Fall 2005 Math 151
Exam 2A: Solutions
Mon, 31/Oct ©2005, Art Belmonte

1. (d) Now $f'(x) = 4e^{4x} - 15e^{3x} + 3e^{x} + \cos x$, whence $f'(0) = 4 - 15 + 3 + 1 = -7$.

2. (b) We have

$$\begin{aligned} F'(x) &= f'(g(x))\,g'(x) \\ F'(2) &= f'(g(2))\,g'(2) \\ &= f'(5)\,g'(2) \\ &= (11)(4) = 44. \end{aligned}$$

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3. (a) Use the fact that $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.

$$\begin{aligned} \lim_{x\to 0} \frac{3x(1+\cos x)}{\sin 4x} &= \lim_{x\to 0} \frac{\frac{3x}{4x}(1+\cos x)}{\frac{\sin 4x}{4x}} \\ &= \lim_{x\to 0} \frac{\frac{3}{4}(1+\cos x)}{\frac{\sin 4x}{4x}} \\ &= \left(\frac{3}{4}\right)(2) = \frac{3}{2} \end{aligned}$$

4. (c) The product rule gives

$$\frac{d}{dx}(x\cos 2x) = (1)(\cos 2x) + (x)(-2\sin 2x) = \cos 2x - 2x\sin 2x.$$

5. (d) We have

$$\begin{aligned} \frac{d}{dt}\left(x^2 y\right) &= \left(2x\frac{dx}{dt}\right)(y) + \left(x^2\right)\left(\frac{dy}{dt}\right) \\ &= (4(-2))(3) + (4)(4) \\ &= 16 - 24 = -8. \end{aligned}$$

6. (d) Solve $y = f(x)$ for x.

$$\begin{aligned} y = f(x) &= \frac{2x+1}{x+5} \\ yx + 5y &= 2x + 1 \\ (y-2)x &= 1 - 5y \\ x &= \frac{1-5y}{y-2} \end{aligned}$$

Thus $f^{-1}(x) = \frac{1-5x}{x-2}$.

7. (c) When $t = -1$, we have

$$\begin{aligned} h'(t) &= 3\left(t^3 - t^2 + t + 1\right)^2\left(3t^2 - 2t + 1\right) \\ &= 3(-1-1-1+1)^2(3+2+1) \\ &= 3(-2)^2(6) = 72. \end{aligned}$$

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8. (e) The limiting expression is a decaying exponential. Therefore, as $x \to -\infty$, we see that $e^{x-1} \to 0$.

9. (a) Use log properties. Be sure to check your answer.

$$\begin{aligned} \log_4 x + \log_4\left(x^2\right) &= 6 \\ 3\log_4 x &= 6 \\ \log_4 x &= 2 \\ x &= 4^2 = 16 \end{aligned}$$

This checks out via substitution into the original equation.

10. (c) By inspection, $f(1) = 4$. Thus $g(4) = 1$. Accordingly,

$$\begin{aligned} g'(4) &= \frac{1}{f'(g(4))} \\ &= \frac{1}{f'(1)} \\ &= \frac{1}{\left(5x^4+2\right)\big|_{x=1}} = \frac{1}{7}. \end{aligned}$$

11. (a) Given position $\mathbf{r}(t) = \left[t^3 - 4t^2 + 2,\ 2t^2 - 3t\right]$, we have $\mathbf{r}(2) = [8 - 16 + 2,\ 8 - 6] = [-6, 2]$.

12. (e) Velocity, the derivative of position, is a vector.

$$\mathbf{v}(t) = \mathbf{r}'(t) = \left[3t^2 - 8t,\ 4t - 3\right]$$

Thus $\mathbf{v}(2) = [12 - 16,\ 8 - 3] = [-4, 5]$. The speed is the *magnitude* of velocity: $\sqrt{16+25} = \sqrt{41}$, a scalar.

13. (b) The acceleration is $\mathbf{a}(t) = \mathbf{v}'(t) = [6t - 8, 4]$. Therefore, $\mathbf{a}(2) = [12 - 8, 4] = [4, 4]$.

