

Multiple Choice: (4 points each)

1. Which of the following gives the trapezoid approximation to $\int_1^9 e^{-x^2} dx$ with 4 intervals?

- a. $e^{-1} + e^{-9} + e^{-25} + e^{-49} + e^{-81}$
 b. $\frac{1}{2}e^{-1} + e^{-9} + e^{-25} + e^{-49} + \frac{1}{2}e^{-81}$
 c. $e^{-1} + 2e^{-9} + 2e^{-25} + 2e^{-49} + e^{-81}$ correctchoice
 d. $2e^{-1} + 2e^{-9} + 2e^{-25} + 2e^{-49} + 2e^{-81}$
 e. $2e^{-1} + 4e^{-9} + 4e^{-25} + 4e^{-49} + 2e^{-81}$

$$\Delta x = \frac{9-1}{4} = 2$$

$$T_4 = \Delta x \left(\frac{1}{2}f(1) + f(3) + f(5) + f(7) + \frac{1}{2}f(9) \right) = 2 \left(\frac{1}{2}e^{-1} + e^{-9} + e^{-25} + e^{-49} + \frac{1}{2}e^{-81} \right) \\ = e^{-1} + 2e^{-9} + 2e^{-25} + 2e^{-49} + e^{-81}$$

2. Compute $\int_1^3 \frac{x}{(x^2-1)^{2/3}} dx$

- a. $-\infty$
 b. $\frac{3}{2}(\sqrt[3]{3} - 1)$
 c. $\frac{3}{2}\sqrt[3]{2}$
 d. 3 correctchoice
 e. ∞

$$u = x^2 - 1 \quad du = 2x dx \quad \frac{1}{2} du = x dx$$

$$\int_1^3 \frac{x}{(x^2-1)^{2/3}} dx = \frac{1}{2} \int \frac{1}{(u)^{2/3}} du = \frac{1}{2} \frac{u^{1/3}}{1/3} = \frac{3}{2} (x^2-1)^{1/3} \Big|_1^3 = \frac{3}{2} (8^{1/3} - 0) = 3$$

3. Compute $\lim_{x \rightarrow 0} \frac{x \sin(x^2) - \sin(x^3)}{x^7}$

- a. $-\frac{1}{6}$ correctchoice
 b. $-\frac{1}{2}$
 c. 0
 d. $\frac{1}{2}$
 e. $\frac{1}{6}$

$$\sin x = x - \frac{x^3}{3!} + \dots \quad \sin(x^2) = x^2 - \frac{x^6}{3!} + \dots \quad \sin(x^3) = x^3 - \frac{x^9}{3!} + \dots$$

$$\lim_{x \rightarrow 0} \frac{x \sin(x^2) - \sin(x^3)}{x^7} = \lim_{x \rightarrow 0} \frac{x \left(x^2 - \frac{x^6}{3!} + \dots \right) - \left(x^3 - \frac{x^9}{3!} + \dots \right)}{x^7} = -\frac{1}{3!} = -\frac{1}{6}$$

4. The region below $y = e^{-x}$ in the first quadrant is rotated about the x -axis. Find the volume of the solid of revolution.

- a. $\frac{\pi}{4}$
 b. $\frac{\pi}{2}$ correctchoice
 c. π
 d. 2π
 e. 4π

$$V = \int_0^{\infty} \pi R^2 dx = \int_0^{\infty} \pi e^{-2x} dx = \left. \frac{\pi e^{-2x}}{-2} \right|_0^{\infty} = 0 - \frac{\pi}{-2} = \frac{\pi}{2}$$

5. A 20 meter chain is hanging down the side of a building. Its mass density is $\rho = 6 + 6 \sin^2(\pi x) \frac{\text{kg}}{\text{m}}$ where x is measured in meters down from the top of the building. Find the total mass of the chain.

- a. 60 kg
 b. 90 kg
 c. 120 kg
 d. 150 kg
 e. 180 kg correctchoice

$$M = \int_0^{20} \rho dx = \int_0^{20} (6 + 6 \sin^2(\pi x)) dx \quad \text{but} \quad \sin^2(\pi x) = \frac{1 - \cos(2\pi x)}{2}$$

$$M = \int_0^{20} \left(6 + 6 \frac{1 - \cos(2\pi x)}{2} \right) dx = \int_0^{20} (9 - 3 \cos(2\pi x)) dx = \left[9x - \frac{3 \sin(2\pi x)}{2\pi} \right]_0^{20} = 180$$

6. Find the total area between $y = 4x$ and $y = x^3$.

- a. 0
 b. 2
 c. 4
 d. 8 correctchoice
 e. 16

$$A = \int_{-2}^0 (x^3 - 4x) dx + \int_0^2 (4x - x^3) dx = \left[\frac{x^4}{4} - 2x^2 \right]_{-2}^0 + \left[2x^2 - \frac{x^4}{4} \right]_0^2 = 0 - [4 - 8] + [8 - 4] - 0 = 8$$

