ANALYSIS OF A MULTIGRID ALGORITHM FOR TIME HARMONIC MAXWELL EQUATIONS

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Abstract. This paper considers a multigrid algorithm suitable for efficient solution of indefinite linear systems arising from finite element discretization of time harmonic Maxwell equations. In particular, a "backslash" multigrid cycle is proven to converge at rates independent of refinement level if certain indefinite block smoothers are used. The method of analysis involves comparing the multigrid error reduction operator with that of a related positive definite multigrid operator. This idea has previously been used in multigrid analysis of indefinite second order elliptic problems. However, the Maxwell application involves a non-elliptic indefinite operator. With the help of a few new estimates, the earlier ideas can still be applied. Some numerical experiments with lowest order Nedelec elements are also reported.

1. Introduction

The purpose of this paper is to study certain multigrid methods for the solution of the discrete equations which result from time harmonic Maxwell equations. Since the introduction of Nedelec elements [22], finite element methods using these **curl**- conforming elements have become a popular choice for discretization of Maxwell equations. An analysis of the finite element method in the time harmonic case and lossless media was provided in [20]. However, the efficient solution of the resulting linear systems has remained a challenge, mainly because of two reasons: the linear systems are indefinite and the differential operator **curl** has a large null space.

For the time harmonic problem, although a multigrid analysis has been lacking, numerical experiments indicating the suitability of certain two-level and multilevel algorithms can be found in literature [3, 4, 23]. Numerical results for parallel preconditioners based on Schwarz overlapping techniques were reported in [23]. Computational experiments with a multigrid V-cycle have been reported [3, 4]. More recently, an analysis for an additive overlapping preconditioner and a two level multiplicative variant was given in [16].

Two works that made recent advances related to development of preconditioners for Maxwell equations, [1] and [17], deserve special mention. Both provided smoothers for use in multigrid V-cycle for the positive definite bilinear form $\mathbf{\Lambda}(\cdot,\cdot)$ defined later in (2.1). These smoothers are based on two different subspace decompositions of the Nedelec space. Our smoothers for the indefinite problem are constructed based on the same decompositions and our analysis makes use of the results in [1] and [17].

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The current paper provides an analysis for a multilevel algorithm. Specifically, we prove that the so-called "backslash cycle" gives a convergent linear iterative method with a convergence rate independent of mesh size, provided the coarse grid is sufficiently fine. The latter restriction stems from the indefiniteness and seems unavoidable both in theory and practice. Fundamentally different solution methods may be needed to overcome this. Nonetheless, in spite of this restriction there are many practical applications (of moderate frequencies) where a multigrid iteration using a relatively fine coarse grid can reduce computational effort significantly.

The analysis we will provide is based on [16] and an earlier paper on multigrid applied to elliptic non-symmetric and indefinite problems [6] (see also [8]). In [6], a perturbation technique to analyze a multigrid algorithm for indefinite or nonsymmetric operators was developed. This involves comparing the error propagation operator of the multigrid algorithm with that of a multigrid algorithm for a corresponding positive definite operator. The difference between these operators was then proved small for elliptic problems that may be nonsymmetric or indefinite. However, our application involves a non-elliptic operator. We will show that techniques in [6] can still be applied. In [16], some fundamental estimates were developed concerning the approximation properties of the discrete solution operator corresponding to the time harmonic Maxwell approximation. These estimates will play an important role in the analysis given here.

The outline of the remainder of the paper is as follows. In Section 2, we define the problem and give the multigrid algorithm. Smoothers are defined and analyzed in Section 3. Convergence estimates for the multigrid algorithm are given in Section 4. Finally, the results of numerical experiments are given in Section 5.

2. The problem and multigrid algorithm.

We set up a model problem arising from time harmonic Maxwell equations and a simple multigrid algorithm in this section. First we establish notation for some spaces and their norms. Let Ω be an open bounded connected polyhedral domain in $\mathbb{R}^3$ and let $L^2(\Omega)$ denote the space of square integrable functions on Ω. We will use $(\cdot,\cdot)_\Omega$ and $\|\cdot\|_{0,\Omega}$ to denote the inner product and norm respectively in $L^2(\Omega)$ or $L^2(\Omega)^3$. The latter will often be abbreviated to $\|\cdot\|$. In the space of vector functions in $L^2(\Omega)^3$ with square integrable $\mathbf{curl}$, tangential traces $\boldsymbol{n}\times\boldsymbol{u}$ on the boundary $\partial\Omega$ are well defined [15], and we define

$$H_0(\mathbf{curl}\,;\Omega) = \{\boldsymbol{u}\in (L^2(\Omega))^3 : \mathbf{curl}\,\boldsymbol{u}\in (L^2(\Omega))^3,\ \boldsymbol{n}\times\boldsymbol{u} = 0 \text{ on } \partial\Omega\}.$$

Here $\boldsymbol{n}$ is the unit outward normal on the boundary $\partial\Omega$. This space is normed with $\|\cdot\|_{\boldsymbol{\Lambda},\Omega} = \boldsymbol{\Lambda}(\cdot,\cdot)^{1/2}$, where

$$\boldsymbol{\Lambda}(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{u},\boldsymbol{v})_\Omega + (\mathbf{curl}\,\boldsymbol{u},\mathbf{curl}\,\boldsymbol{v})_\Omega. \tag{2.1}$$

Analogous definitions hold for $\|\cdot\|_{0,D}$, $(\cdot,\cdot)_D$, and $\|\cdot\|_{\boldsymbol{\Lambda},D}$ in domains D different from Ω. In the notation for function spaces and their norms, when the domain is absent, it is to be taken as Ω; for example, $H_0(\mathbf{curl}) \equiv H_0(\mathbf{curl}\,;\Omega)$.

We restrict our attention to the time harmonic Maxwell equations in a homogeneous lossless media occupying Ω and also assume that the boundary of Ω is adjacent to a perfect conductor. The following equation is a variational system for the electric field $\boldsymbol{U}\in H_0(\mathbf{curl}\,;\Omega)$ given by Maxwell equations [11, 20] in the simple case of unit material properties:

$$\boldsymbol{A}(\boldsymbol{U},\boldsymbol{v}) = (\boldsymbol{F},\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in H_0(\mathbf{curl}\,;\Omega), \tag{2.2}$$

where

$$\boldsymbol{A}(\boldsymbol{U},\boldsymbol{v}) = (\mathbf{curl}\,\boldsymbol{U}, \mathbf{curl}\,\boldsymbol{v}) - \omega^2(\boldsymbol{U},\boldsymbol{v}).$$

The vector $\boldsymbol{F}$, being a constant multiple of electric current has zero divergence, and consequently $\operatorname{div}\boldsymbol{U} = 0$. In (2.2), ω is a real number denoting frequency of propagation. Note that there is a countable set of real values for ω for which (2.2) does not have a unique solution [19]. Throughout this paper we assume that ω is not one of these values and so (2.2) is uniquely solvable.

In our arguments later, we will need the solutions to (2.2) to be regular and hence, we assume that Ω is convex. It is well known [14, 20] that $\boldsymbol{U}, \mathbf{curl}\,\boldsymbol{U} \in (H^1(\Omega))^3$ and there is a constant C_Ω depending only on Ω such that

$$\|\boldsymbol{U}\|_{H^1} + \|\mathbf{curl}\,\boldsymbol{U}\|_{H^1} \le C_\Omega \|\boldsymbol{F}\|. \tag{2.3}$$

In (2.3), $\|\cdot\|_{H^1}$ denotes the norm in $(H^1(\Omega))^3$ and $H^1(\Omega) = \{u \in L^2(\Omega) : \boldsymbol{\nabla} u \in (L^2(\Omega))^3\}$. For later use, let us also denote $H_0^1(\Omega)$ to be the set of functions in $H^1(\Omega)$ which vanish on $\partial\Omega$.

The preconditioner which we shall consider is developed in terms of multilevel approximation subspaces of $H_0(\mathbf{curl})$. We start with a coarse partitioning of Ω into (nonoverlapping) tetrahedra $\mathcal{T}_1 = \{\tau_1^i : \ i = 1, \ldots N_0\}$. This forms a quasi-uniform mesh of mesh size d_1. A nested sequence of shape regular meshes $\mathcal{T}_k$, $k = 2, 3, \ldots,$ can be obtained by successively refining $\mathcal{T}_1$, using, for e.g., techniques given in [2]. For a given tetrahedron τ, let h_τ denote the radius of the largest ball contained in τ and H_τ denote the diameter of τ. By uniformity, we assume that there is a constant ζ not depending on $\mathcal{T}_i$ satisfying

$$\zeta h_\tau \ge H_\tau \quad \text{for all } \tau \in \mathcal{T}_i, \quad i = 1, \ldots, j. \tag{2.4}$$

Our goal is to solve the problem associated with the finest mesh $\mathcal{T}_j$, for some integer $j > 1$. The mesh size of $\mathcal{T}_1$ will be denoted by d_1 and can be taken to be the diameter of the largest tetrahedron. The mesh size of $\mathcal{T}_k$ is essentially $2^{1-k}d_1$.

For theoretical and practical purposes, the coarsest grid in the multilevel algorithm must be sufficiently fine. For $k = 1, \ldots, J$, let M_k denote the lowest order Nedelec finite element subspaces [22] of $H_0(\mathbf{curl})$ (of the first kind) based on $\mathcal{T}_{k+L}$ for some $L \ge 0$. The coarsest approximation subspace M_1 can be made sufficiently accurate by increasing L. Since the meshes are nested, it follows that

$$M_1 \subset M_2 \subset \ldots \subset M_J.$$

The space M_k has a mesh size of $h_k = 2^{1-L-k}d_1 = 2^{1-k}h_1$. Also let W_k be the subspace of continuous scalar functions which are linear in every element of $\mathcal{T}_{k+L}$. In Appendix A, we show how our results can be generalized to higher order Nedelec elements.

It was shown in [20] (see also [21]) that the discrete problem of finding $\boldsymbol{U}_k \in M_k$ satisfying

$$\boldsymbol{A}(\boldsymbol{U}_k,\boldsymbol{v}) = (\boldsymbol{F},\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in M_k, \tag{2.5}$$

has a unique solution provided h_k is small enough. We will assume that h_1 is small enough (or, equivalently L is large enough) so that (2.5) is uniquely solvable for $k = 1, 2, \ldots, J$.

In our analysis, we shall use the projector $\boldsymbol{P}_k : H_0(\mathbf{curl}) \mapsto M_k$ defined by

$$\boldsymbol{A}(\boldsymbol{P}_k \boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{A}(\boldsymbol{u}, \boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in M_k,$$

and the orthogonal L^2-projector $\boldsymbol{Q}_k : (L^2(\Omega))^3 \mapsto M_k$ defined by

$$(\boldsymbol{Q}_k \boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}, \boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in M_k.$$

That $\boldsymbol{P}_k$ is well defined (for $k = 1, 2, \ldots, J$) follows from the unique solvability of (2.5). Let us also introduce, for each k, an operator $\boldsymbol{A}_k : M_k \to M_k$ defined by

$$(\boldsymbol{A}_k \boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{A}(\boldsymbol{u}, \boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in M_k.$$

Problem (2.5), on level J, can be rewritten in the above notation as

$$\boldsymbol{A}_J \boldsymbol{U}_J = \boldsymbol{Q}_J \boldsymbol{F}. \tag{2.6}$$

We describe a simple multigrid algorithm for iteratively computing the solution $\boldsymbol{U}_J$ of (2.6). Given an initial iterate $\boldsymbol{u}_0 \in M_J$, we define a sequence approximating $\boldsymbol{U}_J$ by

$$\boldsymbol{u}_{i+1} = \boldsymbol{Mg}_J(\boldsymbol{u}_i, \boldsymbol{Q}_J \boldsymbol{F}). \tag{2.7}$$

Here $\boldsymbol{Mg}_J(\cdot,\cdot)$ is the map of $M_J \times M_J$ into M_J defined by the following algorithm.

