widetilde{\boldsymbol{w}} - \nabla\varphi\|_{0,B(0;r)} \le C\|\widetilde{\boldsymbol{w}}\|_{0,B(0;r)} + C\|\widetilde{\boldsymbol{w}} - \widetilde{\boldsymbol{z}}\|_{0,B(0;r)} \le C\|\mathring{\boldsymbol{z}}\| \le C\|\boldsymbol{z}\|$$

and

$$\|\widetilde{\boldsymbol{w}} - \nabla\varphi\|_{1,B(0;r)} \le C(\|\widetilde{\boldsymbol{w}}\|_{1,B(0;r)} + \|\widetilde{\boldsymbol{z}}\|) \le C\|\mathbf{curl}\,\mathring{\boldsymbol{z}}\| = C\|\mathbf{curl}\,\boldsymbol{z}\|.$$

We complete the proof by setting $\boldsymbol{w}$ to be the restriction to Ω of $\widetilde{\boldsymbol{w}} - \nabla\varphi$, and ψ the sum of $\sum_{i=1}^{I}\langle \boldsymbol{z}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} q_i$ and the restriction to Ω of $\widetilde{\varphi} - \varphi$. □

3. The discrete problem

Let $\mathcal{T}_h$ be a simplicial mesh of Ω that is regular (c.f., [6]) and let h denote the maximal diameter of the tetrahedra in $\mathcal{T}_h$. As usual, we assume that the tetrahedra are essentially disjoint, i.e., the intersection of two being either an entire face, edge, or vertex.

Fix an integer $k \ge 0$ and let $\mathcal{P}_k(\tau)$ be the space of polynomials of degree at most k restricted to a tetrahedron τ. $\overline{S}_h$ stands for the subspace of $H^1(\Omega)$ consisting of piecewise polynomials with respect to the above mesh of degree at most $k+1$. We denote by $\overline{\boldsymbol{U}}_h$ the Nedelec finite element subspace of $\boldsymbol{H}(\mathbf{curl};\Omega)$ of index k associated with $\mathcal{T}_h$. When $\boldsymbol{x} \in \mathbb{R}^3$ is restricted to a tetrahedron τ, the elements of $\overline{\boldsymbol{U}}_h$ are functions of the form $\boldsymbol{p}(\boldsymbol{x}) + \boldsymbol{r}(\boldsymbol{x})$ with $\boldsymbol{p} \in \mathcal{P}_k(\tau)^3$ and $\boldsymbol{r} \in \mathcal{P}_{k+1}(\tau)^3$ such that $\boldsymbol{r}\cdot\boldsymbol{x} = 0$. Let $\overline{\boldsymbol{V}}_h$ be the Raviart-Thomas finite element subspace of degree k of $\boldsymbol{H}(\mathrm{div};\Omega)$. Restrictions of functions in $\overline{\boldsymbol{V}}_h$ to $\boldsymbol{x}$ in a tetrahedra τ are of the form $\boldsymbol{p}(\boldsymbol{x}) + q(x)\boldsymbol{x}$ where $\boldsymbol{p} \in \mathcal{P}_k^3$ and $q \in \mathcal{P}_k$. For the detailed constructions and the connection between these spaces, we refer to [4, 14, 16, 17]. Our results also hold for the analogous finite element spaces based on cubes.

In what follows, a subspace without overline stands for the corresponding finite element subspace of functions with homogeneous boundary conditions. For example, $\boldsymbol{U}_h = \boldsymbol{H}_0(\mathbf{curl};\Omega)\bigcap\overline{\boldsymbol{U}}_h$.

