

Multiple Choice: (4 points each)

1. Compute $\int_0^{\pi/2} x \cos(3x) dx$

- a. $-\frac{\pi}{3} - \frac{1}{3}$
- b. $-\frac{\pi}{6} - \frac{1}{3}$
- c. $-\frac{\pi}{3} + \frac{1}{3}$
- d. $-\frac{\pi}{6} - \frac{1}{9}$ correctchoice
- e. $-\frac{\pi}{3} + \frac{1}{9}$

Integrate by parts with $u = x$ $dv = \cos(3x) dx$
 $du = dx$ $v = \frac{1}{3} \sin(3x)$

$$\begin{aligned} \int_0^{\pi/2} x \cos(3x) dx &= \left[\frac{x}{3} \sin(3x) - \frac{1}{3} \int \sin(3x) dx \right]_0^{\pi/2} = \left[\frac{x}{3} \sin(3x) + \frac{1}{9} \cos(3x) \right]_0^{\pi/2} \\ &= \frac{\pi}{6} \sin\left(\frac{3\pi}{2}\right) - \frac{1}{9} \cos(0) = -\frac{\pi}{6} - \frac{1}{9} \end{aligned}$$

2. Compute $\lim_{n \rightarrow \infty} \frac{2^n}{1 + 3^n}$

- a. 0 correctchoice
- b. $\frac{1}{2}$
- c. $\frac{1}{1 - \frac{2}{3}}$
- d. $\frac{\frac{1}{2}}{1 - \frac{2}{3}}$
- e. ∞

$$\lim_{n \rightarrow \infty} \frac{2^n}{1 + 3^n} = \lim_{n \rightarrow \infty} \frac{\frac{2^n}{3^n}}{\frac{1}{3^n} + 1} = \frac{0}{1} = 0$$

3. Compute $\int_0^{\pi/2} \sin^3 \theta d\theta$

- a. $-\frac{2}{3}$
- b. $-\frac{1}{3}$
- c. 0
- d. $\frac{1}{3}$
- e. $\frac{2}{3}$ correctchoice

Let $u = \cos \theta$. Then $du = -\sin \theta d\theta$. So:

$$\begin{aligned} \int_0^{\pi/2} \sin^3 \theta d\theta &= \int_0^{\pi/2} (1 - \cos^2 \theta) \sin \theta d\theta = -\int_1^0 (1 - u^2) du \\ &= -\left[u - \frac{u^3}{3} \right]_1^0 = -0 + \left[1 - \frac{1}{3} \right] = \frac{2}{3} \end{aligned}$$

4. Which formula will give the arclength of the curve $y = \sin x$ between $x = 0$ and $x = \pi$?

- a. $L = \int_0^\pi 2\pi x \sqrt{1 + \cos^2 x} dx$
 b. $L = \int_0^\pi \sqrt{1 + \cos^2 x} dx$ correct choice
 c. $L = \int_0^\pi 2\pi \sin x \sqrt{1 + \cos^2 x} dx$
 d. $L = \int_0^\pi 2\pi x \sqrt{1 + \sin^2 x} dx$
 e. $L = \int_0^\pi \sqrt{1 + \sin^2 x} dx$

$$\frac{dy}{dx} = \cos x \quad L = \int_0^\pi \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^\pi \sqrt{1 + \cos^2 x} dx$$

5. Which initial value problem describes the solution to the following problem:

A 100 gal tank is initially filled with sugar water whose concentration is $0.05 \frac{\text{lb sugar}}{\text{gal water}}$. Sugar is added to the tank at the rate of $2 \frac{\text{lb}}{\text{hr}}$ and pure water is added at the rate of $3 \frac{\text{gal}}{\text{hr}}$. The mixture is kept well mixed and drained at the rate of $3 \frac{\text{gal}}{\text{hr}}$. Find the amount of sugar in the tank after t hours.

- a. $\frac{dS}{dt} = 2 - 0.03S, \quad S(0) = 5$ correct choice
 b. $\frac{dS}{dt} = 0.1 - 0.15S, \quad S(0) = 5$
 c. $\frac{dS}{dt} = 3S - 0.02, \quad S(0) = 0.05$
 d. $\frac{dS}{dt} = 0.02 - 3S, \quad S(0) = 5$
 e. $\frac{dS}{dt} = 0.02 - 0.03S, \quad S(0) = 0.05$

$$\frac{dS}{dt} \frac{\text{lb}}{\text{hr}} = 2 \frac{\text{lb}}{\text{hr}} - \frac{S \text{ lb}}{100 \text{ gal}} 3 \frac{\text{gal}}{\text{hr}} = 2 - 0.03S \quad S(0) = 0.05 \frac{\text{lb}}{\text{gal}} 100 \text{ gal} = 5$$

6. Find the solution of the differential equation $\frac{dy}{dx} = 2x(1 + y^2)$ satisfying the initial condition $y(2) = 0$.

