STABILITY OF DISCRETE STOKES OPERATORS IN FRACTIONAL SOBOLEV SPACES

JEAN-LUC GUERMOND[1,‡], JOSEPH E. PASCIAK[1]

ABSTRACT. Using a general approximation setting having the generic properties of finite-elements, we prove uniform boundedness and stability estimates on the discrete Stokes operator in Sobolev spaces with fractional exponents. As an application, we construct approximations for the time-dependent Stokes equations with a source term in $L^p(0,T;\mathbf{L}^q(\Omega))$ and prove uniform estimates on the time derivative and discrete Laplacian of the discrete velocity that are similar to those in Sohr and von Wahl [20].

1. INTRODUCTION

1.1. **Scope of the paper.** The objective of this paper is to construct approximations for the time-dependent Stokes equations with a source term in $L^p(0,T;\mathbf{L}^q(\Omega))$ and to prove uniform estimates on the discrete pressure and the time derivative and discrete Laplacian of the discrete velocity that are similar to those proved by Solonnikov [21] and Sohr and von Wahl [20]. To this purpose we construct a finite-element-like approximate Stokes operator and we prove norm equivalences between the scale of norms which it generates and the usual fractional order Sobolev norms for $-\frac{1}{2} < s < \frac{3}{2}$. The boundary condition under consideration is the homogeneous Dirichlet condition. By working with fractional exponents of the discrete Stokes operator and the Fourier technique in time we avoid the non-Hilbertian $L^p(\mathbf{L}^q)$-framework, which we do not know yet how to handle in finite-element-like discrete settings. This technique yields near optimal counterparts of the estimates of Sohr and von Wahl on the time derivative and discrete Laplacian of the discrete velocity. The main results summarizing the content of the paper are Theorem 4.1, and Theorem 5.1.

The paper is organized as follows. The rest of this section is devoted to introducing notation and recalling the definitions of the Leray projector and the Stokes operator. The discrete finite-element-like setting alluded to above is introduced in §2. Boundedness and invariance properties of the discrete

Date: Draft version: January 11, 2007, Submitted to Journal of Mathematical Fluid Mechanics.

1991 *Mathematics Subject Classification.* 35Q30, 65N35, 76M05.

Key words and phrases. Stokes operator, Finite elements, Navier–Stokes equations, pressure estimates, negative norms.

‡On long leave from LIMSI (CNRS-UPR 3251), BP 133, 91403, Orsay, France.

Leray projector are stated in §3. The discrete Stokes operator is analyzed in §4. The results of §4, in particular Theorem 4.1, are used to analyze the semi-discrete time-dependent Stokes operator in §5. Discrete counterparts of the estimates of Sohr and von Wahl using fractional Sobolev spaces are sated in Theorem 5.1.

The results presented in this paper are part of a research program aiming at characterizing the weak solutions of the three-dimensional Navier–Stokes equations that are suitable in the sense of Scheffer [18]. It has been shown in [10] that, in the three-dimensional torus, weak solutions that are constructed as limit of sequences of finite-element-like Galerkin approximations are suitable. The goal we are pursuing is to eventually extend this result to homogeneous Dirichlet conditions. One important intermediate step on the way are estimates like (5.23) and (5.24). A proof that finite-element-like Galerkin approximations are indeed suitable is reported in [12]. Theorem 5.1 (which is a consequence of Theorem 4.1) is an essential key for proving this result.

1.2. **Notation and conventions.** Let Ω be a connected, open, bounded domain in $\mathbb{R}^d$ (d is the space dimension). The boundary of Ω is assumed to be such that the H^2-regularity property of the Laplace operator holds, i.e., there is $c > 0$ such that

$$\forall v \in H^1_0(\Omega) \cap H^2(\Omega), \qquad \|v\|_{H^2} \le c\|\Delta v\|_{L^2}. \tag{1.1}$$

For instance, Ω convex or Ω of class $\mathcal{C}^{1,1}$ are sufficient conditions for this property to hold, cf. e.g. [9]. The boundary of Ω is denoted by Γ.

We use bold notation to denote the product space with $d-$components in a given space, e.g. $\mathbf{H}^1(\Omega) = (H^1(\Omega))^d$, but no notational distinction is made between $\mathbb{R}$-valued and $\mathbb{R}^d$-valued functions. Whenever E is a normed space, $\|\cdot\|_E$ denotes a norm in E. Whenever E is a Hilbert space, $(\cdot,\cdot)_E$ denotes the scalar product in E. The scalar product in $L^2(\Omega)$ and $\mathbf{L}^2(\Omega)$ is simply denoted by $(\cdot,\cdot)$. Henceforth c is a generic constant. The symbol $c_u(\cdot)$ denotes a generic positive non decreasing function. The symbol $c_l(\cdot)$ denotes a generic positive non increasing function. Both the generic constant c and the generic functions c_u and c_l are independent of the mesh parameter h. The value of c and the exact form of c_u and c_l may vary at each occurrence.

For $0 < s < 1$, the space $H^s(\Omega)$ is defined by the real method of interpolation between $H^1(\Omega)$ and $L^2(\Omega)$, i.e., the so-called K-method of Lions and Peetre [17], see also [16] or [3, Appendix A]. To define $H^s(\Omega)$, we interpolate between $H^1(\Omega)$ and $H^2(\Omega)$ if $1 < s < 2$. We denote $H^s_0(\Omega)$ to be the closure of $\mathcal{C}^\infty_0(\Omega)$ in $H^s(\Omega)$ for $0 < s < 1$ and $\widetilde{H}^s_0(\Omega)$ to be the interpolation space $[H^1_0(\Omega), L^2(\Omega)]_s$ for $0 \le s \le 1$. For $s \in (1,2]$, $\widetilde{H}^s_0(\Omega)$ is defined to be $H^s(\Omega) \cap H^1_0(\Omega)$. Note that the spaces $H^s(\Omega)$ and $H^s_0(\Omega)$ coincide for $0 \le s \le \frac{1}{2}$ with uniformly equivalent norms (see [16, Thm 11.1]). The spaces $H^s(\Omega)$ and $\widetilde{H}^s_0(\Omega)$ coincide for $0 \le s < \frac{1}{2}$ and their norms are equivalent;

i.e., there is $c_1 > 0$ and a non-decreasing function c_u such that

$$c_1\|v\|_{\mathbf{H}^s} \le \|v\|_{\widetilde{\mathbf{H}}^s_0} \le c_u(s)\|v\|_{\mathbf{H}^s}, \qquad \forall v \in \widetilde{\mathbf{H}}^s_0,\ \forall s \in [0, \tfrac{1}{2}), \tag{1.2}$$

with $\lim_{s\to\frac{1}{2}} c_u(s) = \infty$, see [16, Thm 11.7].

For negative s, $\widetilde{H}^s_0(\Omega)$ is the dual of $\widetilde{H}^{-s}_0(\Omega)$. The space $H^{-s}(\Omega)$ for $s > 0$ is defined by duality, i.e.,

$$\|v\|_{H^{-s}} = \sup_{0\neq w\in\mathcal{C}^\infty_0(\Omega)} \frac{(v, w)}{\|w\|_{H^s}}.$$

For $s \in [0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{2})$, H^{-s} coincides with $\widetilde{H}^{-s}_0(\Omega)$.

We define $H^s_{\int=0}(\Omega)$, $s \in [0, 1]$, to be composed of those functions in $H^s(\Omega)$ that are of zero mean. It can be shown that $H^s_{\int=0}(\Omega) = [L^2_{\int=0}(\Omega), H^1_{\int=0}(\Omega)]_s$, for all $s \in [0, 1]$, cf. e.g. [11].

1.3. **The Leray projector.** Following [14, 22] we define

$$\boldsymbol{\mathcal{V}} = \{v \in \boldsymbol{\mathcal{C}}^\infty_0(\Omega);\ \nabla\cdot v = 0\} \tag{1.3}$$

to account for solenoidal vector fields, and we set

$$\mathbf{V}^0 = \overline{\boldsymbol{\mathcal{V}}}^{\mathbf{L}^2}, \qquad \mathbf{V}^1 = \overline{\boldsymbol{\mathcal{V}}}^{\mathbf{H}^1}, \qquad \mathbf{V}^2 = \mathbf{V}^1 \cap \mathbf{H}^2(\Omega). \tag{1.4}$$

The following characterizations of $\mathbf{V}^0$ and $\mathbf{V}^1$ hold, cf. [22],

$$\mathbf{V}^0 = \{v \in \mathbf{L}^2(\Omega);\ \nabla\cdot v = 0;\ v\cdot n|_\Gamma = 0\}, \tag{1.5}$$

$$\mathbf{V}^1 = \{v \in \mathbf{H}^1(\Omega);\ \nabla\cdot v = 0;\ v|_\Gamma = 0\}. \tag{1.6}$$

$\mathbf{V}^0$ is a closed subspace of $\mathbf{L}^2(\Omega)$ and the following well known Helmholtz decomposition holds, see e.g. [14, 22]

$$\mathbf{L}^2(\Omega) = \mathbf{V}^0 \overset{\perp}{\oplus} \nabla H^1_{\int=0}(\Omega). \tag{1.7}$$

We denote by $P : \mathbf{L}^2(\Omega) \longrightarrow \mathbf{V}^0$ the $\mathbf{L}^2$-projection onto $\mathbf{V}^0$ (i.e., the so-called Leray projection).

Lemma 1.1. *There is $c > 0$ such that for all $s \in [0, 1]$,*

$$\forall v \in \widetilde{\mathbf{H}}^s_0(\Omega), \qquad \|Pv\|_{\mathbf{H}^s} \le c\|v\|_{\widetilde{\mathbf{H}}^s_0}. \tag{1.8}$$

Proof. Let $v \in \mathbf{L}^2(\Omega)$. Define $q \in H^1_{\int=0}(\Omega)$ be such that $(\nabla q, \nabla r) = (v, \nabla r)$ for all r in $H^1_{\int=0}(\Omega)$. It is clear that

$$\|q\|_{H^1} \le c\|v\|_{\mathbf{L}^2}.$$

Assume moreover that $v \in \mathbf{H}^1_0(\Omega)$, then q solves the homogeneous Neumann problem

$$\Delta q = \nabla\cdot v, \quad \partial_n q|_\Gamma = 0.$$

Owing to the regularity hypothesis on Ω, the regularity theory of elliptic operators implies

$$\|q\|_{H^2} \le c\|v\|_{\mathbf{H}^1}.$$

Then, by interpolation, we obtain

$$\|q\|_{H^{1+s}} \le c\|v\|_{\widetilde{\mathbf{H}}_0^s}, \quad \forall v \in \widetilde{\mathbf{H}}_0^s(\Omega).$$

Then we define $Pv = v - \nabla q$. The triangle inequality yields $\|Pv\|_{\mathbf{H}^s} \le \|v\|_{\mathbf{H}^s} + \|\nabla q\|_{\mathbf{H}^s} \le c\|v\|_{\widetilde{\mathbf{H}}_0^s}$. □

1.4. **The Stokes operator.** Let us define the unbounded vector-valued Laplace operator $-\Delta : \mathbf{D}(\Delta){:=}\mathbf{H}_0^1(\Omega)\cap\mathbf{H}^2(\Omega) \longrightarrow \mathbf{L}^2(\Omega)$. We introduce the so-called Stokes operator $A : \mathbf{D}(A){:=}\mathbf{V}^2 \longrightarrow \mathbf{V}^0$ by setting $A = -P\Delta|_{\mathbf{V}^2}$. We assume that the domain Ω is such that there is $c > 0$

$$\forall v \in \mathbf{V}^2, \qquad \|v\|_{\mathbf{H}^2} \le c\,\|Av\|_{\mathbf{L}^2}. \tag{1.9}$$

This property holds in two and three dimensions ($d = 2,\, 3$) whenever Ω is convex or of class $\mathcal{C}^{1,1}$, see [6, Thm 6.3].