All of the above spaces have associated degrees of freedom and there are natural interpolation operators corresponding to these degrees of freedom [16, 17]. The interpolation operator $\boldsymbol{\Pi}_h$ for the subspace $\overline{\boldsymbol{U}}_h$ is well defined for $\boldsymbol{H}^1(\Omega)$ vector fields whose $\mathbf{curl}$ is in $\boldsymbol{L}^p(\Omega)$, for any fixed $p > 2$. This follows from Lemma 4.7 in [1] and the Sobolev embedding theorem. In particular, $\mathbf{\Pi}_h$ is defined for $\boldsymbol{H}^1(\Omega)$ vector fields whose $\mathbf{curl}$ belongs to $\boldsymbol{V}_h$. Moreover, the following estimate holds (see, [2])

$$\|\boldsymbol{u} - \mathbf{\Pi}_h \boldsymbol{u}\| \le Ch|\boldsymbol{u}|_1 \tag{3.1}$$

for all $\boldsymbol{u}$ $\boldsymbol{H}^1(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{u}$ is in $\boldsymbol{V}_h$. In (3.1) and the remainder of the paper, C, with or without subscript, denotes a generic constant independent of h, H, and α. The value of C may differ at different occurrences.

In our analysis, we will use the $\boldsymbol{L}^2$-projection $\mathbf{Q}_h^{\boldsymbol{U}} : \boldsymbol{L}^2(\Omega) \to \boldsymbol{U}_h$, onto the finite element space. The following stability and error estimates was suggested in [15]:

$$\|\boldsymbol{u} - \mathbf{Q}_h^{\boldsymbol{U}} \boldsymbol{u}\| + h\|\mathbf{curl}\,\mathbf{Q}_h^{\boldsymbol{U}} \boldsymbol{u}\| \le Ch|\boldsymbol{u}|_1, \quad \text{for all } \boldsymbol{u} \in H^1(\Omega) \tag{3.2}$$

although its proof was not given there. A proof of (3.2) can be developed [13] using the operator $\mathfrak{P}_h$ introduced in [3]. The projector $\mathfrak{P}_h$ was defined locally and replaced integration on the edges with integration on the faces. This produces an interpolation which is well defined on vector fields in $\boldsymbol{H}^1$. By applying a Bramble-Hilbert argument, Lemma 5 of [3] shows that $\|\boldsymbol{u} - \mathfrak{P}_h \boldsymbol{u}\| \le Ch|\boldsymbol{u}|_1$, and $\|\mathbf{curl}\,\mathfrak{P}_h \boldsymbol{u}\| \le C|\boldsymbol{u}|_1$, for all $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$. The first estimate of (3.2) follows from the best approximation property of $\mathbf{Q}_h^{\boldsymbol{U}}$, and the second follows from

$$\|\mathbf{curl}\,\mathbf{Q}_h^{\boldsymbol{U}} \boldsymbol{u}\| \le Ch^{-1}\|(\mathbf{Q}_h^{\boldsymbol{U}} - \mathfrak{P}_h)\boldsymbol{u}\| + C\|\mathbf{curl}\,\mathfrak{P}_h \boldsymbol{u}\|.$$

The finite element approximation to (1.1) is the function $\boldsymbol{u}_h \in \boldsymbol{U}_h$ satisfying

$$\mathbf{A}(\boldsymbol{u}_h, \boldsymbol{w}) = (\boldsymbol{f}, \boldsymbol{w}), \quad \text{for all } \boldsymbol{w} \in \boldsymbol{U}_h. \tag{3.3}$$

The above equation can be written as

$$\mathbf{A}_h^{\boldsymbol{U}} \boldsymbol{u}_h = \boldsymbol{f}_h \equiv \mathbf{Q}_h^{\boldsymbol{U}} \boldsymbol{f}, \tag{3.4}$$

where $\mathbf{A}_h^{\boldsymbol{U}} : \boldsymbol{U}_h \to \boldsymbol{U}_h$ is defined by

$$(\mathbf{A}_h^{\boldsymbol{U}} \boldsymbol{u}, \boldsymbol{w}) = \mathbf{A}(\boldsymbol{u}, \boldsymbol{w}), \quad \text{for all } \boldsymbol{w} \in \boldsymbol{U}_h.$$

4. Overlapping Schwarz preconditioners

In this section, we give two overlapping Schwarz preconditioners for the discrete system corresponding to (3.4). The overlapping Schwarz algorithms as described in [8, 15, 21] are based on two levels of partitioning of Ω. The first is a coarse partitioning into (non-overlapping) tetrahedra $\{\Omega_i : i = 1, \dots, N_0\}$. This forms a mesh $\mathcal{T}_H$ of mesh size H. Next, each Ω_i is further partitioned into finer tetrahedra $\{\tau_i^j : j = 1, 2, \dots, N_i\}$. The fine partitioning gives the fine mesh $\mathcal{T}_h$ of mesh size h. Both $\mathcal{T}_H$ and $\mathcal{T}_h$ are assumed to be regular.

