

MATH 152  
Exam 2  
Spring 2000  
Test Form A  
- Solutions -

Part I is multiple choice. There is no partial credit.

Part II is work out. Show all your work. Partial credit will be given.

You may not use a calculator.

Part I: Multiple Choice (5 points each)

There is no partial credit. You may not use a calculator.

1. Compute  $\int_{-1}^1 \frac{1}{x^6} dx$ .

- a. 0
- b.  $\frac{2}{5}$
- c.  $-\frac{2}{7}$
- d.  $-\frac{2}{5}$
- e. Divergent      correctchoice

$$\int_{-1}^1 \frac{1}{x^6} dx = \int_{-1}^0 \frac{1}{x^6} dx + \int_0^1 \frac{1}{x^6} dx \quad \text{and} \quad \int_0^1 \frac{1}{x^6} dx = \lim_{a \rightarrow 0} \left[ \frac{x^{-5}}{-5} \right]_a^1 = \left[ \frac{-1}{5} \right] - \lim_{a \rightarrow 0^+} \left[ \frac{-1}{5a^5} \right] = \infty$$

Since  $\int_0^1 \frac{1}{x^6} dx$  is divergent, so is  $\int_{-1}^1 \frac{1}{x^6} dx$ .

2. Find the solution to  $\frac{dy}{dx} = -\frac{x}{y}$  satisfying the initial condition  $y(3) = 4$ . Then  $y(0) =$

- a. 0
- b. 1
- c. 3
- d. 5      correctchoice
- e. 6

Separable differential equation. Separate the variables and integrate:

$$y dy = -x dx \quad \Rightarrow \quad \int y dy = -\int x dx \quad \Rightarrow \quad \frac{y^2}{2} = -\frac{x^2}{2} + C$$

Use the initial condition  $x = 3$   $y = 4$ :

$$\frac{4^2}{2} = -\frac{3^2}{2} + C \quad \Rightarrow \quad C = \frac{16}{2} + \frac{9}{2} = \frac{25}{2}$$

The solution is:

$$\frac{y^2}{2} = -\frac{x^2}{2} + \frac{25}{2} \quad \Rightarrow \quad x^2 + y^2 = 25 \quad \Rightarrow \quad y = \sqrt{25 - x^2}$$

So  $y(0) = 5$ .

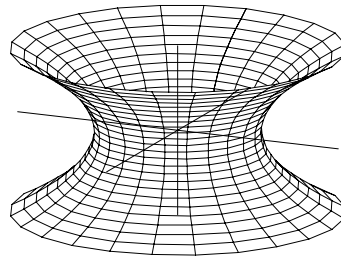
3. A tank contains 500 liters of water with 10 kg of sugar dissolved. Sugar water that contains  $\frac{1}{10}$  kg of sugar per liter of water flows into the tank at the rate of 7 liters per minute. Sugar water that contains  $\frac{1}{20}$  kg of sugar per liter of water flows into the tank at the rate of 8 liters per minute. The solution is kept thoroughly mixed and drains from the tank at the rate of 15 liters per minute. Write down the initial value problem for  $S(t)$ , the amount of sugar in the tank at time  $t$ .

- a.  $\frac{dS}{dt} = 10t + \frac{3}{10} - \frac{S}{500}$  with  $S(0) = 100$   
 b.  $\frac{dS}{dt} = \frac{11}{10} - \frac{3S}{100}$  with  $S(0) = 10$  correctchoice  
 c.  $\frac{dS}{dt} = \frac{3}{20} - \frac{S}{500}$  with  $S(0) = 10$   
 d.  $\frac{dS}{dt} = \frac{3}{20} - \frac{3S}{100}$  with  $S(0) = \frac{1}{50}$   
 e.  $\frac{dS}{dt} = \frac{11}{10}t - \frac{S}{500}$  with  $S(0) = 100$

$$\frac{dS}{dt} = \underbrace{\frac{1 \text{ kg}}{10 \text{ L}} \cdot \frac{7 \text{ L}}{\text{min}} + \frac{1 \text{ kg}}{20 \text{ L}} \cdot \frac{8 \text{ L}}{\text{min}}}_{\text{in}} - \underbrace{\frac{S \text{ kg}}{500 \text{ L}} \cdot \frac{15 \text{ L}}{\text{min}}}_{\text{out}} = \frac{11}{10} - \frac{3S}{100}$$

$$S(0) = 10 \text{ kg}$$

4. Which integral gives the surface area of the “spool” obtained by revolving the curve  $y^2 - x + 1 = 0$  for  $-1 \leq y \leq 1$  about the  $y$ -axis.