It follows from (1.9) that A is closed. Moreover, it is positive and self-adjoint and its inverse is compact. We denote by $(\phi_k, \lambda_k)_{k\ge1}$ the eigenpairs of A so that the family $(\phi_k)_{k\ge1}$ forms an orthonormal basis for $\mathbf{V}^0$. Following [7] we define

$$\mathbf{E} = \left\{ v = \textstyle\sum_{k=1}^N v_k\phi_k;\ N \in \mathbb{N};\ (v_1,\ldots,v_N) \in \mathbb{R}^N \right\}, \tag{1.10}$$

and for all $s \in \mathbb{R}$ we denote by $\mathbf{E}^s$ the completion of $\mathbf{E}$ in the norm

$$\left(\textstyle\sum_{j=1}^\infty \lambda_k^s |v_k|^2\right)^{\frac12} = (A^s v, v)^{\frac12}. \tag{1.11}$$

It is clear that $\mathbf{E}^s = \mathbf{V}^s$, for $s = 0,1,2$. We henceforth introduce the notation $\mathbf{V}^s := \mathbf{E}^s$ for all $s \in \mathbb{R}$ and we set $\|v\|_{\mathbf{V}^s} := (A^s v, v)^{\frac12}$. Using the K-interpolation method, it can be shown also that $\{\mathbf{V}^s\}_{s\in\mathbb{R}}$ forms an Hilbert scale.

2. The discrete setting and preliminaries

We introduce in this section a discrete approximation setting and we prepare the ground for the main result of §4, i.e., Theorem 4.1

2.1. **The discrete setting.** We assume that we have at hand two families of finite-dimensional spaces, $\{\mathbf{X}_h\}_{h>0}$ and $\{M_h\}_{h>0}$ such that $\mathbf{X}_h \subset \mathbf{H}_0^1(\Omega)$ and $M_h \subset L^2_{\int=0}(\Omega)$. To avoid irrelevant technicalities we assume $M_h \subset H^1_{\int=0}(\Omega)$.

The spaces $\{\mathbf{X}_h\}_{h>0}$ and $\{M_h\}_{h>0}$ have approximating properties in the sense that there is a constant c uniform in h such that for all $\forall l,\, s$, $0 \le l \le \min(1, s)$, $l \le s \le 2$,

$$\forall v \in \widetilde{\mathbf{H}}_0^s(\Omega), \qquad \inf_{v_h\in\mathbf{X}_h} \|v - v_h\|_{\widetilde{\mathbf{H}}_0^l} \le c\,h^{s-l}\|v\|_{\widetilde{\mathbf{H}}_0^s} \tag{2.1}$$

$$\forall q \in \mathbf{H}^s_{\int=0}(\Omega), \qquad \inf_{q_h\in M_h} \|q - q_h\|_{L^2} \le c\,h^s\|q\|_{H^s}. \tag{2.2}$$

We moreover assume that the following inverse inequality holds: There is a positive non-decreasing function c_u, uniform in h, such that for all $s \in [0, \frac{3}{2})$

$$\forall v_h \in \mathbf{X}_h, \qquad \|v_h\|_{\widetilde{\mathbf{H}}^s_0} \le c_u(s) h^{-s} \|v_h\|_{\mathbf{L}^2}. \tag{2.3}$$

We assume also that the L^2-projection $\pi_h : \mathbf{L}^2(\Omega) \longrightarrow \mathbf{X}_h$ onto $\mathbf{X}_h$ is stable on $\widetilde{\mathbf{H}}^s_0(\Omega)$, for $0 \le s < \frac{3}{2}$, i.e., there is a positive non-decreasing function c_u, uniform in h, so that

$$\forall v \in \widetilde{\mathbf{H}}^s_0(\Omega), \qquad \|\pi_h v\|_{\widetilde{\mathbf{H}}^s_0} \le c_u(s)\, \|v\|_{\widetilde{\mathbf{H}}^s_0}, \tag{2.4}$$

for all $s \in [0, \frac{3}{2})$.

Remark 2.1. The hypotheses (2.3) is usually satisfied when $\mathbf{X}_h$, and M_h are constructed by using finite elements on quasi-uniform meshes. By redoing carefully the computation in [2, Appendix] we can show that $c_u(s) \sim (1 - 2s)^{-\frac{1}{2}}$. The hypothesis (2.4) can be proved to hold on quasi-uniform meshes also by using Lemma A.3 with $\rho_h = \pi_h$ and $T_h = \pi_h$.

2.2. **Discrete projections and Laplace operator.** Let $E_h : \mathbf{H}^1_0(\Omega) \longrightarrow \mathbf{X}_h$ be the so-called elliptic projection defined by

$$(\nabla E_h x, \nabla x_h) = (\nabla x, \nabla x_h), \quad \forall x \in \mathbf{H}^1(\Omega),\ \forall x_h \in \mathbf{X}_h. \tag{2.5}$$

Lemma 2.1. *There is c independent of h such that $\forall l, s$, $0 \le l \le \min(1, s)$, $l \le s \le 2$,*

$$\|E_h x - x\|_{\widetilde{\mathbf{H}}^l_0} \le c\, h^{s-l} \|x\|_{\widetilde{\mathbf{H}}^s_0}, \quad \forall x \in \widetilde{\mathbf{H}}^s_0(\Omega). \tag{2.6}$$

Proof. This is a standard result when l is integer; see e.g. [8]. The result follows by interpolation for non-integer l in $(0, 1)$. □

We also assume that the family $(\mathbf{X}_h)_{h>0}$ is such that E_h is uniformly $\mathbf{H}^s$-stable for $s \in [1, \frac{3}{2})$, i.e., there is a positive non-decreasing function c_u, independent of h, such that

$$\|E_h v\|_{\widetilde{\mathbf{H}}^s_0} \le c_u(s)\, \|\mathbf{v}\|_{\widetilde{\mathbf{H}}^s_0}, \tag{2.7}$$

for all v in $\widetilde{\mathbf{H}}^s_0(\Omega)$. When the spaces $(\mathbf{X}_h)_{h>0}$ are finite-element-based, this assumption is known to hold under quite weak regularity requirements on the underlying mesh family, see [2] or Lemma A.3 with $\rho_h = \pi_h$ and $T_h = E_h$.

We define the discrete Laplace operator $\Delta_h : \mathbf{X}_h \longrightarrow \mathbf{X}_h$ as follows:

$$(\Delta_h x_h, y_h) = -(\nabla x_h, \nabla y_h), \quad \forall x_h, y_h \in \mathbf{X}_h.$$

Clearly the four operators Δ_h, E_h, π_h, and Δ are related by

$$\Delta_h E_h x = \pi_h \Delta x, \quad \forall x \in \mathbf{D}(\Delta). \tag{2.8}$$

In other words the following diagram commutes:

$$
\begin{array}{ccc}
\mathbf{D}(\Delta) & \xrightarrow{\ \Delta\ } & \mathbf{L}^2(\Omega) \\
\Big\downarrow E_h & & \Big\downarrow \pi_h \\
\mathbf{X}_h & \xrightarrow{\ \Delta_h\ } & \mathbf{X}_h.
\end{array}
\tag{2.9}
$$

The operator $-\Delta_h$ is self-adjoint and positive definite so we can define $(-\Delta_h)^s$ for all $s \in \mathbb{R}$ and the following norm makes sense

$$\|v_h\|_{\mathbf{X}_h^s} := ((-\Delta_h)^s v_h, v_h)^{\frac{1}{2}}, \quad \forall v_h \in \mathbf{X}_h. \tag{2.10}$$

We denote by $\mathbf{X}_h^s$ the vector space $\mathbf{X}_h$ equipped with this norm. $\mathbf{X}_h^s$ is clearly a Hilbert space. The family $\{\mathbf{X}_h^s\}_{s\in\mathbb{R}}$ is a Hilbert scale in the sense of Lions and Peetre [17], [16], [3, Appendix A].

Lemma 2.2. *Under the above assumptions, there is a positive non-increasing function c_l and there is a positive non-decreasing function c_u, both uniform in h, such that for all $s \in (-\frac{3}{2}, \frac{3}{2})$,*

$$\forall v_h \in \mathbf{X}_h, \quad c_l(|s|)\,\|v_h\|_{\widetilde{\mathbf{H}}_0^s} \le \|v_h\|_{\mathbf{X}_h^s} \le c_u(|s|)\,\|v_h\|_{\widetilde{\mathbf{H}}_0^s}. \tag{2.11}$$

For completeness, we sketch a proof of this result in Appendix A.

2.3. **Compatibility between $\mathbf{X}_h$ and M_h.** We assume that $\mathbf{X}_h$ and M_h are compatible in the sense that there is a constant $c > 0$ independent of h such that

$$\forall q_h \in M_h, \qquad \|\pi_h \nabla q_h\|_{\mathbf{L}^2} \ge c\|\nabla q_h\|_{L^2}. \tag{2.12}$$

This inequality can also be equivalently rewritten as

$$\forall q_h \in M_h, \qquad \sup_{0\neq v_h\in\mathbf{X}_h} \frac{(\nabla q_h, v_h)}{\|v_h\|_{\mathbf{L}^2}} \ge c\|\nabla q_h\|_{\mathbf{L}^2}. \tag{2.13}$$

A first consequence of this hypothesis is that $\mathbf{X}_h$ and M_h satisfy the so-called LBB condition, see e.g. [8].

Lemma 2.3. *Assume that (2.1) holds with $l = 0$, $s = 1$, and (2.3) holds with $s = 1$. Then (2.12) implies that there is a constant c independent of h such that*

$$\inf_{q_h\in M_h} \sup_{0\neq v_h\in\mathbf{X}_h} \frac{(q_h, \nabla\cdot v_h)}{\|q_h\|_{L^2}\|v_h\|_{\mathbf{H}^1}} \ge c. \tag{2.14}$$

Proof. See the proof of Lemma 2.1 in [10]. The operator $\mathcal{C}_h$ can be e.g. the Clément interpolation operator [5] or the Scott-Zhang operator [19]. □

Lemma 2.4. *Hypothesis (2.12) holds in either one of the following situations:*

(i) $\mathbf{X}_h$ *is composed of* $\mathbb{P}_1$*–Bubble* H^1*-conforming finite elements and* M_h *is composed of* $\mathbb{P}_1$ H^1*-conforming finite elements (i.e., the so-called MINI element).*

(ii) $\mathbf{X}_h$ *is composed of* $\mathbb{P}_2$ H^1*-conforming finite elements and* M_h *is composed of* $\mathbb{P}_1$ H^1*-conforming finite elements (i.e., the so-called Hood-Taylor element), and no tetrahedron has more than 3 edges on* $\partial\Omega$.

Proof. See the proof of Lemma 2.2 in [10]. □

2.4. **The Discrete Leray projection and Stokes operator.** We now define the space $\mathbf{V}_h$ to be the set of discretely divergence free vectors, i.e.,

$$\mathbf{V}_h = \{v_h \in \mathbf{X}_h;\ (v_h, \nabla q_h) = 0,\ \forall q_h \in M_h\}. \tag{2.15}$$

Then let $P_h : \mathbf{L}^2(\Omega) \longrightarrow \mathbf{V}_h$ be the L^2-projection onto $\mathbf{V}_h$. P_h is a discrete version of the Leray projection.