Along with this partitioning, we assume that we are given another sequence of (overlapping) subdomains Ω'_j $j = 1, \dots, N$ in such a way that $\partial\Omega'_j$

aligns with the h-level mesh. Then each subdomain Ω'_j is also partitioned by tetrahedra in $\mathcal{T}_h$ and the space

$$\boldsymbol{U}_h^j = \boldsymbol{U}_h \cap \boldsymbol{H}_0(\mathbf{curl}\,; \Omega'_j), \quad j = 1, \ldots, N,$$

is again a Nedelec finite element space. In the above definition, we consider $\boldsymbol{H}_0(\mathbf{curl}\,;\Omega'_j)$ as a subset of $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ by identifying functions in $\boldsymbol{H}_0(\mathbf{curl}\,;\Omega'_j)$ with their extension by zero. It is convenient to set $\Omega'_0 = \Omega$ and $\boldsymbol{U}_h^0 = \boldsymbol{U}_H$. Similarly, we define the Lagrange finite element space $S_h^j, j = 0, 1, \ldots, N$ by replacing $\boldsymbol{H}_0(\mathbf{curl}\,;\Omega'_j)$ with $H_0^1(\Omega'_j)$.

We assume throughout this paper that subdomains $\{\Omega'_j\}$ are such that there is a partition of unity $\{\theta_j\}_{j=1}^N$ where the partition functions are piecewise linear with respect to the fine mesh and satisfy

$$\|\nabla\theta_j\|_\infty \leq CH^{-1}, \text{ for } j = 1, \ldots, N.$$

We finally assume that the subdomains $\{\Omega'_j\}$ satisfy a limited overlap property, i.e., each point of Ω is contained in at most n_0 subdomains where n_0 is independent of H and h.

One can, for example, define the overlapping subdomains to be regions associated with vertices of the coarse mesh, i.e., Ω'_j is the interior of the union of the closures of the coarse grid tetrahedra which share the j'th vertex. In this case, the partition of unity functions can be taken to be the nodal finite element basis functions associated with the conforming piecewise linear coarse grid approximation to $H^1(\Omega)$. Alternatively, one can use the classical approach of defining the overlapping subdomains by extending the original coarse grid subdomains $\{\Omega_j\}$ so that

$$\text{dist}(\partial\Omega'_j \cap \Omega, \partial\Omega_j \cap \Omega) \geq \delta H \quad \text{for all } j = 1, \ldots, N. \tag{4.1}$$

Here δ is some constant independent of h and H.

A key property to establish the effectiveness of the overlapping Schwarz preconditioners is the following stability result. Its proof will be given in the next section.

Lemma 4.1. *Suppose that the overlapping subdomains and partition of unity satisfy the conditions above. Then there is a constant C_{stab} such that for all $\boldsymbol{u} \in \boldsymbol{U}_h$, we have a decomposition $\boldsymbol{u} = \sum_{j=0}^N \boldsymbol{u}_j$ with $\boldsymbol{u}_j \in \boldsymbol{U}_h^j$ satisfying*

$$\sum_{j=0}^N \mathbf{A}(\boldsymbol{u}_j, \boldsymbol{u}_j) \leq C_{stab}\mathbf{A}(\boldsymbol{u}, \boldsymbol{u}).$$

The overlapping Schwarz methods uses the solvers on the overlapping subregions $\{\Omega'_j\}$. For $j = 0, 1, \ldots, N$, we define $\mathbf{A}_j : \boldsymbol{U}_h^j \to \boldsymbol{U}_h^j$ by

$$(\mathbf{A}_j\boldsymbol{u}, \boldsymbol{w}) = \mathbf{A}(\boldsymbol{v}, \boldsymbol{w}), \quad \text{for all } \boldsymbol{w} \in \boldsymbol{U}_h^j,$$

and set $\mathbf{Q}_j : \boldsymbol{U}_h \to \boldsymbol{U}_h^j$ to be the $\boldsymbol{L}^2$-projection.