Algorithm 2.1. Set $\boldsymbol{Mg}_1(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{A}_1^{-1} \boldsymbol{w}$. Let $k > 1$ and $\boldsymbol{v}, \boldsymbol{w} \in M_k$. Assuming that $\boldsymbol{Mg}_{k-1}(\cdot,\cdot)$ has been defined, we define $\boldsymbol{Mg}_k(\boldsymbol{v}, \boldsymbol{w})$ as follows:

(1) Set $\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{R}_k(\boldsymbol{w} - \boldsymbol{A}_k \boldsymbol{v})$.
(2) $\boldsymbol{Mg}_k(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{x} + \boldsymbol{Mg}_{k-1}(\boldsymbol{0}, \boldsymbol{Q}_{k-1}(\boldsymbol{w} - \boldsymbol{A}_k \boldsymbol{x}))$.

Here $\boldsymbol{R}_k : M_k \mapsto M_k$ is a linear smoothing operator. Note that in this multigrid algorithm (often called a "backslash cycle") we smooth only as we proceed to coarser grids. Our smoothing operators will always be based on a generalized block Jacobi or block Gauss Seidel iteration. In this case, the Gramm matrix inversions associated with $\boldsymbol{Q}_k$, $k = 2, \ldots, J$ are avoided (see, [5] or [24]). The smoother $\boldsymbol{R}_k$ will be defined in Section 3.

$\boldsymbol{Mg}_J(\cdot,\cdot)$ is a linear map from $M_J \times M_J$ into M_J. Moreover, the scheme is consistent in the sense that $\boldsymbol{v} = \boldsymbol{Mg}_J(\boldsymbol{v}, \boldsymbol{A}_J \boldsymbol{v})$ for all $\boldsymbol{v} \in M_J$. It easily follows that the linear operator $\boldsymbol{E} = \boldsymbol{Mg}_J(\cdot, 0)$ is the error reduction operator for (2.7), that is

$$\boldsymbol{u} - \boldsymbol{u}_{i+1} = \boldsymbol{E}(\boldsymbol{u} - \boldsymbol{u}_i).$$

Error reduction operators for variational multigrid algorithms generally have a product representation (see, e.g. [7]). Let $\boldsymbol{T}_k = \boldsymbol{R}_k \boldsymbol{A}_k \boldsymbol{P}_k$ for $k > 1$ and set $\boldsymbol{T}_1 = \boldsymbol{P}_1$. Let $\boldsymbol{E}_k \boldsymbol{u} = \boldsymbol{u} - \boldsymbol{Mg}_k(\boldsymbol{0}, \boldsymbol{A}_k \boldsymbol{P}_k \boldsymbol{u})$ and $\boldsymbol{E}_0 \equiv \boldsymbol{I}$, the identity operator. Then

$$\boldsymbol{E}_k = \boldsymbol{E}_{k-1}(\boldsymbol{I} - \boldsymbol{T}_k)$$

and

$$\boldsymbol{E} = (\boldsymbol{I} - \boldsymbol{T}_1)(\boldsymbol{I} - \boldsymbol{T}_2) \cdots (\boldsymbol{I} - \boldsymbol{T}_J). \tag{2.8}$$

The product representation of the error operator given above will be a fundamental ingredient in the convergence analysis presented in Section 4.

The above algorithm is a special case of more general multigrid algorithms in that we only use pre-smoothing. Alternatively, we could define an algorithm with just post-smoothing or both pre- and post-smoothing. The analysis of these algorithms is similar to that above and will not be presented. Algorithms with more than one smoothing are not generally advised since the smoothing iteration may be unstable.

Our multigrid analysis is based on perturbation and the estimates for the positive definite case. We define $\widetilde{\boldsymbol{P}}_k$, $\boldsymbol{\Lambda}_k$ and $\widetilde{\boldsymbol{T}}_k$ analogously to $\boldsymbol{P}_k$, $\boldsymbol{A}_k$ and $\boldsymbol{T}_k$ using the form $\boldsymbol{\Lambda}$ instead of $\boldsymbol{A}$.

3. Smoothers

In this section, we consider some smoothers appropriate for the multigrid algorithm (Algorithm 2.1). These smoothers are generalized Jacobi or Gauss-Seidel iterations, based on subspace decompositions of [1] and [17].

First, let us review the decomposition of [1]. For any $k \in \{2, 3, \ldots, J\}$, let $x_{k,i}$, $i = 1, \ldots, N_k^{\mathrm{I}}$, denote the interior vertices of the mesh $\mathcal{T}_{k+L}$. Let $\Omega_{k,i}^{\mathrm{I}}$ denote the interior of the union of the closures of the elements of $\mathcal{T}_{k+L}$ whose boundary contains $x_{k,i}$. Let $M_{k,i}^{\mathrm{I}}$ (resp. $W_{k,i}^{\mathrm{I}}$) denote the functions in M_k (resp. W_k) whose support is contained in $\overline{\Omega}_{k,i}^{\mathrm{I}}$. Then M_k admits the decomposition

$$M_k = \sum_{i=0}^{N_k^{\mathrm{I}}} M_{k,i}^{\mathrm{I}}.$$

Next, consider the decomposition of [17]. Let $\{\boldsymbol{\phi}_{k,i} : i = 1, \ldots n_k^M\}$ and $\{\psi_{k,i} : i = 1, \ldots n_k^W\}$ denote the usual nodal bases of M_k and W_k respectively. Then this decomposition is given by

$$M_k = \sum_{i=0}^{N_k^{\mathrm{II}}} M_{k,i}^{\mathrm{II}},$$

where $M_{k,i}^{\mathrm{II}}$ equals the span of $\boldsymbol{\phi}_{k,i}$ for $i = 1, \ldots n_k^M$, while for $i = n_k^M + j$, $j = 1, \ldots, n_k^W$, it equals the span of $\boldsymbol{\nabla}\psi_{k,j}$, and $N_k^{\mathrm{II}} = n_k^M + n_k^W$. Also let $\Omega_{k,i}^{\mathrm{II}}$ be such that $\overline{\Omega}_{k,i}^{\mathrm{II}}$ equals the support of nonzero functions in $M_{k,i}^{\mathrm{II}}$,

$$\mathring{M}_{k,i}^{\mathrm{I}} = \{\boldsymbol{u} \in M_{k,i}^{\mathrm{I}} \ : \ (\boldsymbol{u}, \boldsymbol{\nabla}\theta)_{\Omega_{k,i}^{\mathrm{I}}} = 0 \text{ for all } \theta \in W_{k,i}^{\mathrm{I}}\} \quad \text{for } i = 1, \ldots, N_k^{\mathrm{I}},$$
$$\mathring{M}_{k,i}^{\mathrm{II}} = M_{k,i}^{\mathrm{II}} \quad \text{for } i = 1, \ldots, n_k^M,$$

and let $\mathring{M}_{k,i}^{\mathrm{II}}$ for $i = n_k^M + 1, \ldots, N_k^{\mathrm{II}}$ be empty.

Our smoothers for the indefinite form are based on the above decompositions. Let $d \in \{\mathrm{I}, \mathrm{II}\}$. Operators $\boldsymbol{Q}_{k,i}^d$, $\boldsymbol{\Lambda}_{k,i}^d$, and $\boldsymbol{A}_{k,i}^d$ are defined analogously to $\boldsymbol{Q}_k$, $\boldsymbol{\Lambda}_k$ and $\boldsymbol{A}_k$, by replacing M_k with $M_{k,i}^d$. The smoothing operators involve local solves on $M_{k,i}^d$, so before we define them we must ensure that the operators $\{\boldsymbol{A}_{k,i}^d\}$ are invertible. That this is the case if h_1 is taken sufficiently small, is a consequence of the Poincaré-Friedrichs type inequality of the next lemma. This inequality will also be important for a subsequent perturbation analysis.

In the remainder of the paper, we adopt the convention of denoting by C or c a generic constant independent of all mesh sizes $\{h_k\}$ and the number of levels J. It will be explicitly stated when such an independence holds only in a range $0 < h_k < H$ for some H (i.e., only for small enough mesh sizes).

Lemma 3.1. *For any* $\boldsymbol{q} \in \mathring{M}_{k,i}^d$, $d \in \{\mathrm{I}, \mathrm{II}\}$, $k = 2, \ldots, J$,

$$\|\boldsymbol{q}\| \le C h_k \|\mathbf{curl}\, \boldsymbol{q}\|. \tag{3.1}$$

Remark 3.1. Note that for discretely divergence free functions on a convex domain, such an inequality is proved in [15]. However, $\Omega^d_{k,i}$ may be non-convex and we need the constant in the inequality to be independent of the shape of the mesh patches. The proof does not follow from a simple scaling argument as the discrete divergence free condition does not carry over under linear mapping unless the transformation is unitary.

Remark 3.2. In the case $d = \text{II}$ and lowest order elements, this inequality is well known [17] and a simple proof can be given by a scaling argument.

Proof. Consider first a tetrahedron τ with a face f contained in the x–y plane with the origin at the barycenter of the face. A function $\boldsymbol{\phi}$ in the lowest order Nedelec edge space on τ with vanishing tangential components on f has the form

$$\boldsymbol{\phi} = (0, 0, \eta) + (\alpha_1, \alpha_2, 0) \times (x, y, z). \tag{3.2}$$

Here η, α_1 and α_2 are constants. Moreover, $\mathbf{curl}\,\boldsymbol{\phi} = 2\boldsymbol{\alpha}$ where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, 0)$. Also note that if $\boldsymbol{a}$ is the vertex of τ not in f, and $\boldsymbol{c}$ is any vertex of f, then the tangential component of $\boldsymbol{\phi}$ along the edge connecting $\boldsymbol{a}$ to $\boldsymbol{c}$ is given by

$$(\eta\, a_3 + (\boldsymbol{\alpha} \times \boldsymbol{c}) \cdot \boldsymbol{a})/|\boldsymbol{a} - \boldsymbol{c}|, \tag{3.3}$$

where a_3 is the z-component of $\boldsymbol{a}$. We will now prove the lemma for decompositions I and II separately.

Case $d = \text{I}$: Let D be the domain formed by a collection of unit sized tetrahedra τ_j, $j = 0, 1, \ldots, N$ meeting at vertex $\boldsymbol{a}$ and let the corresponding approximation spaces (of $H_0(\mathbf{curl}\,; D)$ and $H^1_0(D)$ respectively) be denoted by M'_D and W'_D. Furthermore, let $\mathring{M}'_D = \{\boldsymbol{v} \in M'_D : (\boldsymbol{v}, \boldsymbol{\nabla}\theta)_D = 0, \text{ for all } \theta \in W'_D\}$. If we show that

$$\|\boldsymbol{v}\| \le C\|\mathbf{curl}\,\boldsymbol{v}\| \quad \text{for all } \boldsymbol{v} \in \mathring{M}'_D, \tag{3.4}$$

the required result follows easily by dilation.