We also introduce the mapping $R_h : \mathbf{H}^1(\Omega) \longrightarrow \mathbf{V}_h$ defined by

$$(\nabla R_h v, \nabla v_h) = (\nabla v, \nabla v_h), \quad \forall v_h \in \mathbf{V}_h. \tag{2.16}$$

Lemma 2.5. *Under the hypotheses of Lemma 2.3, there is a constant* c *independent of* h *such that*

$$\forall v \in \mathbf{V}^2, \qquad \|R_h v - v\|_{\mathbf{H}^1} \le c\, h \|v\|_{\mathbf{H}^2}. \tag{2.17}$$

Proof. (2.17) is a standard result; see e.g. [8, 4]. □

We now define the discrete Stokes operator $A_h : \mathbf{V}_h \longrightarrow \mathbf{V}_h$ as follows: For all $u_h \in \mathbf{V}_h$, $A_h u_h$ is the element of $\mathbf{V}_h$ such that

$$(A_h u_h, v_h) = (\nabla u_h, \nabla v_h), \qquad \forall v_h \in \mathbf{V}_h. \tag{2.18}$$

Note that $A_h = -P_h \Delta_h|_{\mathbf{V}_h}$. Observe that the four operators A_h, R_h, P_h, and Δ are related by

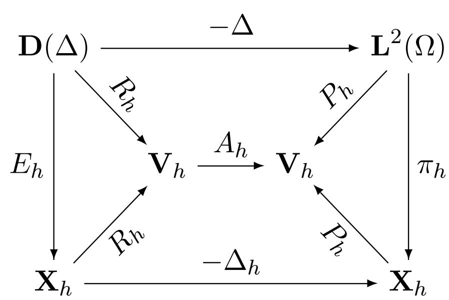
$$A_h R_h v = -P_h \Delta v, \quad \forall v \in D(\Delta). \tag{2.19}$$

An identical argument shows that

$$A_h R_h x_h = -P_h \Delta_h x_h, \quad \forall x_h \in \mathbf{X}_h. \tag{2.20}$$

In other words the following diagram commutes

(2.21)



Since A_h is self-adjoint and positive definite, the operator A_h^s is well defined for all $s \in \mathbb{R}$. We equip the vector space $\mathbf{V}_h$ with the norm

$$\|v_h\|_{\mathbf{V}_h^s} = (A_h^s v_h, v_h)^{\frac{1}{2}}, \tag{2.22}$$

and we denote by $\mathbf{V}_h^s$ the corresponding normed (Hilbert) space. Using the so-called K-interpolation method of Lions and Peetre [17], [16], [3, Appendix A], it is clear that $\{\mathbf{V}_h^s\}_{s\in\mathbb{R}}$ is a scale.

3. Properties of the discrete Leray projection

The goal of this section is to provide a preliminary result concerning P_h that will be used in the proof of Theorem 4.1. The main result of this section is Lemma 3.1.

Lemma 3.1. *There is a positive non-decreasing function c_u, independent of h, such that*

$$\forall v \in \mathbf{H}^s(\Omega), \qquad \|P_h v\|_{\widetilde{\mathbf{H}}^s_0} \le c_u(s)\|v\|_{\widetilde{\mathbf{H}}^s_0}, \qquad \forall s \in [0, \tfrac{1}{2}). \tag{3.1}$$

Proof. Let v be a member of $\mathbf{H}^s(\Omega) = \widetilde{\mathbf{H}}^s_0(\Omega)$ for $0 \le s < \frac{1}{2}$. Let Pv be the $\mathbf{L}^2$-projection of v onto $\mathbf{V}^0$. There is $q \in H^1_{\int=0}(\Omega)$ such that

$$\begin{aligned} &(Pv, l) + (\nabla q, l) = (v, l), && \forall l \in \mathbf{L}^2(\Omega),\\ &(\nabla r, Pv) = 0, && \forall r \in H^1_{\int=0}(\Omega). \end{aligned}$$

The above problem is clearly a well posed mixed problem.

Let $(v_h, q_h) \in \mathbf{X}_h \times M_h$ solve

$$\begin{aligned} &(v_h, l_h) + (\nabla q_h, l_h) = (v, l_h), && \forall l_h \in \mathbf{X}_h,\\ &(\nabla r_h, v_h) = 0, && \forall r_h \in M_h. \end{aligned}$$

This is a stable mixed problem by (2.12). Clearly, $v_h = P_h v$. Thus $P_h v$ and q_h are the mixed approximations of Pv and q, respectively. Owing to (2.13) the approximation theory of mixed problems yields (see e.g. [8, 4])

$$\|Pv - P_h v\|_{\mathbf{L}^2} + \|q - q_h\|_{H^1} \le c(\inf_{w_h \in \mathbf{X}_h} \|Pv - w_h\|_{\mathbf{L}^2} + \inf_{r_h \in M_h} \|q - r_h\|_{H^1}).$$

Since v is in $\mathbf{H}^s(\Omega)$, Lemma 1.1 implies $Pv \in \mathbf{H}^s(\Omega) = \widetilde{\mathbf{H}}^s_0(\Omega)$, $0 \le s < \frac{1}{2}$. The approximation hypotheses (2.1)–(2.2) together with the norm equivalence (1.2) then yield

$$\begin{aligned} \|Pv - P_h v\|_{\mathbf{L}^2} &\le c\, h^s(\|Pv\|_{\widetilde{\mathbf{H}}^s_0} + \|q\|_{H^{s+1}})\\ &\le c\, h^s(c_u(s)\|Pv\|_{\mathbf{H}^s} + \|q\|_{H^{s+1}}) \le c_u(s)\, h^s \|v\|_{\widetilde{\mathbf{H}}^s_0}. \end{aligned}$$

Then using the above approximation result together with the approximation and stability properties of π_h and the inverse inequality (2.3), we infer

$$\begin{aligned} \|P_h v\|_{\widetilde{\mathbf{H}}^s_0} &\le \|P_h v - \pi_h P v\|_{\widetilde{\mathbf{H}}^s_0} + \|\pi_h P v\|_{\widetilde{\mathbf{H}}^s_0}\\ &\le c\, h^{-s}\|P_h v - \pi_h P v\|_{\mathbf{L}^2} + c'\|Pv\|_{\widetilde{\mathbf{H}}^s_0}\\ &\le c\, h^{-s}(\|P_h v - Pv\|_{\mathbf{L}^2} + \|Pv - \pi_h P v\|_{\mathbf{L}^2}) + c'\|Pv\|_{\widetilde{\mathbf{H}}^s_0}\\ &\le c_u(s)\, \|v\|_{\widetilde{\mathbf{H}}^s_0}. \end{aligned} \tag{3.2}$$

This completes the proof. □

Remark 3.1. Observe that the above result does not hold for $s \ge \frac{1}{2}$. Even if v is in $\widetilde{\mathbf{H}}^s_0$, Pv is not in $\widetilde{\mathbf{H}}^s_0$ in general if $s \ge \frac{1}{2}$, i.e., the boundary conditions are lost on Pv (the normal component of Pv is zero, but the tangential component is not zero). On the other hand, observe that $P_h v \in \mathbf{X}_h \subset \mathbf{H}^1_0(\Omega)$. Hence, in general $P_h v \in \widetilde{\mathbf{H}}^s_0$ but $Pv \notin \widetilde{\mathbf{H}}^s_0$ when $s \geq \frac{1}{2}$. This boundary value incompatibility implies that for all $s \geq \frac{1}{2}$, $\|Pv - P_h v\|_{\mathbf{L}^2} \leq c(\epsilon) h^{\frac{1}{2}-\epsilon} \|v\|_{\widetilde{\mathbf{H}}^s_0}$, $\forall \epsilon > 0$, is the best estimate that can be obtained in general.

4. Properties of the discrete Stokes operator

The main result of this section is embodied in Theorem 4.1.

4.1. Stability properties of R_h on $\mathbf{X}_h$.

We first derive a discrete counterpart of (1.9).

Lemma 4.1. *There is c independent of h such that*

$$\|\Delta_h v_h\|_{\mathbf{L}^2} \leq c\,\|A_h v_h\|_{\mathbf{L}^2}, \quad \forall v_h \in \mathbf{V}_h. \tag{4.1}$$

Proof. Let v_h be a member of $\mathbf{V}_h$. Let $(v,p) \in \mathbf{H}^1_0(\Omega) \times L^2_{\int=0}(\Omega)$ be the solution of the Stokes problem with data $A_h v_h$, i.e.,

$$\begin{aligned} &(\nabla v, \nabla l) + (p, \nabla\cdot l) = (A_h v_h, l), && \forall l \in \mathbf{H}^1_0(\Omega),\\ &(\nabla\cdot v, q) = 0, && \forall q \in L^2_{\int=0}(\Omega). \end{aligned}$$

Let $(w_h, r_h) \in \mathbf{X}_h \times M_h$ be the solution to

$$\begin{aligned} &(\nabla w_h, \nabla l_h) + (r_h, \nabla\cdot l_h) = (A_h v_h, l_h), && \forall l_h \in \mathbf{X}_h,\\ &(\nabla\cdot w_h, q_h) = 0, && \forall q_h \in M_h. \end{aligned}$$

Clearly $w_h \in \mathbf{V}_h$ and actually $w_h = v_h$. This means that v_h is the Galerkin approximation to v. The theory of mixed problems together with (2.14) implies

$$\|v - v_h\|_{\mathbf{H}^1} \leq c\,h(\|v\|_{\mathbf{H}^2} + \|p\|_{H^1}) \leq c\,h\|A_h v_h\|_{\mathbf{L}^2}.$$

We then have for $x_h \in \mathbf{X}_h$,

$$\begin{aligned} |(\nabla v_h, \nabla x_h)| &\leq |(\nabla(v_h - v), \nabla x_h)| + |(\Delta v, x_h)|\\ &\leq c(h\|x_h\|_{\mathbf{H}^1} + \|x_h\|_{\mathbf{L}^2})\|A_h v_h\|_{\mathbf{L}^2} \leq c\,\|x_h\|_{\mathbf{L}^2}\|A_h v_h\|_{\mathbf{L}^2}. \end{aligned}$$

Thus,

$$\|\Delta_h v_h\|_{\mathbf{L}^2} = \sup_{0\neq x_h \in \mathbf{X}_h} \frac{(\nabla v_h, \nabla x_h)}{\|x_h\|_{\mathbf{L}^2}} \leq c\,\|A_h v_h\|_{\mathbf{L}^2},$$

which completes the proof of the lemma. □

We now turn our attention to the discrete operator R_h. It is obvious that R_h is stable on $\mathbf{H}^1(\Omega)$. The following lemma shows that it is also stable in $\mathbf{X}_h$ in the $\mathbf{L}^2(\Omega)$-norm.