- a.  $\int_{-1}^1 2\pi(1+y^2)\sqrt{1+4y^2} dy$  correctchoice  
 b.  $\int_{-1}^1 2\pi y\sqrt{1+4y^2} dy$   
 c.  $\int_{-1}^1 \pi y\sqrt{1+2y} dy$   
 d.  $\int_{-1}^1 2\pi(1+4y^2)\sqrt{1+y^2} dy$   
 e.  $\int_{-1}^1 2\pi(1+y^2)\sqrt{1+2y^2} dy$

$$y\text{-integral, } r = x = y^2 + 1, \quad \frac{dx}{dy} = 2y$$

$$A = \int 2\pi r ds = \int_{-1}^1 2\pi(y^2 + 1) \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_{-1}^1 2\pi(y^2 + 1) \sqrt{1 + 4y^2} dy$$

5. Find the  $x$ -coordinate of the centroid (center of mass,  $\bar{x}$ ) of the region between  $y = x^3$  and the  $x$ -axis for  $0 \leq x \leq 2$ .
- $\frac{4}{5}$
  - 4
  - $\frac{8}{5}$  correctchoice
  - $\frac{32}{5}$
  - $\frac{2}{3}$

The area is  $A = \int_0^2 x^3 dx = \left[ \frac{x^4}{4} \right]_0^2 = 4$ .

The first moment is 1<sup>st</sup>-mom =  $\int_0^2 x^4 dx = \left[ \frac{x^5}{5} \right]_0^2 = \frac{32}{5}$ .

The  $x$ -coordinate of the centroid is  $\bar{x} = \frac{1^{\text{st-mom}}}{A} = \frac{32}{5 \cdot 4} = \frac{8}{5}$

6. Find the limit of the sequence  $a_n = \frac{\ln(n^2 + 1)}{\ln(n)}$ .
- 0
  - $\ln 2$
  - 2 correctchoice
  - $e$
  - $\infty$

$$\lim_{n \rightarrow \infty} \frac{\ln(n^2 + 1)}{\ln(n)} \stackrel{\text{IH}}{=} \lim_{n \rightarrow \infty} \frac{\frac{2n}{n^2 + 1}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{2n^2}{n^2 + 1} = \lim_{n \rightarrow \infty} \frac{2}{1 + \frac{1}{n^2}} = 2$$

7. Which of the following series are convergent?

$$(i) \sum_{n=1}^{\infty} \frac{100^n}{n!} \quad (ii) \sum_{n=1}^{\infty} \frac{2^n}{n + 3^n}$$

- both (i) and (ii) correctchoice
- (i) only
- (ii) only
- neither

(i) converges by the Ratio Test:  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{100^{n+1}}{(n+1)!} \frac{n!}{100^n} = \lim_{n \rightarrow \infty} \frac{100}{(n+1)} = 0 < 1$ .

(ii) converges by the Ratio Test:  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{2^{n+1}}{n+1+3^{n+1}} \frac{n+3^n}{2^n} = \frac{2}{3} < 1$

or by comparison to  $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$  which is a geometric series with  $r = \frac{2}{3} < 1$ .

8. Which of the following series are convergent?

$$(i) \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1} \quad (ii) \sum_{n=2}^{\infty} \frac{n}{\ln n}$$

- a. both (i) and (ii)
- b. (i) only
- c. (ii) only
- d. neither correctchoice

(i) diverges by limit comparison with  $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$  which is a  $p$ -series with  $p = \frac{1}{2} < 1$  since

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{\sqrt{n}}{n+1}}{\frac{1}{\sqrt{n}}} = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$$

or by comparison to  $\sum_{n=1}^{\infty} \frac{1}{n+1}$  which is a harmonic series since  $\frac{\sqrt{n}}{n+1} \geq \frac{1}{n+1}$ .