Let $\boldsymbol{\phi} \in M'_D$, and f_j denote the face of τ_j not containing $\boldsymbol{a}$, and $\boldsymbol{c}$ be a vertex of f_j. Let $\boldsymbol{a}$ and $\boldsymbol{c}$ have local coordinate triples $\boldsymbol{a}_j \equiv (a_{j,1}, a_{j,2}, a_{j,3})$ and $\boldsymbol{c}_j$ respectively in the coordinate system on each tetrahedron which has f_j in the $x - y$ plane, and the origin at its barycenter. Then, by (3.2), $\boldsymbol{\phi}$ has the form $\boldsymbol{\phi} = (0, 0, \eta_j) + \boldsymbol{\alpha}_j \times (x, y, z)$. By (3.3), the tangential component of $\boldsymbol{\phi}$ along the edge connecting $\boldsymbol{a}$ to $\boldsymbol{c}$ is given by $(\eta_j\, a_{j,3} + (\boldsymbol{\alpha}_j \times \boldsymbol{c}_j) \cdot \boldsymbol{a}_j)/|\boldsymbol{a}_j - \boldsymbol{c}_j|$. If τ_l is another tetrahedron in D sharing the vertex $\boldsymbol{c}$ then the same quantity is also given by $(\eta_l\, a_{l,3} + (\boldsymbol{\alpha}_l \times \boldsymbol{c}_l) \cdot \boldsymbol{a}_l)/|\boldsymbol{a}_l - \boldsymbol{c}_l|$. Here subscripts l indicate coordinates in the τ_l system. Thus,

$$\eta_l = \frac{\eta_j\, a_{j,3} + (\boldsymbol{\alpha}_j \times \boldsymbol{c}_j) \cdot \boldsymbol{a}_j - (\boldsymbol{\alpha}_l \times \boldsymbol{c}_l) \cdot \boldsymbol{a}_l}{a_{l,3}}. \tag{3.5}$$

Let $\boldsymbol{v} \in \mathring{M}'_D$. We will construct a function $\boldsymbol{\phi}$ in M'_D which satisfies

$$\mathbf{curl}\,\boldsymbol{\phi} = \mathbf{curl}\,\boldsymbol{v} \quad \text{and} \quad \|\boldsymbol{\phi}\| \le C\|\mathbf{curl}\,\boldsymbol{v}\|, \tag{3.6}$$

with C depending only on the quasi-uniformity condition. Note that $\boldsymbol{v} - \boldsymbol{\phi}$ is a gradient of a function in W'_D so

$$\|\boldsymbol{v}\| \le \|\boldsymbol{\phi}\| \le C\|\mathbf{curl}\,\boldsymbol{v}\|,$$

i.e., (3.4) follows if we construct $\boldsymbol{\phi}$ satisfying (3.6).

We define $\boldsymbol{\phi} = \boldsymbol{v} - \mu \boldsymbol{\nabla}\psi_{\boldsymbol{a}}$ where $\psi_{\boldsymbol{a}}$ is the nodal function in W_D' which is one on $\boldsymbol{a}$ and μ is to be determined. Clearly, $\boldsymbol{\nabla}\psi_{\boldsymbol{a}}$ has a local representation of the form

$$\boldsymbol{\nabla}\psi_{\boldsymbol{a}} = (0, 0, \zeta_j)$$

on τ_j with $\zeta_j \neq 0$. We choose μ so that $\eta_0 = 0$ in the above representation of $\boldsymbol{\phi}$. All of the remaining η_j's in the representation of $\boldsymbol{\phi}$ can be determined from the $\boldsymbol{\alpha}_j$'s by (3.5). By quasi-uniformity, $\{a_{l,3}\}$ are uniformly bounded away from zero so magnitudes of the η_l's can be bounded in terms of the $\boldsymbol{\alpha}$'s. Now (3.6) follows by quasi-uniformity and the fact that the $\boldsymbol{\alpha}$'s can be bounded in terms of $\|\mathbf{curl}\, \boldsymbol{v}\|$.

Case $d = \mathrm{II}$: Let τ, f, $\boldsymbol{a}$, and $\boldsymbol{c}$ be as in the beginning of this proof. Then, in the coordinate system there, the nodal basis function $\boldsymbol{\phi}$ of the edge connecting $\boldsymbol{a}$ to $\boldsymbol{c}$ has the representation (3.2). Moreover, if $\boldsymbol{b}$ is an alternate vertex of f, then $\eta = -(\boldsymbol{\alpha} \times \boldsymbol{b}) \cdot \boldsymbol{a}/a_3$. Since η can be bounded by $\boldsymbol{\alpha} = \mathbf{curl}\, \boldsymbol{\phi}/2$, the proof can be finished in the same way as before. □

Proposition 3.1. *There exists an $H > 0$ such that whenever $h_1 \leq H$, any solution $\boldsymbol{p}_{k,i}^d \in M_{k,i}^d$ of the square system*

$$\boldsymbol{A}(\boldsymbol{p}_{k,i}^d, \boldsymbol{v}_{k,i}) = \boldsymbol{A}(\boldsymbol{u}, \boldsymbol{v}_{k,i}) \quad \textit{for all } \boldsymbol{v}_{k,i} \in M_{k,i}^d,$$

satisfies

$$\left\|\boldsymbol{p}_{k,i}^d\right\|_{\boldsymbol{\Lambda},\Omega_{k,i}^d} \leq C \left\|\boldsymbol{u}\right\|_{\boldsymbol{\Lambda},\Omega_{k,i}^d} \tag{3.7}$$

for $\boldsymbol{u} \in M_k$ and for all $i = 1, \ldots, N_k$ and $d \in \{\mathrm{I}, \mathrm{II}\}$. It follows that $\boldsymbol{A}_{k,i}^d$ is nonsingular.

Proof. In the case of decomposition I, the proof proceeds exactly as an analogous result in [16, Lemma 4.2], and we omit it.

In the case $d = \mathrm{II}$, for $i = 1, \ldots, n_k^M$, (3.7) follows for sufficiently small h_k from

$$\|\mathbf{curl}\, \boldsymbol{p}_{k,i}^d\|^2 - \omega^2 \|\boldsymbol{p}_{k,i}^d\|^2 = (\mathbf{curl}\, \boldsymbol{u}, \mathbf{curl}\, \boldsymbol{p}_{k,i}^d) - \omega^2 (\boldsymbol{u}, \boldsymbol{p}_{k,i}^d),$$

by applying the Cauchy-Schwarz inequality on the right hand side, and Lemma 3.1 on the left hand side. For the remaining i, since $\omega > 0$,

$$\|\boldsymbol{p}_{k,i}^d\|^2 = (\boldsymbol{u}, \boldsymbol{p}_{k,i}^d),$$

so (3.7) follows. □

Not only does Proposition 3.1 yield the invertibility of $\boldsymbol{A}_{k,i}^d$, but it also implies that the projection operator, $\boldsymbol{P}_{k,i}^d : M_k \mapsto M_{k,i}$, given by

$$\boldsymbol{A}(\boldsymbol{P}_{k,i}^d \boldsymbol{u}, \boldsymbol{v}_{k,i}) = \boldsymbol{A}(\boldsymbol{u}, \boldsymbol{v}_{k,i}) \quad \text{for all } \boldsymbol{u} \in M_k, \boldsymbol{v}_{k,i} \in M_{k,i}^d, \quad d \in \{\mathrm{I}, \mathrm{II}\}, \tag{3.8}$$

is well defined. Moreover, (3.7) implies

$$\left\|\boldsymbol{P}_{k,i}^d \boldsymbol{u}\right\|_{\boldsymbol{\Lambda},\Omega_{k,i}^d} \leq C \left\|\boldsymbol{u}\right\|_{\boldsymbol{\Lambda},\Omega_{k,i}^d} \tag{3.9}$$

for all $\boldsymbol{u} \in M_k$. Also define $\widetilde{\boldsymbol{P}}_{k,i}^d$ analogously to $\boldsymbol{P}_{k,i}^d$ by replacing $\boldsymbol{A}$ with $\boldsymbol{\Lambda}$ in (3.8).

Now that we have proven the invertibility of $\boldsymbol{A}_{k,i}^d$, we can define the smoothers for the indefinite problem. Jacobi type smoothers $\boldsymbol{J}_k^{\mathrm{I}}$ and $\boldsymbol{J}_k^{\mathrm{II}}$ are given by

$$\boldsymbol{J}_k^d = \gamma \sum_{i=0}^{N_k^d} (\boldsymbol{A}_{k,i}^d)^{-1} \boldsymbol{Q}_{k,i}, \qquad d \in \{\mathrm{I}, \mathrm{II}\}, \tag{3.10}$$

where γ is a scaling factor. Gauss-Seidel type smoothers $\boldsymbol{G}_k^d$ for $d \in \{\mathrm{I}, \mathrm{II}\}$ are defined by the following algorithm:

Algorithm 3.1 (Indefinite Gauss Seidel). Let $\boldsymbol{f}$ be in M_k. We define $\boldsymbol{G}_k^d$ by

(1) Set $\boldsymbol{v}_0 = 0 \in M_k$.
(2) Define $\boldsymbol{v}_i$, for $i = 1, \ldots, N_k^d$, by

$$\boldsymbol{v}_i = \boldsymbol{v}_{i-1} + (\boldsymbol{A}_{k,i}^d)^{-1}\boldsymbol{Q}_{k,i}^d(\boldsymbol{f} - \boldsymbol{A}_k\boldsymbol{v}_{i-1}).$$

(3) Set $\boldsymbol{G}_k^d\boldsymbol{f} = \boldsymbol{v}_{N_k^d}$.

The analogous Jacobi and Gauss-Seidel smoothers were given in [1] and [17] for the positive definite operators $\boldsymbol{\Lambda}_k$. These are denoted here by $\widetilde{\boldsymbol{J}}_k^d$ and $\widetilde{\boldsymbol{G}}_k^d$, and are again defined by (3.10) and Algorithm 3.1 respectively, but with $\boldsymbol{\Lambda}$ in place of $\boldsymbol{A}$. The scaling factor γ in (3.10) is chosen such that the $\boldsymbol{\Lambda}$-norm of $\boldsymbol{I} - \widetilde{\boldsymbol{J}}_k^d\boldsymbol{\Lambda}_k$ is less than or equal to one for $k = 2, \ldots, J$. Such a γ can be chosen independent of J by the limited overlap property of the subspaces.

Remark 3.3. In implementation, the application of the operator $(\boldsymbol{A}_{k,i}^d)^{-1}\boldsymbol{Q}_{k,i}^d$ reduces to solving a linear system involving the stiffness matrix associated with the indefinite form $\boldsymbol{A}(\cdot,\cdot)$ and the Gramm matrix inversion corresponding to $\boldsymbol{Q}_{k,i}^d$ is avoided.

4. Analysis of the multigrid iteration.

In this section we provide an analysis of the multigrid iteration of Section 2. This analysis is based on the product representation of the error operator (2.8). As done in [6] for second order elliptic problems, our analysis is based on perturbation from the uniform multigrid convergence estimates for a related symmetric positive definite problem.

