Fall 2005 Math 151
Exam 1A: Solutions
Mon, 03/Oct ©2005, Art Belmonte

1. (a) Let $\mathbf{v} = [-2, 1]$. Since

$$\mathbf{a} \cdot \mathbf{v} = [1, 2] \cdot [-2, 1] = -2 + 2 = 0$$

we see that $\mathbf{v} = [-2, 1]$ is perpendicular to $\mathbf{a}$.

2. (b) We have $f(f(x)) = f\left(x^2 + x\right)$ $= \left(x^2 + x\right)^2 + \left(x^2 + x\right) = x^4 + 2x^3 + 2x^2 + x$.

3. (a) Now $\mathbf{F} \cdot \mathbf{D} = \|\mathbf{F}\| \, \|\mathbf{D}\| \cos\theta$, whence $\cos\theta = \dfrac{\mathbf{F} \cdot \mathbf{D}}{\|\mathbf{F}\| \, \|\mathbf{D}\|} = \dfrac{[1, 2] \cdot [-3, 4]}{\sqrt{1+4}\sqrt{9+16}} = \dfrac{-3+8}{\sqrt{5}\,(5)} = \dfrac{1}{\sqrt{5}}$.

4. (d) Given $x = 2 + 3t$ and $y = 4 + 12t$, eliminate the parameter t to obtain a Cartesian form of the line. Now $t = \dfrac{x-2}{3}$, whence $y = 4 + 12\left(\dfrac{x-2}{3}\right)$; i.e., $y = 4x - 4$, the slope of which is 4.

5. (e) Now x is a solution of $g(x) = x^3 - 6x + 5 = 12$ if it is a solution of $f(x) = g(x) - 12 = x^3 - 6x - 7 = 0$. Note that $f(2) = 8 - 19 = -11 < 0$ and $f(3) = 27 - 25 = 2 > 0$. Moreover, f (a polynomial) is continuous on $\mathbb{R}$. Therefore, by the Intermediate Value Theorem (IVT) there is a value of $c \in (2, 3) \subset [2, 3]$ such that $f(c) = 0$ and thus $g(c) = 12$. [Indeed, $c \approx 2.9$ via MATLAB's **fzero**, Maple's **fsolve**, or interactively via the TI-89 graphing facility.]

6. (d) We have $\lim\limits_{x\to 2} \dfrac{3x^2 - 12}{x-2} = \lim\limits_{x\to 2} \dfrac{3\left(x^2-4\right)}{(x-2)}$ $= \lim\limits_{x\to 2} \dfrac{3(x-2)(x+2)}{(x-2)} = \lim\limits_{x\to 2} (3(x+2)) = 12$.

7. (e) Differentiate $y = x^3 - x$ to obtain $y' = 3x^2 - 1$. The slope of the tangent line is $y'(2) = 11$. The point-slope formula yields $y - 6 = 11(x-2)$, whence $y = 11x - 16$.

8. (d) Differentiate $f(x) = x^4 - x^3 + 2x + 1$ term-by-term. $f'(x) = 4x^3 - 3x^2 + 2 + 0 = 4x^3 - 3x^2 + 2$.

9. (c) In order for both component expressions of the vector function $\mathbf{r}(t) = \left[\dfrac{1}{\sqrt{2t-10}}, 2t - 10\right]$ to be defined, we require $2t - 10 > 0$ or $t > 5$.

10. (b) Multiply numerator and demonimator by the algebraic conjugate: $\lim\limits_{h\to 0}\left(\dfrac{\sqrt{4+h}-2}{h} \cdot \dfrac{\sqrt{4+h}+2}{\sqrt{4+h}+2}\right)$ $= \lim\limits_{h\to 0} \dfrac{(4+h)-4}{h\left(\sqrt{4+h}+2\right)} = \lim\limits_{h\to 0} \dfrac{1}{\sqrt{4+h}+2} = \dfrac{1}{4}$.

11. (d) Examine where the denominator is zero.

- Now $f(x) = \dfrac{x-1}{x^2-4x+3} = \dfrac{(x-1)}{(x-1)(x-3)}$ for $x \notin \{1, 3\}$. So *candidates* for vertical asymptotes are $x = 1$ and $x = 3$.
- Notice that $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} \dfrac{1}{x-3} = -\dfrac{1}{2} \neq \pm\infty$. Therefore, $x = 1$ is *not* a vertical asymptote.
- However, $\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} \dfrac{1}{x-3} = \dfrac{1}{0^+} = \infty$, we have that $x = 3$ *is* a vertical asymptote.