Lemma 4.2. *There is c independent of h such that*

$$\|R_h x_h\|_{\mathbf{L}^2} \leq c\,\|x_h\|_{\mathbf{L}^2}, \quad \forall x_h \in \mathbf{X}_h. \tag{4.2}$$

Proof. By (2.21), $A_hR_h = -P_h\Delta_h$ when R_h and P_h are restricted to $\mathbf{X}_h$. It follows that $R_h = -A_h^{-1}P_h\Delta_h$ and so (4.2) will follow if we show

$$\|R_h^*v_h\|_{\mathbf{L}^2} \leq c\,\|v_h\|_{\mathbf{L}^2}, \quad \forall v_h \in \mathbf{V}_h.$$

Here $R_h^* : \mathbf{V}_h \to \mathbf{X}_h$ is the adjoint of R_h and is given by $R_h^* = -\Delta_h A_h^{-1}$. The above inequality is equivalent to proving

$$\|\Delta_h v_h\|_{\mathbf{L}^2} \leq c\,\|A_h v_h\|_{\mathbf{L}^2}, \quad \forall v_h \in \mathbf{V}_h,$$

which is exactly (4.1) in Lemma 4.1. □

Remark 4.1. Somewhat similar forms of Lemmas 4.1, 4.2 can also be found in Heywood and Rannacher [13, §4].

4.2. **Comparing $\widetilde{\mathbf{H}}_0^s$- and $\mathbf{V}_h^s$-norms.** The following theorem is the major result of this section.

Theorem 4.1. *There is is positive function $c_l > 0$, non-decreasing for negative arguments and non-increasing for positive arguments, and a positive non-decreasing function $c_u > 0$, both independent of h, such that the following holds for all v_h in $\mathbf{V}_h$:*

(4.3)
$$c_l(s)\|v_h\|_{\widetilde{\mathbf{H}}_0^s} \leq \|v_h\|_{\mathbf{V}_h^s} \leq c_u(|s|)\|v_h\|_{\widetilde{\mathbf{H}}_0^s}, \quad \begin{cases} -\frac{1}{2} < s < \frac{3}{2}, & \textit{lower bound,} \\ -\frac{3}{2} < s < \frac{3}{2}, & \textit{upper bound.} \end{cases}$$

Proof. Step (1). Clearly, for $v_h \in V_h$, (4.1) means $\|v_h\|_{\mathbf{X}_h^2} \leq c\,\|v_h\|_{\mathbf{V}_h^2}$ and $\|v_h\|_{\mathbf{X}_h^0} \leq \|v_h\|_{\mathbf{V}_h^0}$ is evident. The lower bound in (4.3) for $0 \leq s < \frac{3}{2}$ follows by interpolation and (2.11).

Step (2). Applying this bound, we observe that for $-\frac{3}{2} < s \leq 0$,

$$\|v_h\|_{\mathbf{V}_h^s} = \sup_{0\neq\theta\in V_h} \frac{(v_h,\theta)}{\|\theta\|_{\mathbf{V}_h^{-s}}} \leq \frac{1}{c_l(|s|)} \sup_{0\neq\theta\in V_h} \frac{(v_h,\theta)}{\|\theta\|_{\widetilde{\mathbf{H}}_0^{-s}}} \leq c_u(|s|)\|v_h\|_{\widetilde{\mathbf{H}}_0^s}.$$

This is the upper bound for $-\frac{3}{2} < s \leq 0$.

Step (3). To prove the upper bound for $0 \leq s \leq \frac{3}{2}$, we observe that for all $x_h \in \mathbf{X}_h$

$$\|R_hx_h\|_{\mathbf{V}_h^2} = \|A_hR_hx_h\|_{\mathbf{L}^2} = \|P_h\Delta_hx_h\|_{\mathbf{L}^2} \leq \|\Delta_hx_h\|_{\mathbf{L}^2} = \|x_h\|_{\mathbf{X}_h^2}.$$

Moreover $\|R_hx_h\|_{\mathbf{V}_h^0} \leq c\|x_h\|_{\mathbf{X}_h^0}$ owing to Lemma 4.2. By interpolation this gives $\|R_hx_h\|_{\mathbf{V}_h^s} \leq c\|x_h\|_{\mathbf{X}_h^s}$ for $s \in [0,2]$. By applying this result to $v_h \in \mathbf{V}_h$, we infer $\|v_h\|_{\mathbf{V}_h^s} \leq c\|x_h\|_{\mathbf{X}_h^s}$. Then we conclude using the upper bound in (2.11) for $s \in [0, \frac{3}{2})$.

Step (4). Finally, we prove the lower bound for $-\frac{1}{2} < s \leq 0$. Let v_h be in $\mathbf{V}_h$, then

$$\|v_h\|_{\widetilde{\mathbf{H}}_0^s} = \sup_{0\neq x\in\widetilde{\mathbf{H}}_0^{-s}(\Omega)} \frac{(v_h,x)}{\|x\|_{\widetilde{\mathbf{H}}_0^{-s}}} = \sup_{0\neq x\in\widetilde{\mathbf{H}}_0^{-s}} \frac{(v_h,P_hx)}{\|x\|_{\widetilde{\mathbf{H}}_0^{-s}}}.$$

The key estimate in Lemma 3.1 then implies

$$\|v_h\|_{\widetilde{\mathbf{H}}_0^s} \leq c_u(|s|) \sup_{0\neq x\in\mathbf{H}^{-s}} \frac{(v_h, P_h x)}{\|P_h x\|_{\widetilde{\mathbf{H}}_0^{-s}}} \leq c_u(|s|) \sup_{0\neq x_h\in\mathbf{V}_h} \frac{(v_h, x_h)}{\|x_h\|_{\widetilde{\mathbf{H}}_0^{-s}}}$$
$$\leq c_u(|s|)\|v_h\|_{\mathbf{V}_h^s} \sup_{0\neq x_h\in\mathbf{V}_h} \frac{\|x_h\|_{\mathbf{V}_h^{-s}}}{\|x_h\|_{\widetilde{\mathbf{H}}_0^{-s}}} \leq c_u(|s|)\|v_h\|_{\mathbf{V}_h^s}$$

where we used the upper estimate in (4.3) for the last inequality. This completes the proof of the theorem. □

Similarly we have the following

Corollary 4.1. *There is positive non-increasing function $c_l > 0$ and a positive non-decreasing function $c_u > 0$, both independent of h, such that for all $s \in (-\frac{3}{2}, 0]$*

$$c_l(|s|)\|\Delta_h v_h\|_{\widetilde{\mathbf{H}}_0^s} \leq \|A_h v_h\|_{\mathbf{V}_h^s} \leq c_u(|s|)\|\Delta_h v_h\|_{\widetilde{\mathbf{H}}_0^s}, \quad \forall v_h \in \mathbf{V}_h. \tag{4.4}$$

Proof. Let v_h be a member of $\mathbf{V}_h$. By reasoning as in step (1) of the proof of Theorem 4.1, we infer $\|v_h\|_{\mathbf{X}_h^{s+2}} \leq c\|v_h\|_{\mathbf{V}_h^{s+2}}$ for $s \in [-2, 0]$, i.e.,

$$\|\Delta_h v_h\|_{\mathbf{X}_h^s} \leq c\|A_h v_h\|_{\mathbf{V}_h^s}.$$

Using the lower bound in (2.11) yields the desired result for $-\frac{3}{2} \leq s < 0$.

For the upper bound we reason as in step (3) of the proof of Theorem 4.1 and we have $\|R_h x_h\|_{\mathbf{V}_h^{s+2}} \leq c\|x_h\|_{\mathbf{X}_h^{s+2}}$ for all $x_h \in \mathbf{X}_h$ and $s \in [-2, 0]$. By applying this result to $v_h \in \mathbf{V}_h$, we infer $\|A_h v_h\|_{\mathbf{V}_h^s} \leq c\|\Delta v_h\|_{\mathbf{X}_h^s}$. Then we conclude using the upper bound in (2.11) for $s \in (-\frac{3}{2}, 0]$. □

5. The semi-discrete time-dependent Stokes problem

We show in this section an application of Theorem 4.1.

5.1. **Formulation of the problem.** Let $(0, T)$ be a time interval (T is arbitrary). Let $u_0 \in \mathbf{H}$, let $p \in [1, 2]$, $q \in [1, 2]$, and $f \in L^p(0, T; \mathbf{L}^q(\Omega))$, and consider the following non stationary Stokes problem in weak form

$$\begin{cases} \partial_t u - \Delta u + \nabla p = f, & \text{in } \Omega_T \\ \nabla\cdot u = 0, & \text{in } \Omega_T \\ u|_\Gamma = 0, \qquad u|_{t=0} = u_0. \end{cases} \tag{5.1}$$

where $\Omega_T = \Omega\times(0, T)$. It is well known that this classical problem has a unique solution. In particular, if $u_0 = 0$ and $p = q \in (1, \infty)$, it is proved in Solonnikov [21] that the following bound holds

$$\|\nabla p\|_{L^p(\Omega_T)} + \|\partial_t u\|_{L^p(\Omega_T)} + \|\Delta u\|_{L^p(\Omega_T)} \leq c\, \|f\|_{L^p(\Omega_T)}. \tag{5.2}$$

This estimate has been significantly generalized by Sohr and von Wahl [20] to account for different exponents $p \in (1, \infty)$, $q \in (1, \infty)$,

$$\|\nabla p\|_{L^p(0,T;\mathbf{L}^q)} + \|\partial_t u\|_{L^p(0,T;\mathbf{L}^q)} + \|\Delta u\|_{L^p(0,T;\mathbf{L}^q)} \leq c\, \|f\|_{L^p(0,T;\mathbf{L}^q)}. \tag{5.3}$$

These estimates are important to construct weak solutions to the Navier–Stokes equations that are suitable in the sense of Scheffer [18].

The goal we have in mind now is to derive similar estimates using the discrete (finite-element-like) setting introduced above under the assumption $p \in [1,2]$, $q \in [1,2]$. The long term program we are pursuing is to eventually extend the results of [10] to homogeneous Dirichlet conditions. The results of [10] hold in the three-dimensional torus only, i.e., for periodic boundary conditions. Proving a discrete counterpart of (5.3) with Dirichlet conditions is a key step in this program. However, since we have not yet been able to handle the discrete setting associated with Dirichlet boundary conditions using the non-Hilbertian L^p-framework, we are going to reformulate (5.3) using fractional Sobolev spaces. The idea is to use the Fourier transform in time as done in Lions [15, p. 77].