We start with the estimate for the positive definite problem. For operators on M_k, $k = 1, \ldots J$, we will use $\|\cdot\|_{\boldsymbol{\Lambda}}$ to denote the operator norm induced by the vector norm $\boldsymbol{\Lambda}(\cdot,\cdot)^{1/2}$. Set $\widetilde{\boldsymbol{R}}_k$ to be any one of $\widetilde{\boldsymbol{J}}_k^{\mathrm{I}}, \widetilde{\boldsymbol{J}}_k^{\mathrm{II}}, \widetilde{\boldsymbol{G}}_k^{\mathrm{I}}$ and $\widetilde{\boldsymbol{G}}_k^{\mathrm{II}}$. Let $\widetilde{\boldsymbol{T}}_k = \widetilde{\boldsymbol{R}}_k\boldsymbol{\Lambda}_k\widetilde{\boldsymbol{P}}_k$ for $k > 1$ and $\widetilde{\boldsymbol{T}}_1 = \widetilde{\boldsymbol{P}}_1$. Consider Algorithm 2.1 with $\boldsymbol{\Lambda}_k$ in place of $\boldsymbol{A}_k$ and $\widetilde{\boldsymbol{R}}_k$ in place of $\boldsymbol{R}_k$. Its error reduction operator is

$$\widetilde{\boldsymbol{E}} = (\boldsymbol{I} - \widetilde{\boldsymbol{T}}_1)(\boldsymbol{I} - \widetilde{\boldsymbol{T}}_2)\cdots(\boldsymbol{I} - \widetilde{\boldsymbol{T}}_J). \tag{4.1}$$

The following result is contained in [1, Theorems 3.1 and 4.2], [17, Theorem 3.1] and [18, Theorem 5.4].

Theorem 4.1. *The multigrid error reduction operator in the case of the positive definite problem satisfies*

$$\boldsymbol{\Lambda}(\widetilde{\boldsymbol{E}}\boldsymbol{u}, \widetilde{\boldsymbol{E}}\boldsymbol{u}) \le \hat{\delta}^2\boldsymbol{\Lambda}(\boldsymbol{u},\boldsymbol{u}) \quad \text{for all } \boldsymbol{u} \in M_J, \tag{4.2}$$

with $0 < \hat{\delta} < 1$ *independent of* J.

Remark 4.1. Although the results in [1] are formulated only for symmetric smoothers, let us verify that (4.2) holds for the nonsymmetric Gauss-Seidel smoother $\widetilde{\boldsymbol{G}}_k^{\mathrm{I}}$ as well, as stated in Theorem 4.1. Indeed, we can be more general and consider instead the smoothing operator $\widetilde{\boldsymbol{R}}_k$ of a block successive over-relaxation iteration (SOR(α)) with a relaxation parameter $0 < \alpha < 2$ (with the blocks based on $\{\widetilde{M}_{k,i}^d\}$, $d \in \{\mathrm{I}, \mathrm{II}\}$). We appeal to [9, Lemma 2.2], which shows that (4.2) holds for the $\widetilde{\boldsymbol{E}}$ obtained by any $\widetilde{\boldsymbol{R}}_k$, provided

$$\|\boldsymbol{I} - \widetilde{\boldsymbol{R}}_k \boldsymbol{\Lambda}_k\|_{\boldsymbol{\Lambda}} \leq 1, \quad \text{and} \tag{4.3}$$

$$(\underline{\widetilde{\boldsymbol{R}}}_k^{-1} \boldsymbol{u}, \boldsymbol{u}) \leq C\boldsymbol{\Lambda}(\boldsymbol{u}, \boldsymbol{u}), \quad \text{for all } \boldsymbol{u} \in (\boldsymbol{I} - \widetilde{\boldsymbol{P}}_{k-1})M_k, \tag{4.4}$$

where $\underline{\widetilde{\boldsymbol{R}}}_k = \widetilde{\boldsymbol{R}}_k + \widetilde{\boldsymbol{R}}_k^t - \widetilde{\boldsymbol{R}}_k^t \boldsymbol{\Lambda}_k \widetilde{\boldsymbol{R}}_k$. Here $\widetilde{\boldsymbol{R}}_k^t$ is the L^2-adjoint of $\widetilde{\boldsymbol{R}}_k$. That inequality (4.3) holds for the $\widetilde{\boldsymbol{R}}_k$ of SOR(α) follows immediately from the product representation,

$$\boldsymbol{I} - \widetilde{\boldsymbol{R}}_k \boldsymbol{\Lambda}_k = (\boldsymbol{I} - \alpha \widetilde{\boldsymbol{P}}^{\mathrm{I}}_{k,N_k}) \cdots (\boldsymbol{I} - \alpha \widetilde{\boldsymbol{P}}^{\mathrm{I}}_{k,1}).$$

It remains to see that (4.4) holds for this smoother. Techniques in [1] can be used to prove

$$\inf_{\{\boldsymbol{u}_i\}} \sum_{i=1}^{N_k^d} \boldsymbol{\Lambda}(\boldsymbol{u}_i, \boldsymbol{u}_i) \leq C\boldsymbol{\Lambda}(\boldsymbol{u}, \boldsymbol{u}), \quad \text{for all } \boldsymbol{u} \in (\boldsymbol{I} - \widetilde{\boldsymbol{P}}_{k-1})M_k,$$

where the infimum is taken over all decompositions $\boldsymbol{u} = \sum_{i=1}^{N_k^d} \boldsymbol{u}_i$ such that $\boldsymbol{u}_i \in M^d_{k,i}$. It can be shown as in [7, Theorem 2.2] that for the $\widetilde{\boldsymbol{R}}_k$ of SOR(α),

$$(\underline{\widetilde{\boldsymbol{R}}}_k^{-1} \boldsymbol{u}, \boldsymbol{u}) \leq \frac{(1 + c\alpha)^2}{2 - \alpha} \inf_{\{\boldsymbol{u}_i\}} \sum_{i=1}^{N_k^d} \boldsymbol{\Lambda}(\boldsymbol{u}_i, \boldsymbol{u}_i), \quad \text{for all } \boldsymbol{u} \in M_k.$$

Thus, Theorem 4.1 holds for the SOR(α) smoother.

We will analyze the multigrid algorithm by examining the difference between $\boldsymbol{E}$ and $\widetilde{\boldsymbol{E}}$. Let $\boldsymbol{Z}_k = \boldsymbol{T}_k - \widetilde{\boldsymbol{T}}_k$ and suppose we have

$$\|\boldsymbol{Z}_1\|_{\boldsymbol{\Lambda}} \leq \epsilon, \quad \text{and} \tag{4.5}$$

$$\|\boldsymbol{Z}_k\|_{\boldsymbol{\Lambda}} \leq C_1 h_k, \quad \text{for } k = 2, \ldots, J, . \tag{4.6}$$

Then, it can be shown that the difference $\boldsymbol{E}_k - \widetilde{\boldsymbol{E}}_k$ is small by an argument of [6] (see also [8, Lemma 11.1]). We include the argument here for the sake of completeness: First, note that by triangle inequality, the $\boldsymbol{\Lambda}$-norm of $(\boldsymbol{I} - \boldsymbol{T}_k) = (\boldsymbol{I} - \widetilde{\boldsymbol{T}}_k - \boldsymbol{Z}_k)$ is less than or equal to $1 + ch_k$. Therefore,

$$\|\boldsymbol{E}_k\|_{\boldsymbol{\Lambda},\Omega} \leq (1 + c\epsilon) \prod_{i=2}^{k} (1 + ch_i),$$

which can be bounded by a convergent infinite product. Thus $\|\boldsymbol{E}_k\|_{\boldsymbol{\Lambda},\Omega} \leq C$.

To continue, we observe the following recursion:

$$\boldsymbol{E}_k - \widetilde{\boldsymbol{E}}_k = (\boldsymbol{E}_{k-1} - \widetilde{\boldsymbol{E}}_{k-1})(\boldsymbol{I} - \widetilde{\boldsymbol{T}}_k) - \boldsymbol{E}_{k-1} \boldsymbol{Z}_k \tag{4.7}$$

which implies that for $k > 1$,

$$\begin{aligned} \|\boldsymbol{E}_k - \widetilde{\boldsymbol{E}}_k\|_{\boldsymbol{\Lambda},\Omega} &\leq \|\boldsymbol{E}_{k-1} - \widetilde{\boldsymbol{E}}_{k-1}\|_{\boldsymbol{\Lambda},\Omega} \|\boldsymbol{I} - \widetilde{\boldsymbol{T}}_k\|_{\boldsymbol{\Lambda},\Omega} + \|\boldsymbol{E}_{k-1}\|_{\boldsymbol{\Lambda},\Omega} \|\boldsymbol{Z}_k\|_{\boldsymbol{\Lambda},\Omega} \\ &\leq \|\boldsymbol{E}_{k-1} - \widetilde{\boldsymbol{E}}_{k-1}\|_{\boldsymbol{\Lambda},\Omega} + Ch_k. \end{aligned}$$

Repeated application of this inequality shows that the difference $\boldsymbol{E}_k - \widetilde{\boldsymbol{E}}_k$ is small:

$$\|\boldsymbol{E}_J - \widetilde{\boldsymbol{E}}_J\|_{\boldsymbol{\Lambda},\Omega} \leq c(h_1 + \epsilon).$$

Thus, we have proven the following theorem:

Theorem 4.2. *Let $\boldsymbol{E}$ satisfy (2.8) and $\widetilde{\boldsymbol{E}}$ satisfy (4.1). Assume that (4.5) and (4.6) holds. Then there are positive constants C, $\hat{h}_1$ and $\hat{\epsilon}$ depending only on C_1 above such that if $h_1 \le \hat{h}_1$ and $\epsilon \le \hat{\epsilon}$,*

$$\|\boldsymbol{E}\|_{\boldsymbol{\Lambda}} \le \|\widetilde{\boldsymbol{E}}\|_{\boldsymbol{\Lambda}} + C(h_1 + \epsilon).$$

In (4.5) and (4.6), the operator norm of $\boldsymbol{Z}_k$ can be taken to be that of $\boldsymbol{Z}_k : M_J \mapsto M_k$ or $\boldsymbol{Z}_k : M_k \mapsto M_k$, as both norms are equal. The proofs of our main results proceed by verifying (4.5) and (4.6). In verification of (4.5), the nature of the subspace decompositions is immaterial, and a coarse grid estimate of [16] is critical, as seen in the following lemma.

Lemma 4.1. *There exists $H > 0$ such that if $h_1 \le H$ then (4.5) holds with $\epsilon = ch_1$.*

Proof. For $\boldsymbol{u}, \boldsymbol{v} \in M_J$, the following identity holds:

$$\begin{aligned} \boldsymbol{\Lambda}(\boldsymbol{Z}_1\boldsymbol{u}, \boldsymbol{v}) &= \boldsymbol{\Lambda}(\boldsymbol{P}_1\boldsymbol{u} - \boldsymbol{u}, \widetilde{\boldsymbol{P}}_1\boldsymbol{v}) \\ &= \boldsymbol{A}(\boldsymbol{P}_1\boldsymbol{u} - \boldsymbol{u}, \widetilde{\boldsymbol{P}}_1\boldsymbol{v}) + (\omega^2 + 1)(\boldsymbol{P}_1\boldsymbol{u} - \boldsymbol{u}, \widetilde{\boldsymbol{P}}_1\boldsymbol{v}) \\ &= (\omega^2 + 1)(\boldsymbol{P}_1\boldsymbol{u} - \boldsymbol{u}, \widetilde{\boldsymbol{P}}_1\boldsymbol{v}). \end{aligned} \tag{4.8}$$

It is shown in [16], using a duality argument utilizing the regularity assumption, that there exists $H > 0$ such that if $h_1 \le H$ then

$$(\boldsymbol{u} - \boldsymbol{P}_1\boldsymbol{u}, \boldsymbol{w}) \le Ch_1 \|\boldsymbol{u} - \boldsymbol{P}_1\boldsymbol{u}\|_{\boldsymbol{\Lambda}} \|\boldsymbol{w}\|_{\boldsymbol{\Lambda}}$$

for all $\boldsymbol{u} \in M_J$ and $\boldsymbol{w} \in M_1$. Thus, the lemma follows. □