12. (b) Examine $f(x)$ when $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$ gets big.

- For large *positive* values of x we have

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{x-5}{\sqrt{9x^2+2}} \\
&= \lim_{x\to\infty} \frac{x-5}{\sqrt{x^2\left(9+\frac{2}{x^2}\right)}} \\
&= \lim_{x\to\infty} \frac{x-5}{|x|\sqrt{9+\frac{2}{x^2}}} \\
&= \lim_{x\to\infty} \frac{x-5}{x\sqrt{9+\frac{2}{x^2}}} \\
&= \lim_{x\to\infty} \frac{1-\frac{5}{x}}{\sqrt{9+\frac{2}{x^2}}} = \frac{1}{3}.
\end{aligned}$$

Thus $y = \frac{1}{3}$ is a horizontal asymptote.

- For large *negative* values of x we have

$$\begin{aligned}
\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \frac{x-5}{\sqrt{9x^2+2}} \\
&= \lim_{x\to-\infty} \frac{x-5}{\sqrt{x^2\left(9+\frac{2}{x^2}\right)}} \\
&= \lim_{x\to-\infty} \frac{x-5}{|x|\sqrt{9+\frac{2}{x^2}}} \\
&= \lim_{x\to-\infty} \frac{x-5}{-x\sqrt{9+\frac{2}{x^2}}} \\
&= \lim_{x\to-\infty} \frac{1-\frac{5}{x}}{-\sqrt{9+\frac{2}{x^2}}} = -\frac{1}{3}.
\end{aligned}$$

Thus $y = -\frac{1}{3}$ is also a horizontal asymptote!

13. (c) All of the statements are true except "the limit $\lim\limits_{x\to 2} f(x)$ does not exist." From the graph we see that $\lim\limits_{x\to 2^-} f(x) = 3$ and $\lim\limits_{x\to 2^+} f(x) = 3$. Since these one-sided limits agree, we conclude that the two-sided limit $\lim\limits_{x\to 2} f(x) = 3$ exists.

14. Employ the product and quotient rules, respectively.

(a) Given $f(x) = \left(x^2 + x + 7\right)\left(x^3 + 2x^2 + 3x + 1\right)$,

$$\begin{aligned}
f'(x) &= (2x+1)\left(x^3+2x^2+3x+1\right) \\
&\quad + \left(x^2+x+7\right)\left(3x^2+4x+3\right).
\end{aligned}$$

(b) Given $f(x) = \dfrac{x^2 - 1}{x^2 + 4x}$, we have

$$f'(x) = \frac{(x^2 + 4x)(2x) - (x^2 - 1)(2x + 4)}{(x^2 + 4x)^2}.$$

15. (a) The definition of the derivative as the limit of a difference quotient is $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

(b) Applying this definition to $f(x) = \dfrac{1}{3x+2}$ yields

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\frac{1}{3(x+h)+2} - \frac{1}{3x+2}}{h} \\ &= \lim_{h\to 0}\left(\frac{1}{h}\cdot\frac{(3x+2)-(3x+3h+2)}{(3x+3h+2)(3x+2)}\right) \\ &= \lim_{h\to 0}\left(\frac{1}{h}\cdot\frac{-3h}{(3x+3h+2)(3x+2)}\right) \\ &= \lim_{h\to 0}\frac{-3}{(3x+3h+2)(3x+2)} = -\frac{3}{(3x+2)^2}. \end{aligned}$$

16. With $\mathbf{a} = [1, 3]$ and $\mathbf{b} = [1, 1]$, the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\begin{aligned} \text{proj}_{\mathbf{a}}\mathbf{b} &= \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}\right)\frac{\mathbf{a}}{\|\mathbf{a}\|} \\ &= \left(\frac{1+3}{\sqrt{1+9}}\right)\frac{[1,3]}{\sqrt{10}} \\ &= \frac{4}{10}[1,3] = \frac{2}{5}[1,3] = \left[\frac{2}{5}, \frac{6}{5}\right]. \end{aligned}$$

17. Recall that the displacement in meters is given as $s(t) = t^2 - 8t + 18$, where time t is measured in seconds.

(a) The average velocity on the time interval $[3, 4]$ is $v_{\text{avg}} = \dfrac{\Delta s}{\Delta t} = \dfrac{s(4) - s(3)}{4-3} = s(4) - s(3)$
$= (16 - 32 + 18) - (9 - 24 + 18) = 2 - 3 = -1$ m/sec.