Let H be a Hilbert space with norm $\|\cdot\|_H$. Let δ, $1 \le \delta < \infty$, and define $L^\delta(\mathbb{R};H) = \{\psi : \mathbb{R} \ni t \longmapsto \psi(t) \in H;\ \int_{-\infty}^{+\infty} \|\psi(t)\|_H^\delta dt < \infty\}$. For all $\psi \in L^1(\mathbb{R};H)$, denote by $\hat{\psi}(k) = \int_{-\infty}^{+\infty} \psi(t) e^{-2i\pi kt} dt$ for all $k \in \mathbb{R}$. The Fourier transform is extended to the space of tempered distributions with values in H, say $\mathcal{S}'(\mathbb{R};H)$. We shall make use of the following

Lemma 5.1 (Hausdorff-Young Inequality)**.** *There is $c > 0$ such that for all p, $1 \le \delta \le 2$, and for all $\psi \in L^\delta(\mathbb{R};H) \cap L^1(\mathbb{R};H)$,*

$$\|\hat{\psi}\|_{L^{\delta'}(\mathbb{R};H)} \le c\|\psi\|_{L^\delta(\mathbb{R};H)}, \qquad \tfrac{1}{\delta} + \tfrac{1}{\delta'} = 1. \tag{5.4}$$

Following [16, p. 21], we now define,

$$H^\gamma(\mathbb{R};H) = \{v \in \mathcal{S}'(\mathbb{R};H);\ \int_{-\infty}^{+\infty} (1+|k|)^{2\gamma} \|\hat{v}\|_H^2 dk < +\infty\}, \tag{5.5}$$

that we equip with the norm

$$\|v\|_{H^\gamma(\mathbb{R};H)}^2 := \int_{-\infty}^{+\infty} (1+|k|)^{2\gamma} \|\hat{v}\|_H^2 dk \tag{5.6}$$

The space $H^\gamma((0,T);H)$ is composed of those tempered distributions in $\mathcal{S}'((0,T);H)$ that can be extended to $\mathcal{S}'(\mathbb{R};H)$ and whose extension is in $H^\gamma(\mathbb{R};H)$. The norm in $H^\gamma((0,T);H)$ is the quotient norm, i.e.,

$$\|v\|_{H^\gamma((0,T);H)} = \inf_{\substack{\tilde{v}=u \\ \text{a.e. on } (0,T)}} \|\tilde{v}\|_{H^\gamma(\mathbb{R};H)}. \tag{5.7}$$

Henceforth we set

$$s := s(q) := d(\tfrac{1}{q} - \tfrac{1}{2}), \qquad \overline{r} := \tfrac{1}{p} - \tfrac{1}{2}.$$

This definition of s implies that the embedding $\mathbf{H}^s(\Omega) \subset \mathbf{L}^{q'}(\Omega)$ holds, where $\frac{1}{q'} + \frac{1}{q} = 1$. Note that the embedding $\widetilde{\mathbf{H}}_0^s(\Omega) \subset \mathbf{H}^s(\Omega)$ (if $s \in [0,2]$) being continuous implies that the embedding $\mathbf{L}^q(\Omega) \subset \widetilde{\mathbf{H}}_0^{-s}(\Omega)$ is also continuous. The Hausdorff–Young inequality then implies

$$f \in L^p((0,T);\mathbf{L}^q(\Omega)) \subset H^{-r}((0,T);\widetilde{\mathbf{H}}_0^{-s}(\Omega)), \quad \forall r > \overline{r}. \tag{5.8}$$

Our goal now is to derive estimates in spaces like $H^{-r}((0,T);\widetilde{\mathbf{H}}_0^{-s}(\Omega))$.

5.2. **The a priori estimates.** In addition to $f \in L^p((0,T+1);\mathbf{L}^q(\Omega))$, we also assume $u_0 = 0$ and $f \in L^1((0,T+1);\mathbf{H}^{-1}(\Omega))$. These two hypotheses could be avoided at the price of additional irrelevant technicalities. The approximate counterpart of (5.1) is as follows:

$$\begin{cases} \partial_t u_h + A_h u_h = P_h f, & \text{a.e. } t \in (0,T+1) \\ u_h|_{t=0} = 0. \end{cases} \tag{5.9}$$

We start by proving a series of key estimates.

Lemma 5.2. *Assume $s(q) \in [0, \frac{3}{2})$. There is c independent of h so that*

$$\|\partial_t u_h\|_{H^{-r}((0,T);\mathbf{V}_h^{-s})} + \|A_h u_h\|_{H^{-r}((0,T);\mathbf{V}_h^{-s})} \le c, \qquad \forall r > \overline{r}. \tag{5.10}$$

Moreover, if $\|u_h\|_{L^2((0,T);\mathbf{H}_0^1(\Omega))}$ is uniformly bounded, the following uniform estimates also hold:

$$\|\partial_t u_h\|_{H^{\tau-1}((0,T);\mathbf{V}_h^{-\alpha})} + \|u_h\|_{H^{\tau}((0,T);\mathbf{V}_h^{-\alpha})} \le c, \tag{5.11}$$

for all α, $0 \le \alpha \le s \le 1+2\alpha$, and for all $\tau < \overline{\tau} := \frac{1+\alpha}{1+s}(\frac{3}{2} - \frac{1}{p})$; and

$$\|A_h u_h\|_{H^{-\rho}((0,T);\mathbf{V}_h^{-\alpha})} \le c, \tag{5.12}$$

for all α, $2\alpha - 1 \le s \le \alpha$, for all $\rho > \overline{\rho} := \frac{1-\alpha}{1-s}(\frac{1}{p} - \frac{1}{2})$.

Proof. (1) By taking the scalar product of (5.10) with $A_h^{-1}u_h$ we infer (using $\|P_h f\|_{\mathbf{V}_h^{-1}} \le c\|f\|_{\mathbf{H}^{-1}}$)

$$\tfrac{1}{2} d_t \|u_h\|_{\mathbf{V}_h^{-1}}^2 + \|u_h\|_{\mathbf{L}^2}^2 \le \|P_h f\|_{\mathbf{V}_h^{-1}} \|u_h\|_{\mathbf{V}_h^{-1}} \le c\,\|f\|_{\mathbf{H}^{-1}} \|u_h\|_{\mathbf{V}_h^{-1}}$$

Then, since $\|f\|_{L^1(\mathbf{H}^{-1})}$ is bounded, the Gronwall Lemma yields

$$\|u_h\|_{L^\infty(\mathbf{V}_h^{-1})} + \|u_h\|_{L^2(\mathbf{L}^2)} \le c.$$

This bound clearly implies

$$\|u_h\|_{L^p(\mathbf{L}^q)} \le c. \tag{5.13}$$

(2) Extension. We extend u_h and f by zero on $(-\infty,0]$ and $(T+1,+\infty)$, and we slightly abuse the notation by still denoting these extensions by u_h and f_h, respectively. Let $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ be an infinitely smooth function compactly supported on $(-1,T+1)$ and equal to 1 on $[0,T]$. We now set $\tilde{u}_h = \varphi u_h$ and $\tilde{f} = \varphi f + \varphi' u_h$. It is clear that $\tilde{u}_h$ and $\tilde{f}$ are well defined on the time interval $(-\infty,+\infty)$, and (5.13) implies that $\|\tilde{f}\|_{L^p((0,T);\widetilde{\mathbf{H}}_0^{-s})}$ is uniformly bounded. The approximate problem takes the following form in $\mathcal{S}'(\mathbb{R};\mathbf{V}_h)$:

$$\frac{d}{dt}\tilde{u}_h + A_h \tilde{u}_h = P_h \tilde{f}.$$

Then, denoting by $\hat{u}_h$ and $\hat{f}$ the Fourier transform of $\tilde{u}_h$ and $\tilde{f}$, respectively, and upon taking the Fourier transform of the above equation, we obtain

$$2i\pi k\hat{u}_h + A_h\hat{u}_h = P_h\hat{f}. \tag{5.14}$$

(3) Bound (5.11). Let $\alpha \in \mathbb{R}^+$. Testing the above equation with the complex conjugate of $A_h^{-\alpha}\hat{u}_h$ and taking the imaginary part of the result yields

$$2\pi|k|\|\hat{u}_h\|^2_{\mathbf{V}_h^{-\alpha}} \le \|\hat{f}\|_{\widetilde{\mathbf{H}}_0^{-s}}\|A_h^{-\alpha}u_h\|_{\widetilde{\mathbf{H}}_0^{s}}.$$

Using the lower bound in (4.3) for $s \in [0, \frac{3}{2})$, we obtain

$$\begin{aligned}|k|\|\hat{u}_h\|^2_{\mathbf{V}_h^{-\alpha}} &\le c\|\hat{f}\|_{\widetilde{\mathbf{H}}_0^{-s}}\|A_h^{-\alpha}\hat{u}_h\|_{\widetilde{\mathbf{H}}_0^{s}} \le c'\|\hat{f}\|_{\widetilde{\mathbf{H}}_0^{-s}}\|A_h^{-\alpha}\hat{u}_h\|_{\mathbf{V}_h^{s}}\\ &\le c'\|\hat{f}\|_{\widetilde{\mathbf{H}}_0^{-s}}\|\hat{u}_h\|_{\mathbf{V}_h^{s-2\alpha}}.\end{aligned}$$

Assume $\alpha \le s \le 1+2\alpha$, then by interpolation we obtain

$$\|\hat{u}_h\|_{\mathbf{V}_h^{s-2\alpha}} \le \|\hat{u}_h\|^{\gamma}_{\mathbf{V}_h^{-\alpha}}\|\hat{u}_h\|^{1-\gamma}_{\mathbf{V}_h^{1}},$$

where $\gamma = \frac{2\alpha+1-s}{1+\alpha}$. Inserting this inequality in the previous estimate yields

$$|k|\|\hat{u}_h\|^{2-\gamma}_{\mathbf{V}_h^{-\alpha}} \le c'\|\hat{f}\|_{\widetilde{\mathbf{H}}_0^{-s}}\|\hat{u}_h\|^{1-\gamma}_{\mathbf{V}_h^{1}}.$$

This in turn implies

$$|k|^{\frac{2}{2-\gamma}-\mu}\|\hat{u}_h\|^2_{\mathbf{V}_h^{-\alpha}} \le c(1+|k|)^{-\mu}\|\hat{f}\|^{\frac{2}{2-\gamma}}_{\widetilde{\mathbf{H}}_0^{-s}}\|\hat{u}_h\|^{\frac{2(1-\gamma)}{2-\gamma}}_{\mathbf{V}_h^{1}},$$

where $\mu \in [0, \frac{2}{2-\gamma}]$ is still arbitrary. We now integrate over $\mathbb{R}$ with respect to k,

$$\int_{-\infty}^{+\infty}|k|^{\frac{2}{2-\gamma}-\mu}\|\hat{u}_h\|^2_{\mathbf{V}_h^{-\alpha}}dk \le c\,\|\tfrac{1}{(1+|k|)^{\mu}}\|_{L^{\ell}}\|\hat{f}_h\|^{\frac{2}{2-\gamma}}_{L^{\frac{2m}{2-\gamma}}(\widetilde{\mathbf{H}}_0^{-s})}\|\hat{u}_h\|^{\frac{2(1-\gamma)}{2-\gamma}}_{L^{\frac{2n(1-\gamma)}{2-\gamma}}(\mathbf{V}_h^{1})},$$

where $\frac{1}{\ell}+\frac{1}{m}+\frac{1}{n} = 1$. The first integral in the right-hand side is bounded provided $\mu\ell > 1$. Furthermore, we set m and n so that $p' = \frac{2m}{2-\gamma}$ and $2 = \frac{2n(1-\gamma)}{2-\gamma}$. The Hausdorff–Young Inequality together with the embedding $\mathbf{L}^q(\Omega) \subset \widetilde{\mathbf{H}}_0^{-s}(\Omega)$ then implies

$$\int_{-\infty}^{+\infty}|k|^{2\tau}\|\hat{u}_h\|^2_{\mathbf{V}_h^{-\alpha}}dk \le c\,\|\tilde{f}_h\|^{\frac{1}{2-\gamma}}_{L^p((-\infty,+\infty);\mathbf{L}^q)}\|\tilde{u}_h\|^{\frac{(1-\gamma)}{2-\gamma}}_{L^2((-\infty,+\infty);\mathbf{H}^1)},$$

where $\tau = \frac{1}{2-\gamma} - \frac{\mu}{2} < \frac{1}{2}(\frac{2}{2-\gamma} - \frac{1}{\ell})$. Owing to the definition of $\tilde{u}_h$ and $\tilde{f}$, this and u_h being uniformly bounded in $L^2((0,T);\mathbf{H}^1(\Omega)) \subset L^2((0,T);\mathbf{L}^2(\Omega))$ imply

$$\|\partial_t u_h\|_{H^{\tau-1}((0,T);\mathbf{V}_h^{-\alpha})} + \|u_h\|_{H^{\tau}((0,T);\mathbf{V}_h^{-\alpha})} \le c. \tag{5.15}$$

The bound on $\partial_t u_h$ is obtained by using $\widehat{\partial_t u_h} = 2i\pi k\hat{u}_h$.