While verifying (4.6) for specific smoothers, it will be useful to have bounds for the perturbation operators $\boldsymbol{Z}^d_{k,i} : M_J \mapsto M^d_{k,i}$, $d \in \{\mathrm{I}, \mathrm{II}\}$ defined by

$$\boldsymbol{Z}^d_{k,i} = \boldsymbol{P}^d_{k,i} - \widetilde{\boldsymbol{P}}^d_{k,i}.$$

Note that in the case of subspaces of gradients of decomposition II,

$$\boldsymbol{Z}^{\mathrm{II}}_{k,i} = 0 \quad \text{for } i = n^M_k + 1, \ldots, N^{\mathrm{II}}_k.$$

An identity similar to (4.8) can be obtained for $\boldsymbol{Z}^d_{k,i}$:

$$\boldsymbol{\Lambda}(\boldsymbol{Z}^d_{k,i}\boldsymbol{u}, \boldsymbol{v}) = -(\omega^2 + 1)(\boldsymbol{u} - \boldsymbol{P}^d_{k,i}\boldsymbol{u}, \widetilde{\boldsymbol{P}}^d_{k,i}\boldsymbol{v}). \tag{4.9}$$

Lemma 4.2. *There exists $H > 0$ such that if $h_1 \le H$,*

$$(\boldsymbol{u} - \boldsymbol{P}^d_{k,i}\boldsymbol{u}, \boldsymbol{v}_{k,i}) \le Ch_k \|\boldsymbol{u} - \boldsymbol{P}^d_{k,i}\boldsymbol{u}\|_{0,\Omega^d_{k,i}} \|\mathbf{curl}\, \boldsymbol{v}_{k,i}\|_{0,\Omega^d_{k,i}}$$

for all $\boldsymbol{u} \in M_J$ and $\boldsymbol{v}_{k,i} \in M^d_{k,i}$, $d \in \{\mathrm{I}, \mathrm{II}\}$, $k = 2, \ldots, J$.

Proof. In the case $d = \mathrm{I}$, observe that for any $\boldsymbol{u} \in M_J$, $\boldsymbol{u} - \boldsymbol{P}^{\mathrm{I}}_{k,i}\boldsymbol{u}$ is L^2-orthogonal to functions of the form $\boldsymbol{\nabla} w$ for any $w \in W^{\mathrm{I}}_{k,i}$. Decomposing $\boldsymbol{v}_{k,i} = \boldsymbol{\nabla} w + \boldsymbol{x}$ where $w \in W^{\mathrm{I}}_{k,i}$ and $\boldsymbol{x} \in \mathring{M}^{\mathrm{I}}_{k,i}$ and applying Lemma 3.1 gives

$$\begin{aligned} (\boldsymbol{u} - \boldsymbol{P}_{k,i}\boldsymbol{u}, \boldsymbol{v}_{k,i}) = (\boldsymbol{u} - \boldsymbol{P}_{k,i}\boldsymbol{u}, \boldsymbol{x}) &\le Ch_k \|\boldsymbol{u} - \boldsymbol{P}_{k,i}\boldsymbol{u}\|_{0,\Omega_{k,i}} \|\mathbf{curl}\, \boldsymbol{x}\|_{0,\Omega_{k,i}} \\ &= Ch_k \|\boldsymbol{u} - \boldsymbol{P}_{k,i}\boldsymbol{u}\|_{0,\Omega_{k,i}} \|\mathbf{curl}\, \boldsymbol{v}_{k,i}\|_{0,\Omega_{k,i}}. \end{aligned}$$

In the case $d = \mathrm{II}$, the result immediately follows from Cauchy-Schwarz inequality and Lemma 3.1 for $\boldsymbol{v}_{k,i} \in \mathring{M}^{\mathrm{II}}_{k,i}$. For the remaining $\boldsymbol{v}_{k,i} \in M^{\mathrm{II}}_{k,i}$, both sides of the inequality of the lemma are zero. □

The following theorem is our main result.

Theorem 4.3. *In Algorithm 2.1, set $\boldsymbol{R}_k$ to any of the smoothers $\boldsymbol{J}_k^{\mathrm{I}}, \boldsymbol{G}_k^{\mathrm{I}}, \boldsymbol{J}_k^{\mathrm{II}}$ and $\boldsymbol{G}_k^{\mathrm{II}}$ defined earlier. Then there exists an $H > 0$ such that whenever $h_1 \le H$,*

$$\boldsymbol{\Lambda}(\boldsymbol{E}\boldsymbol{u}, \boldsymbol{E}\boldsymbol{u}) \le \delta^2 \boldsymbol{\Lambda}(\boldsymbol{u}, \boldsymbol{u}) \quad \text{for all } \boldsymbol{u} \in M_J,$$

for $\delta = \hat{\delta} + ch_1$. Here $\hat{\delta}$ is less than one (and independent of J) and is given by Theorem 4.1 applied to the corresponding smoother $\widetilde{\boldsymbol{R}}_k$. In addition, c is independent of h_1.

Proof. We apply Theorem 4.2. By Lemma 4.1, we need only verify (4.6) for each of the smoothers $\boldsymbol{J}_k^{\mathrm{I}}, \boldsymbol{G}_k^{\mathrm{I}}, \boldsymbol{J}_k^{\mathrm{II}}$ and $\boldsymbol{G}_k^{\mathrm{II}}$. Since the proof for the case of the latter two smoothers are completely analogous to the case of the smoothers based on decomposition I, we only give the proof for $\boldsymbol{J}_k^{\mathrm{I}}$ and $\boldsymbol{G}_k^{\mathrm{I}}$.

In the case of $\boldsymbol{R}_k = \boldsymbol{J}_k^{\mathrm{I}}$, the perturbation operator $\boldsymbol{Z}_k$, $k > 1$, satisfies

$$\boldsymbol{Z}_k \boldsymbol{u} = \gamma \sum_{i=1}^{N_k} (\boldsymbol{P}_{k,i}^{\mathrm{I}} - \widetilde{\boldsymbol{P}}_{k,i}^{\mathrm{I}}) \boldsymbol{u} = \gamma \sum_{i=1}^{N_k} \boldsymbol{Z}_{k,i}^{\mathrm{I}} \boldsymbol{u},$$

for any $\boldsymbol{u} \in M_k$. By (4.9), Lemma 4.2 and (3.9),

$$\boldsymbol{\Lambda}(\boldsymbol{Z}_{k,i}^{\mathrm{I}} \boldsymbol{u}, \boldsymbol{v}) = (\omega^2 + 1)(\boldsymbol{P}_{k,i}^{\mathrm{I}} \boldsymbol{u} - \boldsymbol{u}, \widetilde{\boldsymbol{P}}_{k,i}^{\mathrm{I}} \boldsymbol{v}) \le ch_k \|\boldsymbol{u}\|_{\boldsymbol{\Lambda}, \Omega_{k,i}^{\mathrm{I}}} \|\boldsymbol{v}\|_{\boldsymbol{\Lambda}, \Omega_{k,i}^{\mathrm{I}}}, \tag{4.10}$$

for any $\boldsymbol{u}, \boldsymbol{v} \in M_k$. Hence,

$$\boldsymbol{\Lambda}(\boldsymbol{Z}_k \boldsymbol{u}, \boldsymbol{v}) \le ch_k \sum_{i=1}^{N_k} \|\boldsymbol{u}\|_{\boldsymbol{\Lambda}, \Omega_{k,i}^{\mathrm{I}}} \|\boldsymbol{v}\|_{\boldsymbol{\Lambda}, \Omega_{k,i}^{\mathrm{I}}}.$$

The inequality (4.6) now easily follows using the limited overlap properties of the domains $\Omega_{k,i}^{\mathrm{I}}$. This completes the proof of the theorem when $\boldsymbol{R}_k = \boldsymbol{J}_k^{\mathrm{I}}$.

Now consider the case $\boldsymbol{R}_k = \boldsymbol{G}_k^{\mathrm{I}}$. As before, it suffices to verify (4.6). Define $\widetilde{\boldsymbol{\mathcal{E}}}_i$ and $\boldsymbol{\mathcal{E}}_i$ by

$$\widetilde{\boldsymbol{\mathcal{E}}}_i = (\boldsymbol{I} - \widetilde{\boldsymbol{P}}_{k,i}^{\mathrm{I}})(\boldsymbol{I} - \widetilde{\boldsymbol{P}}_{k,i-1}^{\mathrm{I}}) \cdots (\boldsymbol{I} - \widetilde{\boldsymbol{P}}_{k,1}^{\mathrm{I}}) \text{ and}$$
$$\boldsymbol{\mathcal{E}}_i = (\boldsymbol{I} - \boldsymbol{P}_{k,i}^{\mathrm{I}})(\boldsymbol{I} - \boldsymbol{P}_{k,i-1}^{\mathrm{I}}) \cdots (\boldsymbol{I} - \boldsymbol{P}_{k,1}^{\mathrm{I}}),$$

and let $\widetilde{\boldsymbol{\mathcal{E}}}_0 = \boldsymbol{\mathcal{E}}_0 = \boldsymbol{I}$. Then the perturbation operator $\boldsymbol{Z}_k : M_k \mapsto M_k$ for this example is

$$\boldsymbol{Z}_k = \boldsymbol{T}_k - \widetilde{\boldsymbol{T}}_k = \widetilde{\boldsymbol{\mathcal{E}}}_{N_k} - \boldsymbol{\mathcal{E}}_{N_k}.$$

We clearly have that

$$\widetilde{\boldsymbol{\mathcal{E}}}_i - \boldsymbol{\mathcal{E}}_i = (\boldsymbol{I} - \widetilde{\boldsymbol{P}}_{k,i}^{\mathrm{I}})(\widetilde{\boldsymbol{\mathcal{E}}}_{i-1} - \boldsymbol{\mathcal{E}}_{i-1}) - \boldsymbol{Z}_{k,i}^{\mathrm{I}} \boldsymbol{\mathcal{E}}_{i-1}.$$

Since the terms on the right are orthogonal with respect to $\boldsymbol{\Lambda}(\cdot,\cdot)$,

$$\|(\widetilde{\boldsymbol{\mathcal{E}}}_i - \boldsymbol{\mathcal{E}}_i)\boldsymbol{u}\|_{\boldsymbol{\Lambda},\Omega}^2 = \|(\boldsymbol{I} - \widetilde{\boldsymbol{P}}_{k,i}^{\mathrm{I}})(\widetilde{\boldsymbol{\mathcal{E}}}_{i-1} - \boldsymbol{\mathcal{E}}_{i-1})\boldsymbol{u}\|_{\boldsymbol{\Lambda},\Omega}^2 + \|\boldsymbol{Z}_{k,i}^{\mathrm{I}} \boldsymbol{\mathcal{E}}_{i-1} \boldsymbol{u}\|_{\boldsymbol{\Lambda},\Omega}^2.$$

It follows from (4.10) that $\left\|\boldsymbol{Z}_{k,i}^{\mathrm{I}} \boldsymbol{v}\right\|_{\boldsymbol{\Lambda},\Omega} \le Ch_k \|\boldsymbol{v}\|_{\boldsymbol{\Lambda},\Omega_{k,i}^{\mathrm{I}}}$. This and the fact that the $\boldsymbol{\Lambda}$-operator norm of $(\boldsymbol{I} - \widetilde{\boldsymbol{P}}_{k,i}^{\mathrm{I}})$ is bounded by one implies that

$$\|(\widetilde{\boldsymbol{\mathcal{E}}}_i - \boldsymbol{\mathcal{E}}_i)\boldsymbol{u}\|_{\boldsymbol{\Lambda},\Omega}^2 \le \|(\widetilde{\boldsymbol{\mathcal{E}}}_{i-1} - \boldsymbol{\mathcal{E}}_{i-1})\boldsymbol{u}\|_{\boldsymbol{\Lambda},\Omega}^2 + Ch_k^2 \|\boldsymbol{\mathcal{E}}_{i-1} \boldsymbol{u}\|_{\boldsymbol{\Lambda},\Omega_{k,i}}^2.$$