- a. $y = \tan(x^2) + 2$
 b. $y = \tan^2(x - 2)$
 c. $y = \tan(x^2 - 4)$ correct choice
 d. $y = \tan(x^2 + \arctan 2)$
 e. $y = \tan^2(x) - \tan^2 2$

$$\int \frac{dy}{1 + y^2} = \int 2x dx \quad \arctan y = x^2 + C \quad \arctan 0 = 4 + C \quad y = \tan(x^2 - 4)$$

7. Compute $\int_1^2 \frac{1}{(x-2)^{2/3}} dx$

- a. $-\infty$
- b. -3
- c. -1
- d. 3 correctchoice
- e. ∞

$$\int_1^2 \frac{1}{(x-2)^{2/3}} dx = [3(x-2)^{1/3}]_1^2 = 3(2-2)^{1/3} - 3(1-2)^{1/3} = 3$$

8. Compute $\lim_{x \rightarrow 0} \frac{\sin(2x) - 2x}{3x^3}$

- a. $-\frac{1}{9}$
- b. -4
- c. $-\frac{4}{9}$ correctchoice
- d. $-\frac{8}{9}$
- e. $-\frac{4}{3}$

$$\lim_{x \rightarrow 0} \frac{\sin(2x) - 2x}{3x^3} = \lim_{x \rightarrow 0} \frac{\left[2x - \frac{(2x)^3}{3!} + \dots\right] - 2x}{3x^3} = \lim_{x \rightarrow 0} \left[-\frac{(2x)^3}{3!3x^3} + \dots\right] = -\frac{8}{6 \cdot 3} = -\frac{4}{9}$$

9. Find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{2^n}{(n+1)^2} (x-3)^n$.

- a. 0
- b. $\frac{1}{2}$ correctchoice
- c. 2
- d. $\frac{1}{3}$
- e. 3

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}(x-3)^{n+1}}{(n+2)^2} \frac{(n+1)^2}{2^n(x-3)^n} \right| = 2|x-3|$$

Convergent if

$$L = 2|x-3| < 1 \text{ or } |x-3| < \frac{1}{2}. \text{ So } R = \frac{1}{2}.$$

10. Which term is incorrect in the following partial fraction expansion?

$$\frac{-10x^2 + 5x^3 - 8x + 1}{(x-1)(x-3)^2(x^2+2)} = \underbrace{\frac{A}{x-1}}_a + \underbrace{\frac{B}{x-3}}_b + \underbrace{\frac{D}{(x-3)^2}}_c + \underbrace{\frac{Ex+F}{x^2+2}}_d$$

e. They are all correct. correctchoice

A linear or linear to a power in the denominator gets a constant in the numerator.
 A quadratic or quadratic to a power in the denominator gets a linear in the numerator.
 So all terms are correct.

11. A vector \vec{u} has length 3. A vector \vec{v} has length 4. The angle between them is 60° . Find $\vec{u} \cdot \vec{v}$.

- a. 6 correctchoice
- b. $\frac{1}{24}$
- c. $\frac{\sqrt{3}}{24}$
- d. 24
- e. $6\sqrt{3}$

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta = 3 \cdot 4 \cdot \cos 60^\circ = \frac{12}{2} = 6$$

12. Find an equation for the plane containing the two lines

$$\begin{aligned} L_1 : \quad x &= 3 + 3t & y &= 1 + 4t & z &= 2 + 5t \\ L_2 : \quad x &= 3 + t & y &= 1 & z &= 2 - t \end{aligned}$$

- a. $-4x - 8y - 4z = 10$
- b. $-4x + 8y - 4z = 10$
- c. $x - 2y + z = 3$ correctchoice
- d. $x + 2y + z = 7$
- e. $x + 2y + z = 10$

$$\begin{aligned} \vec{v}_1 &= (3, 4, 5) & \vec{v}_2 &= (1, 0, -1) & \vec{N} &= \vec{v}_1 \times \vec{v}_2 = (-4, 8, -4) & P &= (3, 1, 2) \\ \vec{N} \cdot (X - P) &= 0 & -4(x - 3) + 8(y - 1) - 4(z - 2) &= 0 \\ -4x + 8y - 4z &= -12 & x - 2y + z &= 3 \end{aligned}$$

Work Out (13 points each)

Show all your work. Partial credit will be given. You may not use a calculator.