By collecting the definitions of γ, l, m, and n, we deduce that the above inequality holds for all τ and α such that

$$\tau < \overline{\tau} := \tfrac{1+\alpha}{1+s}(\tfrac{3}{2} - \tfrac{1}{p}), \quad \text{and} \quad 0 \le \alpha \le s \le 1 + 2\alpha.$$

(4) Bound (5.12). Multiply (5.14) by $A_h^{1-\alpha}\hat{u}_h$ and take the real part to obtain

$$\|A_h\hat{u}_h\|^2_{\mathbf{V}_h^{-\alpha}} \le \|\hat{f}\|_{\widetilde{\mathbf{H}}_0^{-s}}\|A_h^{1-\alpha}\hat{u}_h\|_{\widetilde{\mathbf{H}}_0^{s}} \le \|\hat{f}\|_{\widetilde{\mathbf{H}}_0^{-s}}\|\hat{u}_h\|_{\mathbf{V}_h^{2+s-2\alpha}}.$$

Note again that we used the lower bound in (4.3) for $s \in [0, \frac{3}{2})$. Assume now that $2\alpha - 1 \le s \le \alpha$, then by interpolation we obtain

$$\|\hat{u}_h\|_{\mathbf{V}_h^{2+s-2\alpha}} \le \|\hat{u}_h\|^{1-\delta}_{\mathbf{V}_h^1}\|\hat{u}_h\|^{\delta}_{\mathbf{V}_h^{2-\alpha}},$$

where $\delta = \frac{1+s-2\alpha}{1-\alpha}$. Inserting this inequality in the previous estimate yields

$$\|\hat{A}_h u_h\|^{2-\delta}_{\mathbf{V}_h^{-\alpha}} \le c\|\hat{f}\|_{\mathbf{H}^{-s}}\|\hat{u}_h\|^{1-\delta}_{\mathbf{V}_h^1}.$$

This in turn implies

$$\frac{1}{(1+|k|)^\nu}\|\hat{u}_h\|^2_{\mathbf{V}_h^{-\alpha}} \le c\frac{1}{(1+|k|)^\nu}\|\hat{f}\|^{\frac{2}{2-\delta}}_{\mathbf{H}^{-s}}\|\hat{u}_h\|^{\frac{2(1-\delta)}{2-\delta}}_{\mathbf{V}_h^1},$$

where $\nu \ge 0$ is still arbitrary. By proceeding as in step (3) we finally infer

$$\|A_h u_h\|_{H^{-\rho}((0,T);\mathbf{V}_h^{-\alpha})} \le c, \tag{5.16}$$

for all α and ρ such that

$$\rho > \overline{\rho} := \tfrac{1-\alpha}{1-s}(\tfrac{1}{p} - \tfrac{1}{2}), \quad \text{and} \quad 2\alpha - 1 \le s \le \alpha.$$

(5) The estimate $A_h u_h$ in (5.10) is obtained by using $\alpha = s$ in (5.16), i.e., $\delta = 1$. The estimate on $\partial_t u_h$ in (5.10) is obtained by using $\alpha = s$ in (5.15). □

We are now in measure to conclude by stating the discrete counterpart of the Sohr and von Wahl estimates (5.3). The following Theorem is the main result of this section.

Theorem 5.1. *Assume $s(q) \in [0, \frac{3}{2})$. There is c independent of h so that for all $r > \overline{r}$*

$$\|\Delta_h u_h\|_{H^{-r}((0,T);\widetilde{\mathbf{H}}_0^{-s})} \le c. \tag{5.17}$$

If q is such that $s(q) < \frac{1}{2}$, then

$$\|\partial_t u_h\|_{H^{-r}((0,T);\widetilde{\mathbf{H}}_0^{-s})} \le c. \tag{5.18}$$

Moreover, if $\|u_h\|_{L^2((0,T);\mathbf{H}_0^1(\Omega))}$ is uniformly bounded, the following uniform estimates also hold:

$$\|\partial_t u_h\|_{H^{\tau-1}((0,T);\widetilde{\mathbf{H}}_0^{-\alpha})} + \|u_h\|_{H^{\tau}((0,T);\widetilde{\mathbf{H}}_0^{-\alpha})} \le c, \tag{5.19}$$

for all $\tau < \overline{\tau} = \frac{1+\alpha}{1+s}(1-\overline{r})$ *and all* $\alpha \in [0, \frac{1}{2})$ *such that* $s \in [\alpha, 1+2\alpha]$*; and*

$$\|\Delta_h u_h\|_{H^{-\rho}((0,T);\widetilde{\mathbf{H}}_0^{-\alpha})} \leq c, \tag{5.20}$$

for all $\rho > \overline{\rho} = \frac{1-\alpha}{1-s}\overline{r}$ *and all* $\alpha \in [0, \frac{3}{2})$ *such that* $s \in [2\alpha - 1, \alpha]$.

Proof. The inequality (5.17) is a consequence of the lower bound in (4.4) together with (5.10). The inequality (5.18) is a consequence of the lower bound in (4.3) in Theorem 4.1 together with (5.10). The rest of the proof follows along the same lines. □

Remark 5.1. The hypothesis $f \in L^1((0,T+1);\mathbf{H}^{-1}(\Omega))$ is not really necessary. It is just meant to deduce an easy bound on u_h in $L^2((0,T+1);\mathbf{L}^2(\Omega))$ to guaranty that the extension $\tilde{f}$ is bounded in $L^p(\mathbb{R};\mathbf{H}^{-s}(\Omega))$, see (5.13). This type of bound could be deduced without this hypothesis by invoking more involved arguments. Modulo more technicalities, the hypothesis $u_0 = 0$ can be removed by assuming $u_0 \in \mathbf{D}(A^{2-s})$.

Remark 5.2. Working with fractional exponents of the Stokes operator is not the most elegant way to treat the above problem. It would be more satisfactory to directly deduce $L^p(\mathbf{L}^q)$ estimates, but this necessitate a $L^p(\mathbf{L}^q)$ theory of the resolvent of the finite-element-based Stokes operator that seems unavailable (or of which we are unaware) at the present time.

5.3. **Application to the 3D Navier–Stokes equations.** Note that when applied to the Navier-Stokes equations in three space dimensions, the restriction $s(q) < \frac{1}{2}$ in Theorem 5.1 makes the bound (5.18) somewhat useless. Actually, in this case the above analysis applies with $f = g - u_h\cdot\nabla u_h$ where g is a smooth source and $u_h\cdot\nabla u_h$ is the nonlinear advection term. Since a standard uniform estimate in $L^\infty((0,T);\mathbf{L}^2(\Omega)) \cap L^2((0,T);\mathbf{H}^1_0(\Omega))$ holds on u_h, it comes that $f \in L^p(0,T;\mathbf{L}^q(\Omega))$ where p and q satisfy the equality $\frac{2}{p} + \frac{3}{q} = 4$ and $1 \leq p \leq 2$, $1 \leq q \leq \frac{3}{2}$. The restriction on q yields $\frac{1}{2} \leq s = 3(\frac{1}{q} - \frac{1}{2}) \leq \frac{3}{2}$, which is contradictory with the assumption $s < \frac{1}{2}$ which is needed for the bound (5.18) to hold. This remark is the reason why we have been led to account for the additional uniform estimate $u_h \in L^2((0,T);\mathbf{H}^1_0(\Omega))$ in Theorem 5.1 which gives the more sophisticated estimate (5.19).

Let us illustrate the use of (5.19) in the three-dimensional Navier-Stokes situation. Assume now that $\alpha \in [0, \frac{1}{2})$, then

$$\|\partial_t u_h\|_{H^{\tau-1}(\mathbf{H}^{-\alpha})} + \|u_h\|_{H^{\tau}(\mathbf{H}^{-\alpha})} \leq c, \tag{5.21}$$

for all $\tau < \overline{\tau}$ provided $0 \leq \alpha \leq s \leq 1+2\alpha$ and $s < \frac{3}{2}$. Note that $\overline{\tau} = \frac{1+\alpha}{1+s}(\frac{s}{2} + \frac{1}{4})$ owing to the relation $\frac{2}{p} + \frac{3}{q} = 4$ and the definition $s = 3(\frac{1}{q} - \frac{1}{2})$. Observe that $\frac{1+\alpha}{1+s}(\frac{s}{2} + \frac{1}{4})$ is maximum at $s = \frac{3}{2}$; as a result, (5.21) holds for all $\tau < \frac{2}{5}(1+\alpha)$ for all $\alpha \in [\frac{1}{4}, \frac{1}{2})$.

Let $\epsilon > 0$ and $\epsilon' > 0$ be two small positive numbers, and set $\alpha = \frac{1}{2} - \epsilon$, $s = \frac{3}{2} - \epsilon'$. Assume that ϵ and ϵ' are small enough so that

$$\alpha \le s \le 1 + 2\alpha, \qquad \tfrac{2}{5} - \epsilon < \tfrac{2-\epsilon'}{5-2\epsilon'} < \tfrac{2}{5} + \epsilon. \tag{5.22}$$

Then $\frac{2}{5} + 3\epsilon > 1 - \frac{1+\alpha}{1+s}(\frac{s}{2} + \frac{1}{4})$, and the bound (5.21) can be rewritten as

$$\|u_h\|_{H^{\frac{3}{5}-3\epsilon}((0,T);\mathbf{H}^{-\frac{1}{2}+\epsilon})} + \|\partial_t u_h\|_{H^{-\frac{2}{5}-3\epsilon}((0,T);\mathbf{H}^{-\frac{1}{2}+\epsilon})} \le c. \tag{5.23}$$

Note that (5.23) is slightly better than what the Sohr and von Wahl estimate gives by embedding. Actually, taking $q = \frac{3}{2}$, we conjecture that a discrete version of the inequality (5.3) together with (5.8) would give $\|\partial_t u_h\|_{H^{-\frac{1}{2}-\epsilon}((0,T);\mathbf{H}^{-\frac{1}{2}})} \le c$, for all $\epsilon > 0$, which is clearly weaker than (5.23) when ϵ is close to zero since $-\frac{1}{2} < -\frac{2}{5}$. This is not a surprise since more information on u_h has been used to deduce (5.23).

The estimate (5.23) is a key to extend to homogeneous Dirichlet conditions the results of [10], which for the time being holds only in the three-dimensional torus. An important link still missing in this program is an estimate on the pressure that allows for the convergence of the product $p_h u_h$ in some reasonable sense. To derive such an estimate, set $s = \alpha = \frac{1}{2}$. Then $\frac{1}{p} = 1$, and (5.17) yields

$$\|\Delta_h u_h\|_{H^{-\frac{1}{2}-\epsilon}((0,T);\widetilde{\mathbf{H}}_0^{-\frac{1}{2}})} \le c, \tag{5.24}$$

for all $\epsilon > 0$. This yield a uniform bound on $\|p_h\|_{H^{-\frac{1}{2}-\frac{\epsilon}{2}}((0,T);\mathbf{H}^{\frac{1}{2}})}$ (note in passing that this is coherent with the estimate $p \in L^{1+\epsilon}((0,T);L^{\frac{3+3\epsilon}{1+3\epsilon}}(\Omega))$ given in [20, Thm 3.3]). This shows that it should be possible to pass to the limit on the product $p_h u_h$, thus implying that the result of [10] should hold for Dirichlet boundary conditions. These developments are reported in [12].