Summing over i and obvious manipulations gives

$$\|(\widetilde{\boldsymbol{\mathcal{E}}}_{N_k} - \boldsymbol{\mathcal{E}}_{N_k})\boldsymbol{u}\|_{\boldsymbol{\Lambda},\Omega}^2 \le Ch_k^2 \sum_{i=1}^{N_k} \|\boldsymbol{\mathcal{E}}_{i-1} \boldsymbol{u}\|_{\boldsymbol{\Lambda},\Omega_{k,i}}^2. \tag{4.11}$$

We shall now show that for sufficiently small h_1,

$$\sum_{i=1}^{N_k} \|\boldsymbol{\mathcal{E}}_{i-1}\boldsymbol{u}\|^2_{\boldsymbol{\Lambda},\Omega_{k,i}} \le C\|\boldsymbol{u}\|^2_{\boldsymbol{\Lambda},\Omega}. \tag{4.12}$$

We first note the identity

$$\boldsymbol{I} - \boldsymbol{\mathcal{E}}_i = \sum_{m=1}^{i} \boldsymbol{P}^{\mathrm{I}}_{k,m}\boldsymbol{\mathcal{E}}_{m-1}.$$

Thus, by the arithmetic–geometric mean inequality, the definition of $\boldsymbol{\mathcal{E}}_i$ and the limited interaction property, it follows that

$$\begin{aligned}
\sum_{i=1}^{N_k} \|\boldsymbol{\mathcal{E}}_{i-1}\boldsymbol{u}\|^2_{\boldsymbol{\Lambda},\Omega^{\mathrm{I}}_{k,i}} &\le 2\sum_{i=1}^{N_k} \|\boldsymbol{u}\|^2_{\boldsymbol{\Lambda},\Omega^{\mathrm{I}}_{k,i}} + 2\sum_{i=1}^{N_k} \|\boldsymbol{u} - \boldsymbol{\mathcal{E}}_{i-1}\boldsymbol{u}\|^2_{\boldsymbol{\Lambda},\Omega^{\mathrm{I}}_{k,i}} \\
&\le C\|\boldsymbol{u}\|^2_{\boldsymbol{\Lambda},\Omega} + 2\sum_{i=1}^{N_k} \left\| \sum_{m=1}^{i-1} \boldsymbol{P}^{\mathrm{I}}_{k,m}\boldsymbol{\mathcal{E}}_{m-1}\boldsymbol{u} \right\|^2_{\boldsymbol{\Lambda},\Omega^{\mathrm{I}}_{k,i}} \\
&\le C\left(\|\boldsymbol{u}\|^2_{\boldsymbol{\Lambda},\Omega} + \sum_{m=1}^{N_k}\sum_{i=1}^{N_k} \|\boldsymbol{P}^{\mathrm{I}}_{k,m}\boldsymbol{\mathcal{E}}_{m-1}\boldsymbol{u}\|^2_{\boldsymbol{\Lambda},\Omega^{\mathrm{I}}_{k,i}} \right) \\
&\le C\left(\|\boldsymbol{u}\|^2_{\boldsymbol{\Lambda},\Omega} + \sum_{m=1}^{N_k} \|\boldsymbol{P}^{\mathrm{I}}_{k,m}\boldsymbol{\mathcal{E}}_{m-1}\boldsymbol{u}\|^2_{\boldsymbol{\Lambda},\Omega} \right).
\end{aligned} \tag{4.13}$$

In order to estimate the last term on the right of (4.13), we write

$$\begin{aligned}
\|\boldsymbol{P}^{\mathrm{I}}_{k,m}\boldsymbol{\mathcal{E}}_{m-1}\boldsymbol{u}\|^2_{\boldsymbol{\Lambda},\Omega} &= \|\boldsymbol{\mathcal{E}}_{m-1}\boldsymbol{u}\|^2_{\boldsymbol{\Lambda},\Omega} - \|\boldsymbol{\mathcal{E}}_{m}\boldsymbol{u}\|^2_{\boldsymbol{\Lambda},\Omega} \\
&\quad - 2\boldsymbol{\Lambda}(\boldsymbol{P}^{\mathrm{I}}_{k,m}\boldsymbol{\mathcal{E}}_{m-1}\boldsymbol{u}, (\boldsymbol{I} - \boldsymbol{P}^{\mathrm{I}}_{k,m})\boldsymbol{\mathcal{E}}_{m-1}\boldsymbol{u}).
\end{aligned} \tag{4.14}$$

Now by (4.9),

$$\boldsymbol{\Lambda}(\boldsymbol{P}^{\mathrm{I}}_{k,m}\boldsymbol{\mathcal{E}}_{m-1}\boldsymbol{u}, (\boldsymbol{I} - \boldsymbol{P}^{\mathrm{I}}_{k,m})\boldsymbol{\mathcal{E}}_{m-1}\boldsymbol{u}) = (1+\omega^2)(\boldsymbol{P}^{\mathrm{I}}_{k,m}\boldsymbol{\mathcal{E}}_{m-1}\boldsymbol{u}, (\boldsymbol{I} - \boldsymbol{P}^{\mathrm{I}}_{k,m})\boldsymbol{\mathcal{E}}_{m-1}\boldsymbol{u}),$$

so by Lemma 4.2, we have

$$\|\boldsymbol{P}^{\mathrm{I}}_{k,m}\boldsymbol{\mathcal{E}}_{m-1}\boldsymbol{u}\|^2_{\boldsymbol{\Lambda},\Omega} \le C(\|\boldsymbol{\mathcal{E}}_{m-1}\boldsymbol{u}\|^2_{\boldsymbol{\Lambda},\Omega} - \|\boldsymbol{\mathcal{E}}_{m}\boldsymbol{u}\|^2_{\boldsymbol{\Lambda},\Omega}) + Ch_k^2\|\boldsymbol{\mathcal{E}}_{m-1}\boldsymbol{u}\|^2_{\boldsymbol{\Lambda},\Omega^{\mathrm{I}}_{k,m}}.$$

Summing over m we conclude that

$$\sum_{m=1}^{N_k} \|\boldsymbol{P}^{\mathrm{I}}_{k,m}\boldsymbol{\mathcal{E}}_{m-1}\boldsymbol{u}\|^2_{\boldsymbol{\Lambda},\Omega} \le C(\|\boldsymbol{u}\|^2_{\boldsymbol{\Lambda},\Omega} + h_k^2 \sum_{m=1}^{N_k} \|\boldsymbol{\mathcal{E}}_{m-1}\boldsymbol{u}\|^2_{\boldsymbol{\Lambda},\Omega^{\mathrm{I}}_{k,m}}). \tag{4.15}$$

Clearly (4.15) and (4.13) yield (4.12) for small enough h_1.

Finally, we obtain from (4.12) and (4.11) that for $k > 1$,

$$\|\boldsymbol{Z}_k\|_{\boldsymbol{\Lambda},\Omega} \le Ch_k.$$

The theorem follows from Lemma 4.1 and Theorem 4.2. □

Remark 4.2. The same analysis could be used for the SOR(α) iteration considered in Remark 4.1. In that case,

$$\boldsymbol{\mathcal{E}}_l = (\boldsymbol{I} - \alpha\boldsymbol{P}^d_{k,l})(\boldsymbol{I} - \alpha\boldsymbol{P}^d_{k,l-1})\cdots(\boldsymbol{I} - \alpha\boldsymbol{P}^d_{k,1}).$$

	h \ H	1/2	1/4	1/8	1/16	1/32	Degrees of freedom
	1/4	6	–	–	–	–	108
	1/8	7	7	–	–	–	1176
h	1/16	9	9	8	–	–	10800
	1/32	10	10	9	7	–	92256
	1/64	11	10	10	8	7	762048
	1/128	11	11	10	9	8	6193536

TABLE 5.1. Preconditioned GMRES iteration counts for $\omega = 1$ case. Degrees of freedom at each refinement level are also shown in the last column.

	h \ H	1/2	1/4	1/8	1/16	1/32
	1/4	6	–	–	–	–
	1/8	7	7	–	–	–
h	1/16	9	9	8	–	–
	1/32	10	11	10	8	–
	1/64	11	11	10	10	8
	1/128	12	11	10	10	10

TABLE 5.2. Linear multigrid iteration counts with $\omega = 1$.

Also, by Remark 4.1, Theorem 4.1 holds with the SOR(α) smoother.

5. NUMERICAL RESULTS

Numerical experiments were conducted using lowest order Nedelec elements on cubes. We report results of some of these experiments in this section. First of all, let us note that not only can Algorithm 2.1 be used as a linear solver for (2.6), but it can also be used to develop a preconditioner. Specifically, the operator $\boldsymbol{B}_J : M_J \mapsto M_J$ defined by $\boldsymbol{B}_J \boldsymbol{g} = \boldsymbol{Mg}_J(\boldsymbol{0}, \boldsymbol{g})$ is a preconditioner for $\boldsymbol{A}_J$ in the sense that the inequalities

$$(1-\delta)\,\boldsymbol{\Lambda}(\boldsymbol{u},\boldsymbol{u}) \le \boldsymbol{\Lambda}(\boldsymbol{B}_J\boldsymbol{A}_J\boldsymbol{u},\boldsymbol{u}) \quad \text{and} \\ \boldsymbol{\Lambda}(\boldsymbol{B}_J\boldsymbol{A}_J\boldsymbol{u},\boldsymbol{v}) \le (1+\delta)\boldsymbol{\Lambda}(\boldsymbol{u},\boldsymbol{u})^{1/2}\boldsymbol{\Lambda}(\boldsymbol{v},\boldsymbol{v})^{1/2} \tag{5.1}$$

hold for all $\boldsymbol{u}, \boldsymbol{v} \in M_J$, for sufficiently small coarse mesh sizes. These bounds easily follow from Theorem 4.3 and δ is as in the theorem. They imply that when GMRES in $\boldsymbol{\Lambda}(\cdot,\cdot)$ innerproduct is used to solve (2.6) with $\boldsymbol{B}_J$ as preconditioner, the number of iterations remain bounded independently of refinement level [12, 16]. In this section we will investigate the performance of $\boldsymbol{B}_J$ as a preconditioner for use in GMRES, as well as that of the linear solver $\boldsymbol{Mg}_J(\cdot,\cdot)$ given by Algorithm 2.1.

In all experiments, our computational domain was $\Omega = (0,1)^3$. We only investigate the multigrid algorithm with the smoother $\boldsymbol{G}_k^{\mathrm{I}}$ based on decomposition I. The domain $(0,1)^3$ was meshed by a hierarchy of multilevel uniform cubic meshes. Each mesh is obtained by breaking up every cubic element of a coarser mesh into eight congruent cubes, the coarsest mesh being just $\{\Omega\}$. Clearly, our analysis holds in this situation.

		H				
		1/2	1/4	1/8	1/16	1/32
	1/4	$\star$	–	–	–	–
	1/8	$\star$	35	–	–	–
h	1/16	$\star$	110	23	–	–
	1/32	$\star$	208	48	10	–
	1/64	$\star$	266	62	15	8
	1/128	$\star$	285	67	16	10

TABLE 5.3. Linear multigrid iteration counts for $\omega = 7$. An entry "$\star$" indicates that $\mathbf{\Lambda}$-norm of iterates became larger than 10^{99}, and iterations were stopped.