7. Find an equation of the line perpendicular to the plane $-2x + 3y + z = 7$ which contains the point $P = (3, 2, 1)$

- a. $x = 3 - 2t$ $y = 2 + 3t$ $z = 1 + t$ correctchoice
 b. $x = 3 - 2t$ $y = 2 - 3t$ $z = 1 + t$
 c. $x = -2 + 3t$ $y = 3 + 2t$ $z = 1 + t$
 d. $x = -2 + 3t$ $y = -3 + 2t$ $z = 1 + t$
 e. $x = -2 + 3t$ $y = 3 - 2t$ $z = 1 + t$

The normal to the plane is $\vec{N} = (-2, 3, 1)$. This is the tangent to the line. So

$$X = P + t\vec{N} \quad (x, y, z) = (3, 2, 1) + t(-2, 3, 1) = (3 - 2t, 2 + 3t, 1 + t)$$

8. The Maclaurin series $\sum_{n=1}^{\infty} \frac{(-x)^{n+1}}{n!}$ converges to

- a. xe^{-x}
- b. $-xe^{-x}$
- c. $\frac{1-e^{-x}}{x}$
- d. $\frac{e^{-x}-1}{x}$
- e. $x - xe^{-x}$ correctchoice

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \quad -xe^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^{n+1}}{n!} = -x + \sum_{n=1}^{\infty} \frac{(-x)^{n+1}}{n!}$$

$$\sum_{n=1}^{\infty} \frac{(-x)^{n+1}}{n!} = x - xe^{-x}$$

9. Find the area of the triangle whose **edges** are

$$\vec{u} = (2, -1, 2), \quad \vec{v} = (0, 1, 2) \quad \text{and} \quad \vec{w} = (2, -2, 0).$$

- a. $\frac{1}{2}\sqrt{6}$
- b. 3 correctchoice
- c. 6
- d. 18
- e. 36

$$\vec{u} \times \vec{v} = (2, -1, 2) \times (0, 1, 2) = \begin{vmatrix} i & j & k \\ 2 & -1 & 2 \\ 0 & 1 & 2 \end{vmatrix} = i(-2-2) - j(4-0) + k(2-0) = (-4, -4, 2)$$

$$A = \frac{1}{2}|\vec{u} \times \vec{v}| = \frac{1}{2}\sqrt{16+16+4} = \frac{1}{2}\sqrt{36} = 3$$

10. The substitution $x = (\tan\theta)^2$ turns the integral $\int_0^1 \frac{dx}{(1+x)\sqrt{x}}$ into

- a. $\int_0^{\pi/4} 2 d\theta$ correctchoice
- b. $\int_0^{\pi/4} 2 \sec \theta d\theta$
- c. $\int_0^{\pi/2} 2 d\theta$
- d. $\int_0^{\pi/2} \frac{d\theta}{2 \sec \theta}$
- e. $\int_0^{\pi/2} \frac{2 d\theta}{\sec \theta}$

$$dx = 2 \tan \theta \sec^2 \theta d\theta \quad 1+x = 1 + \tan^2 \theta = \sec^2 \theta \quad \sqrt{x} = \tan \theta$$

$$x=0 \Rightarrow \tan \theta = 0 \Rightarrow \theta = 0 \quad x=1 \Rightarrow \tan \theta = 1 \Rightarrow \theta = \frac{\pi}{4}$$

$$\int_0^1 \frac{dx}{(1+x)\sqrt{x}} = \int_0^{\pi/4} \frac{2 \tan \theta \sec^2 \theta}{\sec^2 \theta \tan \theta} d\theta = \int_0^{\pi/4} 2 d\theta$$

11. Compute $\int_{-1}^2 \frac{5}{x^2 - x - 6} dx$

- a. 0
- b. $-2\ln 4$ correctchoice
- c. $\ln 4$
- d. $2\ln 4$
- e. ∞

$$\frac{5}{x^2 - x - 6} = \frac{A}{x-3} + \frac{B}{x+2} \Rightarrow 5 = A(x+2) + B(x-3)$$

$$x = -2 : 5 = B(-5) \Rightarrow B = -1$$

$$x = 3 : 5 = A(5) \Rightarrow A = 1$$

$$\begin{aligned} \int_{-1}^2 \frac{5}{x^2 - x - 6} dx &= \int_{-1}^2 \left(\frac{1}{x-3} + \frac{-1}{x+2} \right) dx = \left[\ln|x-3| - \ln|x+2| \right]_{-1}^2 \\ &= \left[\ln|-1| - \ln|4| \right] - \left[\ln|-4| - \ln|1| \right] = -2\ln 4 \end{aligned}$$

12. The function $y(x)$ satisfies the differential equation $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{1+x^2}$ and the initial condition $y(0) = 0$. What is $y(1)$?