The additive Schwarz preconditioner $\mathbf{B}_a : \boldsymbol{U}_h \to \boldsymbol{U}_h$ is defined by

$$\mathbf{B}_a = \sum_{j=0}^{N} \mathbf{A}_j^{-1}\mathbf{Q}_j. \tag{4.2}$$

The symmetric multiplicative Schwarz preconditioner $\mathbf{B}_m : \boldsymbol{U}_h \to \boldsymbol{U}_h$ is defined as follows. For a given $\boldsymbol{g} \in \boldsymbol{U}_h$, we let $\mathbf{B}_m \boldsymbol{g} = \boldsymbol{u}^N \in \boldsymbol{U}_h$, where the $\boldsymbol{u}^N$ is defined by the iteration $\boldsymbol{u}^{-N-1} = 0$, and

$$\boldsymbol{u}^j = \boldsymbol{u}^{j-1} - \mathbf{A}_{|j|}^{-1}\,\mathbf{Q}_{|j|}\,(\boldsymbol{g} - \mathbf{A}_h \boldsymbol{u}^{j-1}),\ j = -N, -N+1, \ldots, N. \tag{4.3}$$

In practice, one can replace $\mathbf{A}_j^{-1}$ by preconditioner for $\mathbf{A}_j$ in either algorithm and still get robust preconditioners for the operator $\mathbf{A}_h^{\boldsymbol{U}}$. The results for the termwise preconditioned algorithm easily follow [5] from those for (4.2) and (4.3) which we give below.

The following theorem provides the upper bound for the conditioner number of the additive and multiplicative Schwarz preconditioners. Its proof is well known (cf.[5, 19]) and follows from the assumptions on the overlapping subdomains and Lemma 4.1.

Theorem 4.1. *Under the assumption of Lemma 4.1, for any $\boldsymbol{u} \in \boldsymbol{U}_h$, we have*

$$C_{stab}^{-1}\,\mathbf{A}(\boldsymbol{u},\boldsymbol{u}) \le \mathbf{A}(\mathbf{B}_a\,\mathbf{A}_h^{\boldsymbol{U}}\boldsymbol{u},\boldsymbol{u}) \le n_0\mathbf{A}(\boldsymbol{u},\boldsymbol{u}),$$

and

$$(C_{stab}\,n_0^2)^{-1}\,\mathbf{A}(\boldsymbol{u},\boldsymbol{u}) \le \mathbf{A}(\mathbf{B}_m\,\mathbf{A}_h^{\boldsymbol{U}}\boldsymbol{u},\boldsymbol{u}) \le \mathbf{A}(\boldsymbol{u},\boldsymbol{u}).$$

Remark 4.1. The above theorem guarantees that the condition number for the preconditioned system remains bounded independently of h and H. This means that, for example, a preconditioned conjugate gradient iteration using these preconditioners is guaranteed to converge at a rate which can be bounded independently of h and H.

Remark 4.2. The theorem suggests that the additive method has a smaller condition number than the multiplicative. In practice this is not the case. In numerical experiments, it is observed that the multiplicative method has a smaller condition number.

5. Proof of Lemma 4.1

In this section, we will give a proof of Lemma 4.1. To do this, we pick an arbitrary $\boldsymbol{u} \in \boldsymbol{U}_h$ and let $\boldsymbol{u} = \boldsymbol{z} + \nabla\varphi$ be its continuous Helmholtz decomposition. Splitting $\boldsymbol{z} = \boldsymbol{w} + \nabla\psi$ as in Lemma 2.2 gives

$$\boldsymbol{u} = \boldsymbol{w} + \nabla p, \tag{5.1}$$

where $\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega)$ and $p = \varphi + \psi \in H^1(\Omega)$ satisfy

$$\|\boldsymbol{w}\| + \|p\|_1 \le C\|\boldsymbol{u}\|, \quad \text{and} \quad |\boldsymbol{w}|_1 \le C\|\mathbf{curl}\,\boldsymbol{u}\|. \tag{5.2}$$