(In particular, a Poincaré-Friedrichs inequality like that of Lemma 3.1 is obvious for uniform cubic meshes.)

The linear system (2.6) is solved on a fine ($k = J$) mesh of mesh size h using one of the two above mentioned iterative methods. The coarse solves of the multigrid algorithm are done on a coarse ($k = 1$) mesh of mesh size H. All coarse solves were done by direct methods of UMFPACK2.2 [10]. The right hand side of (2.6) was chosen so that the true solution equals the interpolant of $\boldsymbol{U}(x, y, z) = [y(1-y)z(1-z), yx(1-x)z(1-z), x(1-x)y(1-y)]$. We report iteration counts for a set of combinations of h and H. The starting iterate was always zero. When the linear multigrid solver is used, the stopping criterion was that the $\mathbf{\Lambda}$-norm of error be reduced by a factor of 10^{-6}. The stopping criterion for GMRES was that the $\mathbf{\Lambda}$-norm of the residual (pre-multiplied by $\boldsymbol{B}_J$) was reduced by a factor of 10^{-6}. GMRES was set to restart after 50 iterations.

We start with the case $\omega = 1$. GMRES iteration counts are reported in Table 5.1. The preconditioner appears to be uniform, as iteration counts never exceeded 11 for all combinations of h and H we considered. For comparison, the case $h = 1/128$ without preconditioner did not converge even after 5000 iterations.

Iteration counts obtained using linear multigrid solver are reported in Table 5.2, and these are in accordance with Theorem 4.3. Although in the case $\omega = 1$, the algorithm gives uniform iteration counts for all choices of H considered, this is no longer the case for a higher wave number, as seen in Table 5.3. This is again in accordance with Theorem 4.3, as its conclusion holds only whenever the coarse mesh is sufficiently fine.

We have also considered the performance of $\boldsymbol{B}_J$ as a preconditioner in GMRES for the case of a higher wave number $\omega = 10$. It is a good preconditioner only for smaller coarse mesh sizes, as Table 5.4 shows. In other (unreported) experiments, the linear multigrid algorithm for this wave number only converged for one of the combinations of h and H considered. Theoretically, (5.1) guarantees that $\boldsymbol{B}_J$ is a good preconditioner only when $\boldsymbol{Mg}_J(\cdot,\cdot)$ is a good contraction. Nonetheless, our experiments indicate that the coarse mesh size at which $\boldsymbol{B}_J$ becomes a good preconditioner is larger than that required for $\boldsymbol{Mg}_J(\cdot,\cdot)$ to be a good contraction. Similar observations have been made in studies of multigrid algorithms for the Helmholtz equation [13]. This may be an argument in favor of using GMRES preconditioned with multigrid as a solution strategy, rather than the linear multigrid solver. However, we must also keep in mind that if too large a mesh

	H: 1/2	1/4	1/8	1/16	1/32
h = 1/4	$3^\times$	–	–	–	–
1/8	$2^\times$	37	–	–	–
1/16	$3^\times$	48	18	–	–
1/32	$2^\times$	$78^\times$	22	16	–
1/64	$2^\times$	$78^\times$	21	17	9
1/128	$2^\times$	$79^\times$	21	16	10

TABLE 5.4. Preconditioned GMRES iteration counts for $\omega = 10$ case. An entry of the form $n^\times$ indicates that although the residual of n-th GMRES iterate met the stopping criterion, the iterate differed from the true solution by more than 10^{-3} in $\mathbf{\Lambda}$-norm.

$\omega = 1$

	H: 1/2	1/4	1/8	1/16
h = 1/4	0.32	—	—	—
1/8	0.40	0.40	—	—
1/16	0.42	0.42	0.42	—
1/32	0.42	0.42	0.42	0.42
1/64	0.42	0.42	0.42	0.42

$\omega = 5$

	H: 1/2	1/4	1/8	1/16
h = 1/4	7.93	—	—	—
1/8	9.67	0.92	—	—
1/16	10.01	0.65	0.60	—
1/32	10.06	0.58	0.44	0.43
1/64	10.07	0.58	0.43	0.43

TABLE 5.5. Numerical convergence rates for the linear multigrid iteration.

size is used, GMRES may find the residual too small and stop, even though the iterate is far from the true solution (see entries $n^\times$).

We conclude by providing numerical convergence rates for the linear multigrid iteration, which also confirm our theoretical results. Entries of Table 5.5 provide estimates for $\|\boldsymbol{I} - \boldsymbol{B}_J \boldsymbol{A}_J\|_{\mathbf{\Lambda}}$ obtained by means of the power method in the cases $\omega = 1$ and $\omega = 5$ for a few combinations of h and H. We see that the only difference between the two cases is that larger ω requires a smaller coarse grid size. Note, though, that once the coarse grid is small enough, both cases give rise to approximately the same reduction rates.

APPENDIX A

Here we will indicate how the main result of this paper can be generalized to higher order Nedelec spaces (of the first kind). Let M_k be defined with r-th order Nedelec spaces on each tetrahedron and W_k be the corresponding conforming approximation space with polynomials of degree at most $r + 1$. The algorithms and definitions of subspace decompositions and smoothers generalize in an obvious way for the case of decomposition I. As we shall see, Case II can also be generalized provided a suitable choice of nodal basis is made,

Case I: First of all note that Theorem 4.1 holds with the higher order spaces, as shown in [1]. The only proof in the previous sections that depended on the order of the spaces is that of Lemma 3.1. We will now prove that the inequality of the lemma holds for higher order spaces as well. We start by considering the set S of all possible quasi-uniform tetrahedral meshes contained in the unit ball with at least one vertex on the unit sphere and every element having the origin as a vertex. Each element in S is represented by a list of vertices and a list of tetrahedra (a tetrahedron number to vertex number list). We can assign labels to the members of S so that two members have the same label if and only if they have the same tetrahedra to vertex list. Quasi-uniformity implies that the number of labels can be bounded in terms of ζ appearing in (2.4). Let R_l be the subset of elements of S with the l'th label.

Any subdomain $\Omega^{\mathrm{I}}_{k,i}$ can be dilated and translated to an element of S. Thus, it suffices to prove that (3.4) holds for each D in R_l with constant independent of D. The general result holds taking the minimum of these constants over $\{R_l\}$.

Clearly, each domain $D \in R_l$ has the same number of vertices, say m. We can define a distance on R_l by using any norm on the vertex set, e.g., the Euclidean norm on $\mathbb{R}^{3m}$. It follows from quasi-uniformity, that R_l is a closed and bounded set in this norm and hence compact.

Let D be in R_l. Denote the corresponding approximation spaces (of $H_0(\mathbf{curl}\,; D)$ and $H^1_0(D)$ respectively) by M'_D and W'_D, and set $\mathring{M}'_D = \{\boldsymbol{q} \in M'_D : (\boldsymbol{q}, \boldsymbol{\nabla}\theta)_D = 0, \text{ for all } \theta \in W'_D\}$. Let

$$\mathcal{I}(D) = \inf_{\boldsymbol{q}\in\mathring{M}'_D} \frac{\|\mathbf{curl}\,\boldsymbol{q}\|_{0,D}}{\|\boldsymbol{q}\|_{0,D}}. \tag{A.1}$$

Note that since D is simply connected with a connected boundary, if $\mathbf{curl}\,\boldsymbol{q} = 0$ and $\boldsymbol{q} \in M'_D$ then $\boldsymbol{q}$ is a gradient of a function in W'_D. It follows that $\mathcal{I}(D) > 0$ for any $D \in R_l$. Thus, to prove that (3.4) holds uniformly for $D \in R_l$, it suffices to show that $\mathcal{I}(D)$ is continuous.

Suppose p and q are vertex sets of two meshes in R_l, with corresponding domains D_p and D_q, respectively. Let $\epsilon > 0$ be given. For $s \in \{p, q\}$, let $\{\boldsymbol{e}^s_i\}_{i=1}^{n_l}$ denote a nodal basis for M'_{D_s}. We identify functions in the above spaces with their extension by zero to the unit ball B. Let $\boldsymbol{z} \in \mathring{M}'_{D_p}$ be a function with $\|\boldsymbol{z}\|_{0,B} = 1$ for which the infimum in (A.1) is attained and let

$$\boldsymbol{z} = \sum_{i=1}^{n_l} c_i \boldsymbol{e}^p_i \quad \text{and} \quad \boldsymbol{z}' = \sum_{i=1}^{n_l} c_i \boldsymbol{e}^q_i.$$

By quasi-uniformity, it is easy to see that if $|p - q|$ is small enough (depending on ϵ), $\|\boldsymbol{z} - \boldsymbol{z}'\|_{\boldsymbol{\Lambda},B} \le \epsilon$. Note that $\boldsymbol{z}'$ is, in general, not in $\mathring{M}'_{D_q}$. Define $\psi' \in W'_{D_q}$ by

$$(\boldsymbol{\nabla}\psi', \boldsymbol{\nabla}\phi)_B = (\boldsymbol{z}', \boldsymbol{\nabla}\phi)_B \quad \text{for all } \phi \in W'_{D_q}.$$

Then $\boldsymbol{z}'' = \boldsymbol{z}' - \boldsymbol{\nabla}\psi'$ is in $\mathring{M}'_{D_q}$. Moreover, if $|p - q|$ is small enough, it can easily be shown that $\|\boldsymbol{z}'' - \boldsymbol{z}'\|_{\boldsymbol{\Lambda},B} \le \epsilon$, so

$$\|\boldsymbol{z} - \boldsymbol{z}''\|_{\boldsymbol{\Lambda},B} \le 2\epsilon.$$

Consequently,

$$\mathcal{I}(q) - \mathcal{I}(p) \le \frac{\|\mathbf{curl}\,\boldsymbol{z}''\|_{0,D_q}}{\|\boldsymbol{z}''\|_{0,D_q}} - \mathcal{I}(p) \le C\epsilon.$$

Interchanging the roles of p and q in the above argument we also get that $\mathcal{I}(p) - \mathcal{I}(q) \le C\epsilon$. Thus, $\mathcal{I}(p)$ is continuous on R_l. This finishes the proof of Lemma 3.1 when $d = \mathrm{I}$.

Case II: Smoothing algorithms of this type can be generalized to higher order spaces provided a suitable choice of nodal basis is made. Note that there choices of nodal basis for which Lemma 3.1 does not hold. We will provide one example of a nodal basis for which our analysis generalizes.

Once a set of degrees of freedom for M_k is defined, a corresponding nodal basis immediately follows. The particular choice of the degrees of freedom we have in mind consists of edge, face, and tetrahedral moments. For any domain D, let $P_l(D)$ denote the set of polynomials of degree at most l and let $\mathcal{P}_l(D)$ denote any basis for $P_l(D)$. For every interior edge e and interior face f of the k-th level mesh, define the edge and face moments

$$\alpha_e^p(\boldsymbol{u}) = \int_e p(\boldsymbol{u}\cdot\boldsymbol{t})\,dt, \quad \alpha_f^{\boldsymbol{q}}(\boldsymbol{u}) = \int_f \boldsymbol{q}\cdot(\boldsymbol{u}\times\boldsymbol{n})\,ds,$$

for $p \in \mathcal{P}_{r-1}(e)$ and $\boldsymbol{q} \in (\mathcal{P}_{r-2}(f))^2$. The tetrahedral moments are defined by mapping to the reference tetrahedron $\hat{\tau}$ bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x+y+z = 1$. Let $\mathcal{R}_{r-3}$ be the set of all vector polynomials that are monomials of degree at most $r-3$ in one coordinate direction and zero in others, e.g., $\boldsymbol{r} = (x^i y^j z^k, 0, 0)$ is in $\mathcal{R}_{r-3}$. For every tetrahedron τ in the k-th level mesh, define the tetrahedral moments

$$\alpha_\tau^{\boldsymbol{r}}(\boldsymbol{u}) = \int_{\widehat{\tau}} \boldsymbol{r}\cdot\widehat{\boldsymbol{u}}\,dx,$$

where $\boldsymbol{r} \in \mathcal{R}_{r-3}$, $\widehat{\boldsymbol{u}}(\widehat{\boldsymbol{x}}) = B^t\boldsymbol{u}(\boldsymbol{x})$, and B is the matrix in the affine correspondence $\widehat{\tau} \overset{B\widehat{\boldsymbol{x}}+b}{\mapsto} \tau$. The edge, face, and tetrahedral moments defined above form a set of degrees of freedom for M_k and define a corresponding nodal basis $\mathcal{B}$ for M_k.