- a. $\frac{\pi}{4}$
- b. $\frac{\pi}{2}$
- c. $\frac{1}{2}$
- d. $\frac{1}{\sqrt{2}}$ correctchoice
- e. 1

$$\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{1+x^2} \Rightarrow \sin^{-1}y = \tan^{-1}x + C \text{ but } y(0) = 0 \Rightarrow C = 0$$

$$\text{So } y = \sin(\tan^{-1}x) \Rightarrow y(1) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

13. Find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{(x-3)^{2n}}{4^n}$.

- a. $\frac{1}{4}$
- b. $\frac{1}{2}$
- c. 2 correctchoice
- d. 3
- e. 4

This is a geometric series with ratio $r = \frac{(x-3)^2}{4}$.

It converges if $\left| \frac{(x-3)^2}{4} \right| < 1$ or $|x-3| < 2$. So $R = 2$.

Work Out (12 points each)

Show all your work. Partial credit will be given. You may not use a calculator.

14. Compute $\int_0^{\pi/2} (x - \pi) \sin x dx$

$$u = x - \pi \quad dv = \sin x dx$$

$$du = dx \quad v = -\cos x$$

$$\int (x - \pi) \sin x dx = -(x - \pi) \cos x + \int \cos x dx = -(x - \pi) \cos x + \sin x + C$$

$$\begin{aligned} \int_0^{\pi/2} (x - \pi) \sin x dx &= \left[-(x - \pi) \cos x + \sin x \right]_0^{\pi/2} \\ &= \left[-\left(\frac{\pi}{2} - \pi\right) \cos \frac{\pi}{2} + \sin \frac{\pi}{2} \right] - \left[-(-\pi) \cos 0 + \sin 0 \right] = 1 - \pi \end{aligned}$$

15. The curve $x = t^2$, $y = \frac{1}{3}t^3 - t$ for $0 \leq t \leq 1$ is rotated about the y -axis. Find the area of the surface of revolution.

$$r = x = t^2 \quad \frac{dx}{dt} = 2t \quad \frac{dy}{dt} = t^2 - 1$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (2t)^2 + (t^2 - 1)^2 = 4t^2 + t^4 - 2t^2 + 1 = t^4 + 2t^2 + 1 = (t^2 + 1)^2$$

$$\begin{aligned} L &= \int_0^1 2\pi r \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^1 2\pi t^2 \sqrt{(t^2 + 1)^2} dt = \int_0^1 2\pi t^2 (t^2 + 1) dt \\ &= 2\pi \int_0^1 (t^4 + t^2) dt = 2\pi \left[\frac{t^5}{5} + \frac{t^3}{3} \right]_0^1 = \frac{16\pi}{15} \end{aligned}$$

16. A 150 lb iron ball is hanging down the side of a building at the bottom of a 20 ft chain which weighs $5 \frac{\text{lb}}{\text{ft}}$. What is the total work done to lift the ball and chain to the top of the building?

$$W_{\text{ball}} = FD = 150 \cdot 20 = 3000 \text{ ft-lb}$$

The piece of chain of length dy a distance y from the top, weighs $dF = 5 dy$ lb and must be lifted a distance $D = y$.

$$W_{\text{chain}} = \int_0^{20} D dF = \int_0^{20} y 5 dy = 5 \frac{y^2}{2} \Big|_0^{20} = \frac{5 \cdot 20^2}{2} = 1000 \text{ ft-lb}$$

$$W_{\text{total}} = W_{\text{ball}} + W_{\text{chain}} = 4000 \text{ ft-lb}$$

17. Determine if each of the following series converges or diverges. Say why.
Be sure to name or quote the test(s) you use and check out all requirements of the test.

a. $\sum_{n=0}^{\infty} \frac{n^2(-5)^n}{(n+2)!}$

Circle one: Converges Diverges

Explain:

Ratio Test $a_n = \frac{n^2(-5)^n}{(n+2)!}$ $a_{n+1} = \frac{(n+1)^2(-5)^{n+1}}{(n+3)!}$

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2(-5)^{n+1}}{(n+3)!} \frac{(n+2)!}{n^2(-5)^n} \right| = 5 \lim_{n \rightarrow \infty} \frac{(n+1)^2}{n^2(n+3)} = 0 < 1$$

b. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

Circle one: Converges Diverges

Explain:

Integral Test $u = \ln n$ $du = \frac{1}{n} dn$

$$\int_2^{\infty} \frac{1}{n(\ln n)^2} dn = \int \frac{1}{u} du = [\ln u] = [\ln \ln u]_2^{\infty} = [\infty] - [\ln \ln 2] = \infty$$

Integral diverges \Rightarrow Series diverges

c. $\sum_{n=2}^{\infty} (-1)^n \frac{2n+1}{n^2+3}$

Circle one: Converges Diverges

Explain:

Alternating Series Test $(-1)^n \Rightarrow$ alternating,

$$\frac{2n+1}{n^2+3} \text{ decreasing, } \lim_{n \rightarrow \infty} \frac{2n+1}{n^2+3} = \lim_{n \rightarrow \infty} \frac{\frac{2}{n} + \frac{1}{n^2}}{1 + \frac{3}{n^2}} = 0$$