(ii) diverges by the  $n^{\text{th}}$ -Term Divergence Test:  $\lim_{n \rightarrow \infty} \frac{n}{\ln n} \stackrel{L'H}{=} \lim_{n \rightarrow \infty} \frac{1}{\frac{1}{n}} = \lim_{n \rightarrow \infty} n = \infty \neq 0$ .

9. Compute  $\sum_{n=0}^{\infty} \frac{5^{n+1}}{4^n}$ .

- a. -20
- b.  $\frac{20}{9}$
- c. 20
- d. 25
- e. divergent correctchoice

This is a geometric series whose ratio is  $r = \frac{5}{4} > 1$ . So the series diverges.

10. The series  $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

- a. is absolutely convergent.
- b. is convergent but not absolutely convergent. correctchoice
- c. is divergent to  $+\infty$ .
- d. is divergent to  $-\infty$ .
- e. is divergent but not to  $\pm\infty$ .

$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$  converges by the Alternating Series Test since  $\frac{1}{\sqrt{n}}$  is decreasing and  $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$

The related absolute series  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$  is a divergent  $p$ -series since  $p = \frac{1}{2} < 1$ .

So the original series  $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$  is not absolutely convergent.

Part II: Work Out (10 points each)

Show all your work. Partial credit will be given.

You may not use a calculator.

11. Find the solution of  $\frac{dy}{dx} - 3y = 6e^{-x}$  satisfying  $y(0) = 5$ . Solve for  $y$ .

$$I = e^{\int(-3)dx} = e^{-3x}$$

$$e^{-3x} \frac{dy}{dx} - 3e^{-3x}y = 6e^{-3x}e^{-x} \Rightarrow \frac{d}{dx}(e^{-3x}y) = 6e^{-4x} \Rightarrow e^{-3x}y = \int 6e^{-4x} dx = \frac{6e^{-4x}}{-4} + C$$

$$e^{-3x}y = -\frac{3}{2}e^{-4x} + C \Rightarrow y = -\frac{3}{2}e^{-x} + Ce^{3x}$$

$$x = 0, \quad y = 5 \Rightarrow e^{0}5 = -\frac{3}{2}e^0 + C \Rightarrow C = \frac{13}{2}$$

$$e^{-3x}y = -\frac{3}{2}e^{-4x} + \frac{13}{2} \Rightarrow y = -\frac{3}{2}e^{-x} + \frac{13}{2}e^{3x}$$

12. Consider the series  $S = \sum_{n=1}^{\infty} \left( \frac{n}{n+1} - \frac{n+1}{n+2} \right)$

- a. Write out the third partial sum  $S_3 = \sum_{n=1}^3 \left( \frac{n}{n+1} - \frac{n+1}{n+2} \right)$  and add it up.

$$S_3 = \left( \frac{1}{2} - \frac{2}{3} \right) + \left( \frac{2}{3} - \frac{3}{4} \right) + \left( \frac{3}{4} - \frac{4}{5} \right) = \frac{1}{2} - \frac{4}{5} = \frac{5-8}{10} = -\frac{3}{10}$$

- b. Compute the sum of the infinite series  $S = \sum_{n=1}^{\infty} \left( \frac{n}{n+1} - \frac{n+1}{n+2} \right)$ .

$$S_k = \sum_{n=1}^k \left( \frac{n}{n+1} - \frac{n+1}{n+2} \right) = \left( \frac{1}{2} - \frac{2}{3} \right) + \left( \frac{2}{3} - \frac{3}{4} \right) + \dots + \left( \frac{k}{k+1} - \frac{k+1}{k+2} \right) = \frac{1}{2} - \frac{k+1}{k+2}$$

$$S = \lim_{k \rightarrow \infty} S_k = \lim_{k \rightarrow \infty} \left( \frac{1}{2} - \frac{k+1}{k+2} \right) = \frac{1}{2} - 1 = -\frac{1}{2}$$