The basis $\mathcal{B}$ is divided into edge basis functions, face basis functions, and interior basis functions: An edge basis function $\boldsymbol{\phi}_e^p$ corresponding to an interior edge e has all of the above defined degrees of freedom equal to zero except $\alpha_e^p(\boldsymbol{\phi}_e^p) = 1$ for some polynomial $p \in \mathcal{P}_{r-1}(e)$. Similarly, a face basis function $\boldsymbol{\phi}_f^{\boldsymbol{q}}$ has all its moments zero except $\alpha_f^{\boldsymbol{q}}(\boldsymbol{\phi}_f^{\boldsymbol{q}}) = 1$ for some interior face f and some $\boldsymbol{q} \in \mathcal{P}_{r-2}(f)^2$. Finally, we have interior basis functions $\boldsymbol{\phi}_\tau^{\boldsymbol{r}}$ supported on τ such that all its moments are zero except $\alpha_\tau^{\boldsymbol{r}}(\boldsymbol{\phi}_\tau^{\boldsymbol{r}}) = 1$ for some $\boldsymbol{r} \in \mathcal{R}_{r-3}$. Thus,

$$\begin{aligned}\mathcal{B} =& \{\boldsymbol{\phi}_e^p : \text{ for all interior mesh edges } e \text{ and } p \in \mathcal{P}_{r-1}(e)\} \cup \\ & \{\boldsymbol{\phi}_f^{\boldsymbol{q}} : \text{ for all interior mesh faces } f \text{ and } \boldsymbol{q} \in (\mathcal{P}_{r-2}(f))^2\} \cup \\ & \{\boldsymbol{\phi}_\tau^{\boldsymbol{r}} : \text{ for all } \boldsymbol{r} \in \mathcal{R}_{r-3} \text{ and all mesh tetrahedra } \tau\}.\end{aligned}$$

Our analysis generalizes to the case when M_k is decomposed as

$$M_k = \sum_{\boldsymbol{\phi}\in\mathcal{B}} \operatorname{span}(\boldsymbol{\phi}) \oplus \sum_i \operatorname{span}(\boldsymbol{\nabla}\psi_{k,i}),$$

where $\{\psi_{k,i}\}$ is a local nodal basis for W_k. To show this, we first of all note that Theorem 4.1 holds for this decomposition as can be seen by following the arguments of [1]. The only other ingredient in our analysis that requires generalization is Lemma 3.1. We now show that $\|\boldsymbol{\phi}\|_{0,\Omega} \le Ch_k\|\mathbf{curl}\,\boldsymbol{\phi}\|_{0,\Omega}$ for all $\boldsymbol{\phi} \in \mathcal{B}$.

It suffices to prove that there is a $\hat{C} > 0$ such that

$$\|\widehat{\boldsymbol{\phi}}\|_{0,\widehat{\tau}} \le \hat{C}\|\mathbf{curl}\,\widehat{\boldsymbol{\phi}}\|_{0,\widehat{\tau}}, \tag{A.2}$$

for all $\boldsymbol{\phi} \in \mathcal{B}$ with $\hat{C}$ independent of τ. Here, as before, $\widehat{\boldsymbol{\phi}}(\widehat{\boldsymbol{x}}) = B^t\, \boldsymbol{\phi}(\boldsymbol{x})$, for $\boldsymbol{x} \in \tau$ for some τ on which $\boldsymbol{\phi}$ is nonzero. Clearly, (3.1) follows from (A.2) by quasi-uniformity and standard affine equivalence arguments since $\|\boldsymbol{\phi}\|_{0,\tau} \le C h_k^{3/2} \|\widehat{\boldsymbol{\phi}}\|_{0,\widehat{\tau}}$ and $\|\mathbf{curl}\, \widehat{\boldsymbol{\phi}}\|_{0,\widehat{\tau}} \le C h_k^{-1/2} \|\mathbf{curl}\, \boldsymbol{\phi}\|_{0,\tau}$.

We prove (A.2) for each type of basis function. First, we consider $\widehat{\boldsymbol{\phi}}_e^p$. Let $L_{\widehat{e}}$ denote the space of functions $\boldsymbol{v}$ in the r-th order Nedelec space on $\widehat{\tau}$ for which all edge, face, and tetrahedral moments are zero except those associated to the edge $\widehat{e}$ which is the image of e. Clearly, $\widehat{\boldsymbol{\phi}}_e^p$ is in $L_{\widehat{e}}$. For any nonzero function $\widehat{\boldsymbol{\phi}} \in L_{\widehat{e}}$, there exists a $p \in P_{r-1}(\widehat{f})$, on a face $\widehat{f}$ adjacent to $\widehat{e}$, such that

$$0 \ne (\widehat{\boldsymbol{\phi}} \cdot \boldsymbol{t}, p)_{\partial \widehat{f}} = (p, \mathbf{curl}\, \widehat{\boldsymbol{\phi}} \cdot \boldsymbol{n})_{\widehat{f}} - (\boldsymbol{\nabla} p \times \boldsymbol{n}, \widehat{\boldsymbol{\phi}})_{\widehat{f}} = (p, \mathbf{curl}\, \widehat{\boldsymbol{\phi}} \cdot \boldsymbol{n})_{\widehat{f}},$$

where $\boldsymbol{n}$ is the outward unit normal on $\widehat{f}$ and $\boldsymbol{t}$ is a unit tangent vector on $\partial \widehat{f}$ (appropriately oriented). Since the left hand side is nonzero, $\mathbf{curl}\, \widehat{\boldsymbol{\phi}} \ne 0$. Thus by the finite dimensionality of $L_{\widehat{e}}$, (A.2) holds for all $\widehat{\boldsymbol{\phi}} \in L_{\widehat{e}}$ and hence holds for $\widehat{\boldsymbol{\phi}}_e^p$.

Next, let us show (A.2) for a mapped face basis function $\widehat{\boldsymbol{\phi}}_f^{\boldsymbol{q}}$. Let $L_{\widehat{f}}$ denote the subspace of the r-th order Nedelec space on $\widehat{\tau}$ for which all edge, face, and tetrahedral moments are zero, except for moments on face $\widehat{f}$. Clearly, $\widehat{\boldsymbol{\phi}}_f^{\boldsymbol{q}}$ is in $L_{\widehat{f}}$. For any nonzero $\widehat{\boldsymbol{\phi}} \in L_{\widehat{f}}$, there is a $\boldsymbol{q} \in P_{r-2}(\widehat{\tau})^3$ such that

$$0 \ne (\widehat{\boldsymbol{\phi}} \times \boldsymbol{n}, \boldsymbol{q})_{\partial \widehat{\tau}} = (\mathbf{curl}\, \widehat{\boldsymbol{\phi}}, \boldsymbol{q})_{\widehat{\tau}} - (\widehat{\boldsymbol{\phi}}, \mathbf{curl}\, \boldsymbol{q})_{\widehat{\tau}} = (\mathbf{curl}\, \widehat{\boldsymbol{\phi}}, \boldsymbol{q})_{\widehat{\tau}}.$$

Thus, $\mathbf{curl}\, \widehat{\boldsymbol{\phi}} \ne 0$ for all $\widehat{\boldsymbol{\phi}} \in L_{\widehat{f}}$ and (A.2) follows for $\widehat{\boldsymbol{\phi}}_f^{\boldsymbol{q}}$.

Finally, consider an interior basis function $\boldsymbol{\phi}_\tau^{\boldsymbol{r}}$. Obviously, all face and edge moments of $\widehat{\boldsymbol{\phi}}_\tau^{\boldsymbol{r}}$ are zero. We will now only show that for $\boldsymbol{r} = (x^i y^j z^k, 0, 0)$, $\mathbf{curl}\, \widehat{\boldsymbol{\phi}}_\tau^{\boldsymbol{r}} \ne 0$, as the argument is similar for other $\boldsymbol{r} \in \mathcal{R}_{r-3}$. We argue by contradiction. If $\mathbf{curl}\, \widehat{\boldsymbol{\phi}}_\tau^{\boldsymbol{r}} = 0$ then $\widehat{\boldsymbol{\phi}}_\tau^{\boldsymbol{r}} = \boldsymbol{\nabla} \psi$ for some $\psi \in P_{r-2}(\widehat{\tau})$. Moreover, since face and edge moments of $\widehat{\boldsymbol{\phi}}_\tau^{\boldsymbol{r}}$ are zero, ψ can be chosen such that $\psi|_{\partial \widehat{\tau}} = 0$. Therefore,

$$1 = (\widehat{\boldsymbol{\phi}}_\tau^{\boldsymbol{r}}, \boldsymbol{r})_{\widehat{\tau}} = -(\psi, \operatorname{div} \boldsymbol{r})_{\widehat{\tau}}.$$

If $i = 0$, i.e., $\boldsymbol{r} = (y^j z^k, 0, 0)$, then $\operatorname{div} \boldsymbol{r} = 0$, which is a contradiction. If $i \ge 1$, then by the definition of $\boldsymbol{\phi}_\tau^{\boldsymbol{r}}$, $(\widehat{\boldsymbol{\phi}}_\tau^{\boldsymbol{r}}, \widetilde{\boldsymbol{r}}) = 0$ for $\widetilde{\boldsymbol{r}} = (0, i x^{i-1} y^{j+1} z^k/(j+1), 0)$. However, $(\widehat{\boldsymbol{\phi}}_\tau^{\boldsymbol{r}}, \widetilde{\boldsymbol{r}})_{\widehat{\tau}} = (\widehat{\boldsymbol{\phi}}_\tau^{\boldsymbol{r}}, \boldsymbol{r})_{\widehat{\tau}}$, which is a contradiction. Therefore, $\mathbf{curl}\, \widehat{\boldsymbol{\phi}}_\tau^{\boldsymbol{r}} \ne 0$, and (A.2) follows for the interior basis functions as well. Thus, we have shown that Lemma 3.1 holds for the nodal basis functions of $\mathcal{B}$.

It is easy to see that there are various other choices of nodal bases for which the lemma does not hold. For instance, in the case $r \ge 4$ the function $\boldsymbol{\nabla}(\lambda_1 \lambda_2 \lambda_3 \lambda_4)$ (where λ_i are the barycentric coordinates of a tetrahedron τ) is an example of an interior basis function for which Lemma 3.1 does not hold. Another example is the function $\lambda_i \boldsymbol{\nabla} \lambda_j + \lambda_j \boldsymbol{\nabla} \lambda_i$ in the case $r = 2$. This function has only one nonzero edge moment, so may be a candidate for an edge basis function. However, it has nonzero face moments. Our analysis does not hold for decompositions based on such basis functions and it is not clear if the associated indefinite multigrid method is convergent.

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