14. Use product and chain rules, as applicable.

    (a) Given $f(x) = \frac{\cos 2x}{\sin 3x + \tan 4x}$, we have

$$f'(x) = \frac{(\sin 3x + \tan 4x)(-2\sin 2x) - (\cos 2x)\left(3\cos 3x + 4\sec^2 4x\right)}{(\sin 3x + \tan 4x)^2}.$$

    (b) Given $f(x) = e^{\sqrt{x^2+3x+4}} = \exp\left(\left(x^2+3x+4\right)^{1/2}\right)$, we have

$$f'(x) = \exp\left(\left(x^2+3x+4\right)^{1/2}\right)\left(\frac{1}{2}\left(x^2+3x+4\right)^{-1/2}\right)(2x+3)$$

or $f'(x) = \frac{(2x+3)\,e^{\sqrt{x^2+3x+4}}}{2\sqrt{x^2+3x+4}}$.

15. Implicitly differentiate $y^5 + 3x^2y^3 + x^3 + 5 = 0$ with respect to x, then substitute the data $(x, y) = (2, -1)$.

$$\begin{aligned} 5y^4\frac{dy}{dx} + \left((6x)\left(y^3\right) + \left(3x^2\right)\left(3y^2\frac{dy}{dx}\right)\right) + 3x^2 &= 0 \\ \text{Substitute data: } 5\frac{dy}{dx} + \left(-12 + 36\frac{dy}{dx}\right) + 12 &= 0 \\ 41y' &= 0 \\ \text{At stated point: } y' &= 0 \end{aligned}$$

The tangent line is a horizontal line: $y = -1$.

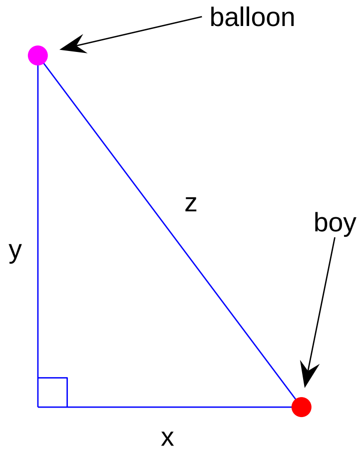
16. Let $f(x) = x^5 + x^3 + 1$. Then $f'(x) = 5x^4 + 3x^2$. Given $x_1 = -1$, Newton's method yields

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= -1 - \frac{-1}{5+3} = -1 + \frac{1}{8} = -\frac{7}{8} = -0.875. \end{aligned}$$

17. The quadratic approximation of $f(x) = \sqrt[4]{x} = x^{\frac{1}{4}}$ near 16 is $Q(x) = \frac{f(16)}{0!} + \frac{f'(16)}{1!}(x-16)^1 + \frac{f''(16)}{2!}(x-16)^2$. Now $f'(x) = \frac{1}{4}x^{-3/4}$ and $f''(x) = -\frac{3}{16}x^{-7/4}$. Thus

$$\begin{aligned} Q(x) &= 2 + \frac{1}{32}(x-16) - \frac{3}{2 \cdot 2^4 \cdot 2^7}(x-16)^2 \\ &= 2 + \frac{1}{32}(x-16) - \frac{3}{4096}(x-16)^2. \end{aligned}$$

18. Here is a diagram depicting the situation.



- Place the boy at the origin of the xy-plane as he passes under the balloon. At time $t = 0$, the balloon is at the point $(0, 45)$. For $t \geq 0$, the boy's position is $x = 15t$ and the balloon's position is $y = 45 + 5t$. Note that $dx/dt = 15$ and $dy/dt = 5$ for all t.
- Let z be the distance (through the air) between the balloon and the boy. By the Pythagorean Theorem, $z^2 = x^2 + y^2$. Differentiate with respect to t.

$$\begin{aligned} 2z\frac{dz}{dt} &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\ \frac{dz}{dt} &= \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{z} \end{aligned}$$

- When $t = 3$ seconds, we have $x = 45$, $y = 60$, $\frac{dx}{dt} = 15$, $\frac{dy}{dt} = 5$, and

$$z = \sqrt{45^2 + 60^2} = 15\sqrt{3^2 + 4^2} = 75.$$

Therefore,

$$\begin{aligned} \frac{dz}{dt} &= \frac{(45)(15) + (60)(5)}{75} \\ &= \frac{15(45+20)}{15(5)} = \frac{65}{5} \\ &= 13 \text{ ft/sec}. \end{aligned}$$