13. Find the arc length of the parametric curve  $x = t^3$ ,  $y = 3t^2$  between  $t = 0$  and  $t = \sqrt{12}$ .

$$t\text{-integral} \quad \frac{dx}{dt} = 3t^2 \quad \frac{dy}{dt} = 6t$$

$$L = \int_0^{\sqrt{12}} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} dt = \int_0^{\sqrt{12}} \sqrt{(3t^2)^2 + (6t)^2} dt = \int_0^{\sqrt{12}} \sqrt{9t^4 + 36t^2} dt = \int_0^{\sqrt{12}} 3t\sqrt{t^2 + 4} dt$$

$$\text{Let } u = t^2 + 4 \quad \text{Then } du = 2t dt \quad \text{or} \quad \frac{1}{2} du = t dt$$

$$L = \frac{3}{2} \int \sqrt{u} du = u^{3/2} = (t^2 + 4)^{3/2} \Big|_0^{\sqrt{12}} = 16^{3/2} - 4^{3/2} = 4^3 - 2^3 = 64 - 8 = 56$$

14. Consider the series  $S = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ .

- a. Prove the series  $S = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$  converges. Be sure to name the convergence test you use.

We apply the Integral Test: Let  $u = \ln n$ . Then  $du = \frac{1}{n} dn$  So

$$\int_2^{\infty} \frac{1}{n(\ln n)^2} dn = \int_{n=2}^{\infty} \frac{1}{u^2} du = \frac{-1}{u} = \left[ \frac{-1}{\ln n} \right]_2^{\infty} = 0 - \frac{-1}{\ln 2} = \frac{1}{\ln 2}$$

Since the integral converges, the series converges.

- b. If you approximate the series  $S$  by the partial sum  $S_9 = \sum_{n=2}^9 \frac{1}{n(\ln n)^2}$ , find an upper bound for the error  $|R_9| = |S - S_9|$  in the approximation and justify your estimate.

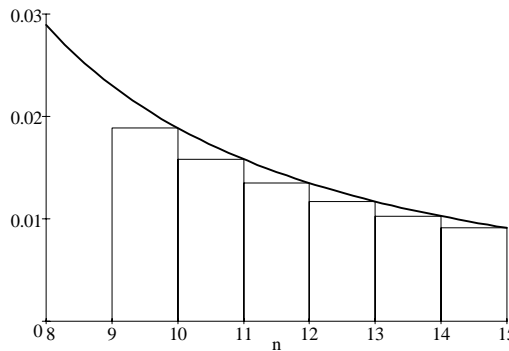
The remainder is

$$R_9 = S - S_9 = \sum_{n=10}^{\infty} \frac{1}{n(\ln n)^2}$$

The function  $\frac{1}{n(\ln n)^2}$  is decreasing.

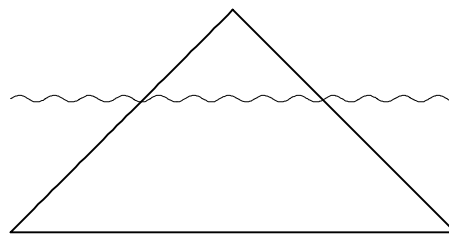
The remainder is bounded by

$$R_9 < \int_9^{\infty} \frac{1}{n(\ln n)^2} dn = \left[ \frac{-1}{\ln n} \right]_9^{\infty} \\ = 0 - \frac{-1}{\ln 9} = \frac{1}{\ln 9}$$



15. A triangular plate is suspended vertically in water as shown. The height of the triangle is 5 ft. Its base is 10 ft, and it is submerged 3 ft into the water. Find the force on one face of the plate.

$$(\rho g = 62.5 \frac{\text{lb}}{\text{ft}^3})$$



Measure  $y$  down from the tip of the triangle. So the water is at  $2 \leq y \leq 5$ . Let  $w$  be the width of the triangle at position  $y$ . So by similar triangles

$$\frac{w}{y} = \frac{10}{5} = 2 \quad \Rightarrow \quad w = 2y$$

Consequently, the area of a slice of height  $dy$  is

$$dA = w dy = 2y dy$$

The height of the water above this slice is  $h = y - 2$ . So the force on the plate is

$$F = \int_2^5 \rho g h dA = \rho g \int_2^5 (y - 2) 2y dy = \rho g \int_2^5 (2y^2 - 4y) dy = \rho g \left[ 2 \frac{y^3}{3} - 2y^2 \right]_2^5 \\ = \rho g \left( \frac{250}{3} - 50 \right) - \rho g \left( \frac{16}{3} - 8 \right) = 36 \rho g = 2250$$