Appendix A. Proof of Lemma 2.2

Lemma 2.2. *There is a non-increasing function $c_l > 0$ and a non-decreasing function $c_u > 0$, both uniform in h, such that for all $s \in (-\frac{3}{2}, \frac{3}{2})$,*

$$c_l(|s|)\, \|v_h\|_{\widetilde{\mathbf{H}}_0^s} \le \|v_h\|_{\mathbf{X}_h^s} \le c_u(|s|)\, \|v_h\|_{\widetilde{\mathbf{H}}_0^s}, \quad \forall v_h \in \mathbf{X}_h. \tag{A.1}$$

To prove the above lemma, we shall use the following two lemmas.

Lemma A.1. *There is constant $c > 0$ such that for all $s \in (\frac{1}{2}, \frac{3}{2})$,*

$$\forall v \in \widetilde{\mathbf{H}}_0^s, \quad \|v\|_{\widetilde{\mathbf{H}}_0^s} \le c \sup_{0\ne x \in \mathbf{H}^{2-s}(\Omega)\cap \mathbf{H}_0^1(\Omega)} \frac{\langle \nabla v, \nabla x\rangle}{\|x\|_{\mathbf{H}^{2-s}}}. \tag{A.2}$$

Here the brackets represent the duality paring between $\widetilde{\mathbf{H}}_0^{-1+s}(\Omega)$ and $\widetilde{\mathbf{H}}_0^{1-s}(\Omega)$.

Lemma A.2. *There is a non-decreasing function $c_u > 0$, uniform in h, such that for all $s \in [0, \frac{1}{2})$*

$$\|E_h(v)\|_{\widetilde{\mathbf{H}}_0^{1-s}} \le c_u(s)\|v\|_{\widetilde{\mathbf{H}}_0^{1-s}}, \quad \forall v \in \widetilde{\mathbf{H}}_0^{1-s}(\Omega). \tag{A.3}$$

Proof of Lemma 2.2. Step (1): The case $s \in [0, 1]$ is given in Bank and Dupont [1, Lemma 1]. (The upper bound is a consequence of $\pi_h : \mathbf{L}^2(\Omega) \longrightarrow \mathbf{X}_h^0$ and $\pi_h : \mathbf{H}_0^1(\Omega) \longrightarrow \mathbf{X}_h^1$ being stable (see (2.4)). The lower bound is a consequence of the natural injection $I : \mathbf{X}_h^0 \longrightarrow \mathbf{L}^2(\Omega)$, and $I : \mathbf{X}_h^1 \longrightarrow \mathbf{H}_0^1(\Omega)$ being stable.)

Step (2): The lower bound for $1 + s$, $s \in (0, \frac{1}{2})$ is verified as follows. Let $v_h \in \mathbf{X}_h$. Owing to (A.2), (A.3), and step (1), we get

$$\begin{aligned}
\|v_h\|_{\widetilde{\mathbf{H}}_0^{1+s}} &\le c \sup_{0\neq w\in\widetilde{\mathbf{H}}_0^{1-s}(\Omega)} \frac{\langle \nabla v_h, \nabla w\rangle}{\|w\|_{\widetilde{\mathbf{H}}_0^{1-s}}} = c \sup_{0\neq w\in\widetilde{\mathbf{H}}_0^{1-s}(\Omega)} \frac{(\nabla v_h, \nabla E_h w)}{\|w\|_{\widetilde{\mathbf{H}}_0^{1-s}}} \\
&\le c_u(s) \sup_{0\neq w\in\widetilde{\mathbf{H}}_0^{1-s}(\Omega)} \frac{(\nabla v_h, \nabla E_h w)}{\|E_h w\|_{\widetilde{\mathbf{H}}_0^{1-s}}} \le c_u(s) \sup_{0\neq w_h\in\mathbf{X}_h} \frac{(\nabla v_h, \nabla w_h)}{\|w_h\|_{\widetilde{\mathbf{H}}_0^{1-s}}} \\
&\le c_u(s) \sup_{0\neq w_h\in\mathbf{X}_h} \frac{(\nabla v_h, \nabla w_h)}{\|w_h\|_{\mathbf{X}_h^{1-s}}}
\end{aligned}$$

But we also have

$$\begin{aligned}
\sup_{0\neq w_h\in\mathbf{X}_h} \frac{(\nabla v_h, \nabla w_h)}{\|w_h\|_{\mathbf{X}_h^{1-s}}} &= \sup_{0\neq w_h\in\mathbf{X}_h} \frac{(-\Delta_h v_h, w_h)}{\|(-\Delta_h)^{\frac{1-s}{2}} w_h\|_{\mathbf{L}_2}} \\
&= \sup_{0\neq w_h\in\mathbf{X}_h} \frac{((-\Delta_h)^{\frac{1+s}{2}} v_h, w_h)}{\|w_h\|_{\mathbf{L}_2}} = \|(-\Delta_h)^{\frac{1+s}{2}} v_h\|_{\mathbf{L}_2} = \|v_h\|_{\mathbf{X}_h^{1+s}},
\end{aligned}$$

which combined with the previous bound yields the desired result.

Step (3): We next prove the upper bound for $1 + s$, $s \in (0, \frac{1}{2})$. We use Step (1) to conclude

$$\begin{aligned}
\|v_h\|_{\mathbf{X}_h^{1+s}} &= \sup_{0\neq w_h\in\mathbf{X}_h} \frac{(\nabla v_h, \nabla w_h)}{\|w_h\|_{\mathbf{X}_h^{1-s}}} \le \sup_{0\neq w_h\in\mathbf{X}_h} \frac{\|\nabla v_h\|_{\widetilde{\mathbf{H}}_0^{s}} \|\nabla w_h\|_{\widetilde{\mathbf{H}}_0^{-s}}}{\|w_h\|_{\mathbf{X}_h^{1-s}}} \\
&\le c \sup_{0\neq w_h\in\mathbf{X}_h} \frac{\|v_h\|_{\widetilde{\mathbf{H}}_0^{1+s}} \|w_h\|_{\widetilde{\mathbf{H}}_0^{1-s}}}{\|w_h\|_{\widetilde{\mathbf{H}}_0^{1-s}}} \le c \, \|v_h\|_{\widetilde{\mathbf{H}}_0^{1+s}}.
\end{aligned}$$

Note that this also shows that c_u in (A.1) is uniformly bounded on $[0, \frac{3}{2}]$.

Step (4): We now consider the case $-\frac{3}{2} < s < 0$. Using the lower bound just proved for $0 < -s < \frac{3}{2}$ yields

$$\|v_h\|_{\mathbf{X}_h^{s}} = \sup_{0\neq w_h\in\mathbf{X}_h} \frac{(v_h, w_h)}{\|w_h\|_{\mathbf{X}_h^{-s}}} \le c_u(s) \sup_{0\neq w_h\in\mathbf{X}_h} \frac{(v_h, w_h)}{\|w_h\|_{\widetilde{\mathbf{H}}_0^{-s}}} \le c_u(s) \, \|v_h\|_{\widetilde{\mathbf{H}}_0^{s}}.$$

Finally, applying (2.4) and the upper bound just proved for $0 < -s < \frac{3}{2}$ gives

$$\begin{aligned}\|v_h\|_{\widetilde{\mathbf{H}}_0^s} &= \sup_{0\neq w\in\widetilde{\mathbf{H}}_0^{-s}} \frac{(v_h,w)}{\|w\|_{\widetilde{\mathbf{H}}_0^{-s}}} \le c_u(s) \sup_{0\neq w\in\widetilde{\mathbf{H}}_0^{-s}} \frac{(v_h,\pi_h w)}{\|\pi_h w\|_{\widetilde{\mathbf{H}}_0^{-s}}} \\ &\le c_u(s) \sup_{0\neq w_h\in\mathbf{X}_h} \frac{(v_h,w_h)}{\|w_h\|_{\widetilde{\mathbf{H}}_0^{-s}}} \le c_u(s) \sup_{0\neq w_h\in\mathbf{X}_h} \frac{(v_h,w_h)}{\|w_h\|_{\mathbf{X}_h^{-s}}} \le c_u(s)\|v_h\|_{\mathbf{X}_h^s}.\end{aligned}$$

This completes the proof. □

Proof of Lemma A.1. Step (1): Let $s \in [0,\frac{1}{2})$ and let $v \in \widetilde{\mathbf{H}}_0^{1+s}(\Omega)$ and set $f = -\Delta v$. Clearly $f \in \widetilde{\mathbf{H}}_0^{-1+s}(\Omega)$ and elliptic regularity implies that there is a constant $c > 0$, independent of s, such that $c\,\|v\|_{\mathbf{H}^{1+s}} \le \|f\|_{\widetilde{\mathbf{H}}_0^{-1+s}}$. Hence

$$\begin{aligned}c\,\|v\|_{\mathbf{H}^{1+s}} \le \|f\|_{\widetilde{\mathbf{H}}_0^{-1+s}} &= \sup_{0\neq\phi\in\widetilde{\mathbf{H}}_0^{1-s}(\Omega)} \frac{\langle f,\phi\rangle}{\|\phi\|_{\widetilde{\mathbf{H}}_0^{1-s}}} \\ &= \sup_{0\neq\phi\in\widetilde{\mathbf{H}}_0^{1-s}(\Omega)} \frac{\langle -\Delta v,\phi\rangle}{\|\phi\|_{\widetilde{\mathbf{H}}_0^{1-s}}} = \sup_{0\neq\phi\in\widetilde{\mathbf{H}}_0^{1-s}(\Omega)} \frac{\langle \nabla v,\nabla\phi\rangle}{\|\phi\|_{\widetilde{\mathbf{H}}_0^{1-s}}}.\end{aligned}$$

Note that the last equality holds because $\boldsymbol{\mathcal{C}}_0^\infty(\Omega)$ is dense in $\widetilde{\mathbf{H}}_0^{1-s}(\Omega)$ for $s \in [0,\frac{1}{2})$.