(b) The instantaneous velocity when $t = 3$ is

$$v(3) = s'(3) = (2t - 8)\big|_{t=3} = -2 \text{ m/sec.}$$

(c) The particle is at rest when its velocity is zero. Thus

$$0 = v(t) = s'(t) = 2t - 8 \text{ implies } t = 4 \text{ sec.}$$

(d) The particle is moving to the right when its velocity is positive. Hence $v(t) = 2t - 8 > 0$ implies $t > 4$ sec.

(e) The total distance traveled by the particle during the first five seconds is the sum of the distance it travels to the left plus the distance it travels to the right.

$$\begin{aligned} \text{total distance} &= |s(4) - s(0)| + |s(5) - s(4)| \\ &= |2 - 18| + |(25 - 40 + 18) - 2| \\ &= 16 + 1 = 17 \text{ m.} \end{aligned}$$

18. For the piecewise-defined function $f(x) = \begin{cases} -2x + a, & \text{if } x < 2 \\ ax^2, & \text{if } x \geq 2 \end{cases}$ to be continuous (on $\mathbb{R}$), it must be continuous at each real number.

- First off, f is a piecewise-polynomial function and thus continuous on $(-\infty, 2) \cup (2, \infty)$. So the crux of the problem is to examine continuity at $x = 2$.
- Observe that $f(2) = 4a$ is defined.
- The left-hand limit is $\lim_{x\to 2^-}(-2x + a) = a - 4$.
- The right-hand limit is $\lim_{x\to 2^+} ax^2 = 4a$.
- We match the one-sided limits and solve for a. Now $a - 4 = 4a$ implies $-3a = 4$ and thus $a = -\frac{4}{3}$.
- CHECK: With this value of a, we have $f(x) = \begin{cases} -2x - \frac{4}{3}, & \text{if } x < 2 \\ -\frac{4}{3}x^2, & \text{if } x \geq 2 \end{cases}$, for which $\lim_{x\to 2} f(x) = -5\frac{1}{3} = -\frac{16}{3} = f(2)$, making f continuous at the last point, $x = 2$, as required. Thus f is continuous on $\mathbb{R}$.

NOTES

1. Here is an alternative solution of #15.

   (a) The definition of the derivative as the limit of a difference quotient is $f'(x) = \lim_{w\to x}\dfrac{f(w) - f(x)}{w - x}$.

   (b) Applying this definition to $f(x) = \dfrac{1}{3x+2}$ yields

$$\begin{aligned} f'(x) &= \lim_{w\to x}\frac{\frac{1}{3w+2} - \frac{1}{3x+2}}{w - x} \\ &= \lim_{w\to x}\left(\frac{1}{(w-x)}\cdot\frac{(3x+2)-(3w+2)}{(3w+2)(3x+2)}\right) \\ &= \lim_{w\to x}\left(\frac{1}{(w-x)}\cdot\frac{-3(w-x)}{(3w+2)(3x+2)}\right) \\ &= \lim_{w\to x}\frac{-3}{(3w+2)(3x+2)} = -\frac{3}{(3x+2)^2}. \end{aligned}$$

2. In #17a, many students computed the average velocity over the time interval $3 \leq t \leq 4$ by averaging the instantaneous velocities at the endpoints; i.e., $\dfrac{v(3) + v(4)}{2}$. While this *coincidentally* gives the correct answer *in this instance*, this approach does NOT work in general for an arbitrary velocity function. Here is a counterexample.

   - Let's say that $v(t) = \begin{cases} 10, & 3 \leq t < 3.8; \\ 20, & 3.8 \leq t \leq 4 \end{cases}$. That is, for the first eight-tenths of the time interval the velocity is 10 and for the last two-tenths it is 20. The average velocity over the time interval $3 \leq t \leq 4$ is obtained as a *weighted* average:

$$v_{\text{avg}} = \left(\frac{8}{10}\right)(10) + \frac{2}{10}(20) = 8 + 4 = 12.$$

   In particular, it is NOT $\dfrac{v(3) + v(4)}{2} = \dfrac{10 + 20}{2} = 15$, the average of the instantaneous velocities at the two endpoints! What *is* in fact true in general is that $v_{\text{avg}} = \dfrac{1}{b - a}\displaystyle\int_a^b v(t)\,dt = \dfrac{s(b) - s(a)}{b - a}$, which is the continuous analog of this weighted average idea. (You'll see this at the beginning of Math 152.)