13. Compute $\int \frac{\sqrt{x^2 - 1}}{x} dx$

$$x = \sec \theta \quad dx = \sec \theta \tan \theta d\theta \quad \sqrt{x^2 - 1} = \sqrt{\sec^2 \theta - 1} = \tan \theta$$

$$\int \frac{\sqrt{x^2 - 1}}{x} dx = \int \frac{\tan \theta}{\sec \theta} \sec \theta \tan \theta d\theta$$

$$= \int \tan^2 \theta d\theta$$

$$= \int 1 - \sec^2 \theta d\theta$$

$$= \theta - \tan \theta$$

$$= \operatorname{arcsec} x - \sqrt{x^2 - 1} + C$$

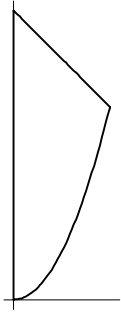
14. The parametric curve given by $x = t^2$, $y = \frac{2}{3}t^3$, $z = \frac{1}{4}t^4$ for $0 \leq t \leq 2$ is rotated about the y -axis. Find the area of the surface of revolution.

HINT: Factor the quantity in the square root.

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 2t^2, \quad \frac{dz}{dt} = t^3$$

$$\begin{aligned} A &= \int_0^2 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \\ &= \int_0^2 2\pi t^2 \sqrt{(2t)^2 + (2t^2)^2 + (t^3)^2} dt \\ &= \int_0^2 2\pi t^2 \sqrt{4t^2 + 4t^4 + t^6} dt \\ &= \int_0^2 2\pi t^2 \sqrt{t^2(2 + t^2)^2} dt = \int_0^2 2\pi t^3 (2 + t^2) dt = 2\pi \int_0^2 (2t^3 + t^5) dt \\ &= 2\pi \left[\frac{t^4}{2} + \frac{t^6}{6} \right]_0^2 \\ &= 2\pi \left[\frac{16}{2} + \frac{64}{6} \right] = 16\pi \left(1 + \frac{4}{3} \right) = \frac{112}{3}\pi \end{aligned}$$

15. The region in the first quadrant between the curves $y = x^2$ and $y = 6 - x$ is rotated about the y -axis. Find the volume of the solid of revolution.

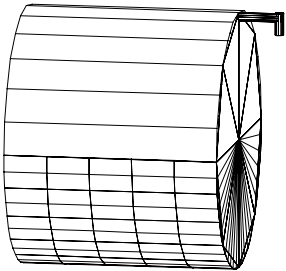


Use an x -integral with cylinders. $h = 6 - x - x^2$ $r = x$

To find the right endpoint, we solve $x^2 = 6 - x$, or $x^2 + x - 6 = 0$ or $(x - 2)(x + 3) = 0$. In the first quadrant $x = 2$.

$$\begin{aligned} V &= \int_0^2 2\pi r h dx = \int_0^2 2\pi x(6 - x - x^2) dx = 2\pi \int_0^2 (6x - x^2 - x^3) dx \\ &= 2\pi \left[3x^2 - \frac{x^3}{3} - \frac{x^4}{4} \right]_0^2 = 2\pi \left[12 - \frac{8}{3} - 4 \right]_0^2 = \frac{32}{3}\pi \end{aligned}$$

16. A water tank has the shape of a circular cylinder laying on its side. It is 3 ft in radius and 5 ft long. It is half full of water. How much work is needed to pump the water out a spout at the top? (The weight density of water is $\rho g = 64.5 \frac{\text{lb}}{\text{ft}^3}$ but you may leave your answer as a multiple of ρg .)



We put the origin at the center of a circular end with y measured downward. So the water at height y must be lifted a distance $D = y + 3$.

To know the weight of a slab of water at height y , we must know its volume. Its length is 5. Its width is $2x$. Its thickness is dy . By the Pythagorean theorem $x = \sqrt{9 - y^2}$. So the weight is

$$dF = \rho g dV = \rho g 10x dy = \rho g 10\sqrt{9 - y^2} dy.$$

So the work is

$$\begin{aligned} W &= \int D dF = \int_0^3 (y + 3)\rho g 10\sqrt{9 - y^2} dy \\ &= 10\rho g \int_0^3 y\sqrt{9 - y^2} dy + 30\rho g \int_0^3 \sqrt{9 - y^2} dy \end{aligned}$$

The first integral is a simple substitution. The second integral is the area of a quarter circle of radius 3.

$$\begin{aligned} W &= 10\rho g \left[-\frac{1}{3}(9 - y^2)^{3/2} \right]_0^3 + 30\rho g \frac{1}{4}\pi(3)^2 \\ &= 10\rho g \frac{1}{3}(9)^{3/2} + \frac{135}{2}\rho g \pi = \rho g \left(90 + \frac{135}{2}\pi \right) \\ &= 64.5 \left(90 + \frac{135}{2}\pi \right) \end{aligned}$$