Step (2): Let $s \in [0,\frac{1}{2})$. For all $w \in \widetilde{\mathbf{H}}_0^{1-s}(\Omega)$, define $x(w) \in \mathbf{H}_0^1(\Omega)$ solving

$$\langle \nabla x(w),\nabla y\rangle = (w,y)_{\widetilde{\mathbf{H}}_0^{1-s}}, \qquad \text{for all } y \in \mathbf{H}_0^1(\Omega)$$

where $(\cdot,\cdot)_{\widetilde{\mathbf{H}}_0^{1-s}}$ is the scalar product in $\widetilde{\mathbf{H}}_0^{1-s}(\Omega)$. Elliptic regularity implies that there is c, independent of s, such that

$$\|x(w)\|_{\widetilde{\mathbf{H}}_0^{1+s}} \le c\|w\|_{\widetilde{\mathbf{H}}_0^{1-s}}, \qquad \forall w \in \widetilde{\mathbf{H}}_0^{1-s}(\Omega).$$

Let $v \in \boldsymbol{\mathcal{C}}_0^\infty(\Omega)$. Then

$$\begin{aligned}\|v\|_{\widetilde{\mathbf{H}}_0^{1-s}} &= \sup_{0\neq w\in\widetilde{\mathbf{H}}_0^{1-s}(\Omega)} \frac{(v,w)_{\widetilde{\mathbf{H}}_0^{1-s}}}{\|w\|_{\widetilde{\mathbf{H}}_0^{1-s}}} = \sup_{0\neq w\in\widetilde{\mathbf{H}}_0^{1-s}(\Omega)} \frac{\langle\nabla v,\nabla x(w)\rangle}{\|w\|_{\widetilde{\mathbf{H}}_0^{1-s}}} \\ &\le c \sup_{0\neq w\in\widetilde{\mathbf{H}}_0^{1-s}(\Omega)} \frac{\langle\nabla v,\nabla x(w)\rangle}{\|x(w)\|_{\widetilde{\mathbf{H}}_0^{1+s}}} \\ &\le c \sup_{0\neq x\in\widetilde{\mathbf{H}}_0^{1+s}(\Omega)} \frac{\langle\nabla v,\nabla x\rangle}{\|x\|_{\widetilde{\mathbf{H}}_0^{1+s}}}.\end{aligned}$$

The desired inequality follows by density since $s \in [0,\frac{1}{2})$. This completes the proof of the lemma. □

Proof of Lemma A.2. Let $v \in \boldsymbol{\mathcal{C}}_0^\infty(\Omega)$ and let $s \in [0, \frac{1}{2})$. Clearly $E_h(v) \in \mathbf{H}_0^1(\Omega) \subset \widetilde{\mathbf{H}}_0^{1-s}(\Omega)$. Owing to (A.2) and (2.7), we have

$$\begin{aligned}
\|E_h v\|_{\mathbf{H}^{1-s}} &\leq c \sup_{0\neq x\in\widetilde{\mathbf{H}}_0^{1+s}(\Omega)} \frac{(\nabla E_h v, \nabla x)}{\|x\|_{\widetilde{\mathbf{H}}_0^{1+s}}} \\
&\leq c \sup_{0\neq x\in\widetilde{\mathbf{H}}_0^{1+s}} \frac{(\nabla v, \nabla E_h x)}{\|x\|_{\widetilde{\mathbf{H}}_0^{1+s}}} \\
&\leq c \|v\|_{\widetilde{\mathbf{H}}_0^{1-s}} \sup_{0\neq x\in\widetilde{\mathbf{H}}_0^{1+s}(\Omega)} \frac{\|E_h x\|_{\widetilde{\mathbf{H}}_0^{1+s}}}{\|x\|_{\widetilde{\mathbf{H}}_0^{1+s}}} \leq c_u(s)\|v\|_{\widetilde{\mathbf{H}}_0^{1-s}}.
\end{aligned}$$

Then use the fact that $\boldsymbol{\mathcal{C}}_0^\infty(\Omega)$ is dense in $\widetilde{\mathbf{H}}_0^{1-s}(\Omega)$ to conclude. □

Lemma A.3. *Assume that the family $\{\mathbf{X}_h\}_{h>0}$ is such that there is a non decreasing function $c_u(s) > 0$, $s \in [0, \frac{1}{2})$, so that*

$$\|v_h\|_{\widetilde{\mathbf{H}}_0^s} \leq c_u(s)\, h^{-s} \|v_h\|_{\mathbf{L}^2}, \quad \forall v_h \in \mathbf{X}_h + \partial_{x_1}\mathbf{X}_h + \ldots + \partial_{x_d}\mathbf{X}_h,\ \forall s \in [0, \tfrac{1}{2}),$$

and there is a linear operator $\rho_h : \widetilde{\mathbf{H}}_0^s(\Omega) \longrightarrow \mathbf{X}_h$ and a constant c_1, independent of h and s, such that

$$\|\rho_h v\|_{\widetilde{\mathbf{H}}_0^s} \leq c_1 \|v\|_{\widetilde{\mathbf{H}}_0^s}, \quad \|(\rho_h - 1)v\|_{\mathbf{L}^2} \leq c_1 h^s \|v\|_{\widetilde{\mathbf{H}}_0^s}, \quad \forall v \in \widetilde{\mathbf{H}}_0^s(\Omega).$$

Let $T_h : \mathbf{Z} \subset \mathbf{H}^{1+s}(\Omega) \longrightarrow \mathbf{X}_h$ be a linear operator, where $\mathbf{Z}$ is a closed subspace of $\mathbf{H}^{1+s}(\Omega)$. Assume that the family $\{T_h\}_{h>0}$ is such that there is a constant c_2, uniform in h and s, so that

$$\|T_h u - u\|_{\mathbf{H}^1} \leq c_2\, h^s \|u\|_{\mathbf{H}^{1+s}}, \qquad \forall u \in \mathbf{Z}, \quad \forall s \in [0, \tfrac{1}{2}).$$

Then there is a non decreasing function $c'_u(s) > 0$, $s \in [0, \frac{1}{2})$, so that

$$\|T_h u\|_{\mathbf{H}^{1+s}} \leq c'_u(s)\, \|u\|_{\mathbf{H}^{1+s}}, \qquad \forall u \in \mathbf{Z}, \quad s \in [0, \tfrac{1}{2}). \tag{A.4}$$

Proof. We proceed as in [2, Appendix]. The norm in $\mathbf{H}^{1+s}(\Omega)$ can be defined by

$$\|v\|_{\mathbf{H}^{1+s}} = \|v\|_{\mathbf{H}^1} + \sum_{i=1}^{d} \|\partial_{x_i} v\|_{\mathbf{H}^s}$$

Then $\|T_h u\|_{\mathbf{H}^{1+s}} \leq \|T_h u\|_{\mathbf{H}^1} + \sum_{i=1}^d \|\partial_{x_i} T_h u\|_{\mathbf{H}^s}$ and for all $i \in \{1, \ldots d\}$,

$$\begin{aligned}
\|\partial_{x_i} T_h u\|_{\mathbf{H}^s} &\leq c(\|\pi_h \partial_{x_i} u\|_{\widetilde{\mathbf{H}}_0^s} + \|\pi_h \partial_{x_i} u - \partial_{x_i} T_h u\|_{\widetilde{\mathbf{H}}_0^s}) \\
&\leq c(c_1 \|\partial_{x_i} u\|_{\widetilde{\mathbf{H}}_0^s} + c_u(s) h^{-s} \|\pi_h \partial_{x_i} u - \partial_{x_i} T_h u\|_{\mathbf{L}^2}) \\
&\leq c_u(s)(\|\partial_{x_i} u\|_{\mathbf{H}^s} + h^{-s}(\|(\pi_h - 1)\partial_{x_i} u\|_{\mathbf{L}^2} + \|\partial_{x_i}(u - T_h u)\|_{\mathbf{L}^2})) \\
&\leq c_u(s)(\|\partial_{x_i} u\|_{\mathbf{H}^s} + h^{-s}(h^s \|\partial_{x_i} u\|_{\widetilde{\mathbf{H}}_0^s} + h^s \|u\|_{\mathbf{H}^{1+s}})) \\
&\leq c_u(s) \|u\|_{\mathbf{H}^{1+s}}.
\end{aligned}$$

This concludes the proof. □

Remark A.1. The inverse inequality hypothesis is reasonable if $\mathbf{X}_h$ is a finite element constructed on a quasi-uniform mesh. In this case, by redoing carefully the computation in [2, Appendix] we infer that there is c, uniform in both h and s such that

$$\|v_h\|_{\mathbf{H}^s} \leq c\,(1-2s)^{-\frac{1}{2}} h^{-s} \|v_h\|_{\mathbf{L}^2}, \quad \forall v_h \in \mathbf{X}_h + \partial_{x_1}\mathbf{X}_h + \ldots + \partial_{x_d}\mathbf{X}_h,\ \forall s \in [0, \tfrac{1}{2}).$$

That is to say the inverse inequality hypothesis holds with $c_u(s) \sim (1-2s)^{-\frac{1}{2}}$.

References

[1] Randolph E. Bank and Todd Dupont. An optimal order process for solving finite element equations. *Math. Comp.*, 36(153):35–51, 1981.

[2] James H. Bramble, Joseph E. Pasciak, and Jinchao Xu. The analysis of multigrid algorithms with nonnested spaces or noninherited quadratic forms. *Math. Comp.*, 56(193):1–34, 1991.

[3] James H. Bramble and Xuejun Zhang. The analysis of multigrid methods. In *Handbook of numerical analysis, Vol. VII*, Handb. Numer. Anal., VII, pages 173–415. North-Holland, Amsterdam, 2000.

[4] F. Brezzi and M. Fortin. *Mixed and Hybrid Finite Element Methods*. Springer-Verlag, New York, NY, 1991.

[5] P. Clément. Approximation by finite element functions using local regularization. *RAIRO, Anal. Num.*, 9:77–84, 1975.

[6] Monique Dauge. Stationary Stokes and Navier-Stokes systems on two- or three-dimensional domains with corners. I. Linearized equations. *SIAM J. Math. Anal.*, 20(1):74–97, 1989.

[7] A. V. Fursikov. *Optimal control of distributed systems. Theory and applications*, volume 187 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2000.

[8] V. Girault and P.-A. Raviart. *Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms*. Springer Series in Computational Mathematics. Springer-Verlag, Berlin, Germany, 1986.

[9] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[10] J.-L. Guermond. Finite-element-based Faedo-Galerkin weak solutions to the Navier-Stokes equations in the three-dimensional torus are suitable. *J. Math. Pures Appl. (9)*, 85(3):451–464, 2006.

[11] J.-L. Guermond. The LBB condition in fractional sobolev spaces and applications. *IMA, Numer. Anal.*, 2006, Submitted.

[12] J.-L. Guermond. Faedo-galerkin weak solutions of the navier–stokes equations with dirichlet boundary conditions are suitable. *J. Math. Pures Appl. (9)*, 2007, Submitted.

[13] John G. Heywood and Rolf Rannacher. Finite element approximation of the non-stationary Navier-Stokes problem. I. Regularity of solutions and second-order error estimates for spatial discretization. *SIAM J. Numer. Anal.*, 19(2):275–311, 1982.

[14] O. A. Ladyzhenskaya. *The mathematical theory of viscous incompressible flow*. Second English edition, revised and enlarged. Translated from the Russian by Richard A. Silverman and John Chu. Mathematics and its Applications, Vol. 2. Gordon and Breach Science Publishers, New York, 1969.

[15] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires*, volume 1. Dunod, Paris, France, 1969.

[16] J.-L. Lions and E. Magenes. *Problèmes aux limites non homogènes et applications*, volume 1. Dunod, Paris, France, 1968.

[17] J.-L. Lions and J. Peetre. Sur une classe d'espaces d'interpolation. *Inst. Hautes Études Sci. Publ. Math.*, 19:5–68, 1964.

[18] V. Scheffer. Hausdorff measure and the Navier-Stokes equations. *Comm. Math. Phys.*, 55(2):97–112, 1977.

[19] R.L. Scott and S. Zhang. Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Math. Comp.*, 54(190):483–493, 1990.

[20] Hermann Sohr and Wolf von Wahl. On the regularity of the pressure of weak solutions of Navier-Stokes equations. *Arch. Math. (Basel)*, 46(5):428–439, 1986.

[21] V. A. Solonnikov. Estimates of the solution of a certain initial-boundary value problem for a linear nonstationary system of Navier-Stokes equations. *Zap. Naučn. Sem. Leningrad. Otdel Mat. Inst. Steklov. (LOMI)*, 59:178–254, 257, 1976.

[22] R. Temam. *Navier–Stokes Equations*, volume 2 of *Studies in Mathematics and its Applications*. North-Holland, 1977.

[1]Department of Mathematics, Texas A&M University 3368 TAMU, College Station, TX 77843-3368, USA.

E-mail address: guermond@math.tamu.edu, pasciak@math.tamu.edu