Since $\boldsymbol{w} \in \boldsymbol{H}^1(\Omega)$ and $\mathbf{curl}\,\boldsymbol{w} \in \boldsymbol{V}_h$, we can apply $\mathbf{\Pi}_h$ to both sides of (5.1) to get

$$\boldsymbol{u} = \mathbf{\Pi}_h \boldsymbol{w} + \nabla p^h \tag{5.3}$$

for some $p^h \in S_h$. We will decompose $\mathbf{\Pi}_h \boldsymbol{w}$ and p^h separately.

Using the domain decomposition theory for the Dirichlet form (cf. [9, 19]), we obtain a decomposition $p^h = \sum_{j=0}^N p_j$ with $p_j \in S_j$ satisfying

$$\sum_{j=0}^N \mathbf{A}(\nabla p_j, \nabla p_j) = \alpha \sum_{j=0}^N \|\nabla p_j\|^2 \le C\mathbf{A}(\nabla p^h, \nabla p^h) \le C\mathbf{A}(\boldsymbol{u}, \boldsymbol{u}). \tag{5.4}$$

Here, the last inequality follows from (3.1) and

$$\begin{aligned} \|\nabla p^h\| &= \|\boldsymbol{w} + \nabla p - \mathbf{\Pi}_h \boldsymbol{w}\| \le \|\nabla p\| + Ch|\boldsymbol{w}|_1 \\ &\le \|\nabla p\| + Ch\|\mathbf{curl}\,\boldsymbol{u}\| \le C\|\boldsymbol{u}\|. \end{aligned}$$

To deal with $\mathbf{\Pi}_h \boldsymbol{w}$ in (5.3), we first eliminate the low frequency components by subtracting $\mathbf{Q}_H^{\boldsymbol{U}} \boldsymbol{w}$ from $\boldsymbol{w}$, and get

$$\mathbf{\Pi}_h \boldsymbol{w} = (\mathbf{\Pi}_h \boldsymbol{w} - \mathbf{Q}_H^{\boldsymbol{U}} \boldsymbol{w}) + \mathbf{Q}_H^{\boldsymbol{U}} \boldsymbol{w} \equiv \boldsymbol{w}^h + \boldsymbol{w}_0, \tag{5.5}$$

By (3.1), (3.2) and (5.2), $\boldsymbol{w}_0$ and $\boldsymbol{w}^h$ satisfy,

$$\mathbf{A}(\boldsymbol{w}_0, \boldsymbol{w}_0) \le \alpha\|\boldsymbol{w}\| + C|\boldsymbol{w}|_1 \le C\mathbf{A}(\boldsymbol{u}, \boldsymbol{u}), \tag{5.6}$$

$$\|\boldsymbol{w}^h\| \le \|\mathbf{\Pi}_h \boldsymbol{w} - \boldsymbol{w}\| + \|\boldsymbol{w} - \mathbf{Q}_H^{\boldsymbol{U}} \boldsymbol{w}\| \le CH|\boldsymbol{w}|_1 \le CH\|\mathbf{curl}\,\boldsymbol{u}\|. \tag{5.7}$$

Alternatively, we have the bound

$$\begin{aligned} \|\boldsymbol{w}^h\| &\le \|\mathbf{\Pi}_h \boldsymbol{w} - \boldsymbol{w}\| + \|\boldsymbol{w} - \mathbf{Q}_H^{\boldsymbol{U}} \boldsymbol{w}\| \\ &\le C(h\|\mathbf{curl}\,\boldsymbol{u}\| + \|\boldsymbol{w}\|) \le C\|\boldsymbol{u}\|. \end{aligned} \tag{5.8}$$

Finally, by (5.3) and (3.2),

$$\begin{aligned} \|\mathbf{curl}\,\boldsymbol{w}^h\| &\le \|\mathbf{curl}\,\mathbf{\Pi}_h \boldsymbol{w}\| + \|\mathbf{curl}\,\mathbf{Q}_H^{\boldsymbol{U}} \boldsymbol{w}\| \\ &\le \|\mathbf{curl}\,\boldsymbol{u}\| + C|\boldsymbol{w}|_1 \le C\|\mathbf{curl}\,\boldsymbol{u}\|. \end{aligned} \tag{5.9}$$

The remainder $\boldsymbol{w}^h$ is decomposed in a classical way. We use the partition of unity $\{\theta_j\}_{j=1}^N$ introduced earlier and define $\boldsymbol{w}_j = \mathbf{\Pi}_h(\theta_j \boldsymbol{w}^h)$, for $j = 1, \dots, N$.

Using the fact that the partition functions $\{\theta_j\}$ are piecewise linear with respect to the fine grid mesh, it can be shown (cf., Lemma 4.5 in [21]) that

$$\|\mathbf{\Pi}_h(\theta_j \boldsymbol{w}^h)\| \le C\|\theta_j \boldsymbol{w}^h\| \qquad \text{and}$$
$$\|\mathbf{curl}\,\mathbf{\Pi}_h(\theta_j \boldsymbol{w}^h) \le C\|\mathbf{curl}\,\theta_j \boldsymbol{w}^h\|.$$

The argument given there uses the property that $\theta_j \boldsymbol{w}^h$ is a piecewise polynomial of fixed order.

Thus, we have

$$\|\boldsymbol{w}_j\| \le C\|\theta_j \boldsymbol{w}^h\| \le C\|\boldsymbol{w}^h\|_{\boldsymbol{L}^2(\Omega_j')},$$

and

$$\begin{aligned}\|\mathbf{curl}\,\boldsymbol{w}_j\| &\le C\|\mathbf{curl}\,\theta_j\boldsymbol{w}^h\|\\ &\le C(\|\nabla\theta_j\|_{L^\infty}\|\boldsymbol{w}^h\|_{\boldsymbol{L}^2(\Omega_j')}+\|\mathbf{curl}\,\boldsymbol{w}^h\|_{\boldsymbol{L}^2(\Omega_j')})\\ &\le C(H^{-1}\|\boldsymbol{w}^h\|_{\boldsymbol{L}^2(\Omega_j')}+\|\mathbf{curl}\,\boldsymbol{w}^h\|_{\boldsymbol{L}^2(\Omega_j')}).\end{aligned}$$

The above inequalities and the limited overlap property of the subdomains imply that

$$\begin{aligned}\sum_{j=1}^{N}\mathbf{A}(\boldsymbol{w}_j,\boldsymbol{w}_j) &\le C((\alpha+H^{-2})\|\boldsymbol{w}^h\|^2+\|\mathbf{curl}\,\boldsymbol{w}^h\|^2)\\ &\le C(\alpha\|\boldsymbol{u}\|+\|\mathbf{curl}\,\boldsymbol{u}\|^2)=CA(\boldsymbol{u},\boldsymbol{u}).\end{aligned}\tag{5.10}$$

The last inequality above followed from applying (5.7) and (5.8).

Finally, setting $\boldsymbol{u}_j=\boldsymbol{w}_j+\nabla p_j$ gives the desired decomposition of $\boldsymbol{u}$. Indeed, combining (5.4), (5.6), and (5.10) shows that

$$\begin{aligned}\sum_{j=0}^{N}\mathbf{A}(\boldsymbol{u}_j,\boldsymbol{u}_j) &\le 2\mathbf{A}(\boldsymbol{w}_0,\boldsymbol{w}_0)+2\sum_{j=1}^{N}\mathbf{A}(\boldsymbol{w}_j,\boldsymbol{w}_j)+2\sum_{j=0}^{N}\mathbf{A}(\nabla p_j,\nabla p_j)\\ &\le C\mathbf{A}(\boldsymbol{u},\boldsymbol{u}).\end{aligned}$$

This completes the proof of Lemma 4.1.

6. Numerical results

In this section we report the results of numerical experiments confirming and illustrating the theory of previous sections. All of the computations to be described use lowest order Nedelec elements on cubes.

The domain Ω is defined to be the three-dimensional domain $(0,1)^3/[0,1/2]^3$. On this domain, the solenoidal component of the Helmholtz decomposition is generally not in $\boldsymbol{H}^1(\Omega)$.

We take the coarse grid to be the 7 cubes of size $[0,1/2]^3$, whose union is the closure of Ω. Ω is meshed uniformly by cubic elements of size h. Overlapping subdomains are constructed by adjoining just enough fine elements to the coarse elements so that (4.1) holds.

Equation (1.1) with various α was solved using the preconditioned Conjugate Gradient method. For the additive and multiplicative preconditioners, the Conjugate Gradient method without preconditioning was used to solve the discrete problems on the coarse mesh and on the subdomains. The condition numbers of the preconditioned system as a function of h were obtained by using a Lanczos technique [12].

In table Table 6.1 and Table 6.2, we report the condition numbers of the preconditioned system as a function of h for various values of α using the additive Schwarz preconditioner (4.2) with $\delta=0.1$ and $\delta=0.2$, respectively. The results are uniform with respect to α and h. Note that larger values of δ yield better preconditioners.

α	10^{-4}	10^{-3}	10^{-2}	10^{-1}	1	10	10^2	10^3	10^4
$h = 1/4$	7.18	7.18	7.18	7.18	7.18	7.20	7.24	7.74	7.95
$h = 1/8$	7.78	7.78	7.77	7.77	7.71	7.20	7.01	7.05	7.07
$h = 1/16$	13.17	13.17	13.17	13.16	13.11	12.38	7.00	7.00	7.00
$h = 1/32$	13.24	13.24	13.24	13.23	13.18	12.43	7.01	7.00	7.00
$h = 1/64$	13.26	13.26	13.26	13.24	13.19	12.44	7.01	7.00	7.00

TABLE 6.1. Condition numbers of $\mathbf{B}_a \mathbf{A}_h^{\boldsymbol{U}}$ with $\delta = 0.1$.

α	10^{-4}	10^{-3}	10^{-2}	10^{-1}	1	10	10^2	10^3	10^4
$h = 1/4$	7.18	7.18	7.18	7.18	7.18	7.20	7.24	7.74	7.95
$h = 1/8$	7.78	7.78	7.77	7.77	7.71	7.20	7.01	7.05	7.07
$h = 1/16$	7.95	7.95	7.95	7.95	7.90	7.27	6.97	7.00	7.00
$h = 1/32$	7.91	7.91	7.91	7.91	7.86	7.26	6.98	7.00	7.00
$h = 1/64$	8.80	8.80	8.80	8.80	8.76	7.94	6.98	7.00	7.00

TABLE 6.2. Condition numbers of $\mathbf{B}_a \mathbf{A}_h^{\boldsymbol{U}}$ with $\delta = 0.2$.

α	10^{-4}	10^{-3}	10^{-2}	10^{-1}	1	10	10^2	10^3	10^4
$h = 1/4$	1.02	1.02	1.02	1.02	1.02	1.004	1.00025	1.005	1.008
$h = 1/8$	1.08	1.08	1.08	1.08	1.07	1.05	1.001	1.0007	1.005
$h = 1/16$	1.34	1.34	1.34	1.34	1.33	1.25	1.06	1.0002	1.002
$h = 1/32$	1.35	1.35	1.35	1.35	1.34	1.26	1.06	1.0002	1.
$h = 1/64$	1.35	1.35	1.35	1.35	1.34	1.26	1.09	1.00032	1.

TABLE 6.3. Condition numbers of $\mathbf{B}_m \mathbf{A}_h^{\boldsymbol{U}}$ with $\delta = 0.1$.

The condition numbers of the preconditioned system using multiplicative preconditioner (4.3) with $\delta = 0.1$ are given in Table 6.3. The multiplicative preconditioner performs better than the additive preconditioner in terms of the condition numbers. Indeed the condition numbers for large α end up being very close to one.

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