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Nonlinear Stability for Multidimensional Fourth-Order Shock Fronts

PETER HOWARD & CHANGBING HU

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Abstract

We consider the question of stability for planar wave solutions that arise in multidimensional conservation laws with only fourth-order regularization. Such equations arise, for example, in the study of thin films, for which planar waves correspond to fluid coating a pre-wetted surface. An interesting feature of these equations is that both compressive, and undercompressive, planar waves arise as solutions (compressive or undercompressive with respect to asymptotic behavior relative to the un-regularized hyperbolic system), and numerical investigation by Bertozzi, Münch, and Shearer indicates that undercompressive waves can be nonlinearly stable. Proceeding with pointwise estimates on the Green's function for the linear fourth-order convection–regularization equation that arises upon linearization of the conservation law about the planar wave solution, we establish that under general spectral conditions, such as appear to hold for shock fronts arising in our motivating thin films equations, compressive waves are stable for all dimensions $d \geqq 2$ and undercompressive waves are stable for dimensions $d \geqq 3$. (In the special case $d = 1$, compressive waves are stable under a very general spectral condition.) We also consider an alternative spectral criterion (valid, for example, in the case of constant-coefficient regularization), for which we can establish nonlinear stability for compressive waves in dimensions $d \geqq 3$ and undercompressive waves in dimensions $d \geqq 5$. The case of stability for undercompressive waves in the thin films equations for the critical dimensions $d = 1$ and $d = 2$ remains an interesting open problem.

1. Introduction

We consider the multidimensional regularized conservation law

$$u_t + \sum_{j=1}^{d} f^j(u)_{x_j} = -\sum_{jklm} \left(b^{jklm}(u) u_{x_j x_k x_l} \right)_{x_m},$$

$$u(0,x) = u_0(x);$$
$$u_0(\pm\infty) = u_\pm, \tag{1.1}$$

where u, f^j, $b^{jklm} \in \mathbb{R}$, $x \in \mathbb{R}^d$, for some dimension $d \geqq 2$ of the space variable, and $t > 0$. Using the detailed pointwise Green's function estimates developed in [6] for the linear fourth-order convection–regularization equation that arises upon linearization of (1.1) about a planar viscous shock front $\bar{u}(x_1)$, $\bar{u}(\pm\infty) = u_\pm$, we establish that under general spectral conditions (defined below), such as appear to hold for shock fronts arising in our motivating thin films equations, compressive waves are stable for all dimensions $d \geqq 2$ and undercompressive waves are stable for dimensions $d \geqq 3$. (In the special case $d = 1$, compressive waves are stable under a very general spectral condition.) We also consider an alternative spectral criterion, valid for example in the case of constant-coefficient regularization, for which we can establish nonlinear stability for compressive waves in dimensions $d \geqq 3$ and undercompressive waves in dimensions $d \geqq 5$. The case of stability for undercompressive waves in the thin films equations for the critical dimensions $d = 1$ and $d = 2$ remains an interesting open problem.

Throughout the paper, we will refer to the following fundamental assumptions on (1.1) and the planar wave solution $\bar{u}(x_1 - st)$:

(H0) (regularity) $f^j, b^{jklm} \in C^2(\mathbb{R}), \quad b^{1111}(\bar{u}(x_1)) \geqq b_0 > 0$.
(H1) (non-sonicity) $\partial_u f^1(u_\pm) \neq s$.
(H2) $\sum_{jklm} b^{jklm}(\bar{u}(x_1))\xi_j\xi_k\xi_l\xi_m \geqq \theta|\xi|^4$ for all $\xi \in \mathbb{R}^d$ and some $\theta > 0$.

We observe that owing to the generality of f^1, we may shift without loss of generality to a moving coordinate system for which $s = 0$, a convention we take throughout the analysis.

Conservation laws of form (1.1) that satisfy hypotheses (H0)–(H2) arise, for example, in the study of thin film flows, in which the height $h(t,x)$ of a film moving along an inclined plane can, under certain circumstances, be modeled by equations with fourth-order smoothing only, such as

$$h_t + (h^2 - h^3)_x = -(h^3 h_{xxx})_x, \quad x \in \mathbb{R};$$
$$h_t + (h^2 - h^3)_{x_1} = -\nabla \cdot (h^3 \nabla \Delta h), \quad x \in \mathbb{R}^2, \tag{1.2}$$

(see [2] and the references therein). In this setting, the onset of fingering instabilities is a critical issue and spectral stability has been considered in [3]. However, to date, no results on nonlinear stability for such equations have been established. An interesting feature of these equations is that both compressive and undercompressive planar waves arise as solutions (compressive or undercompressive with respect to asymptotic behavior relative to the un-regularized hyperbolic system), and numerical investigation by BERTOZZI, MÜNCH, & SHEARER indicates that undercompressive waves can be nonlinearly stable [2].

It is well known that for $d = 1$ solutions, $u(t,x)$ of (1.1) initialized by $u(0,x)$ near a standing wave solution $\bar{u}(x)$ will not generally approach $\bar{u}(x)$ time asymptotically, but rather will approach a translate of $\bar{u}(x)$ determined by the amount of mass (measured by $\int_{\mathbb{R}}(u(0,x) - \bar{u}(x))dx$) carried into the shock as well as the amount of mass convected along outgoing characteristics to the far field. For $d = 1$, a local tracking function $\delta(t)$ will serve to approximate this shift at each time t. In particular, the perturbation can be defined by the relation $v(t, x) = u(t, x) - \bar{u}(x - \delta(t))$, where $\delta(t)$ is chosen so that at each time t the shapes of $u(t, x)$ and $\bar{u}(x)$ are compared, rather than their locations. (See, for example, [10].) We remark that the notation δ for shift is standard at this point for analyses in the current framework and should not be confused with a Dirac delta function.

In the case $d \geqq 2$, the shift from the planar shock front $\bar{u}(x_1)$ depends additionally on the transverse variable $\tilde{x} = (x_2, x_3, \ldots, x_d)$. In this case, $u(t, x)$ does not approach a shifted wave asymptotically (the shift goes to 0 as $t \to \infty$), but these *ripples* along the shock layer slow asymptotic convergence, hindering the nonlinear analysis. We thus proceed, similarly as in the case $d = 1$, by introducing the perturbation $v(t, x)$, defined through

$$v(t, x) = u(t, x) - \bar{u}(x_1 - \delta(t, \tilde{x})),$$

to arrive at the perturbation equation (closely following the notation of [9])

$$(\partial_t - L)v = (\partial_t - L)(\bar{u}_{x_1}(x_1)\delta(t, \tilde{x})) + \sum_{m=1}^{d} (Q^m + R^m + S^m)_{x_m}, \qquad (1.3)$$

where Q, R, S are continuously differentiable functions of their arguments, and

$$\begin{aligned} Lv &:= -\sum_{jklm} (b^{jklm}(x_1) v_{x_j x_k x_l})_{x_m} - \sum_{j=1}^{d} (a_j(x_1) v)_{x_j} \\ a_j(x_1) &:= \partial_u f^j(\bar{u}(x_1)) + \partial_u b^{111j}(\bar{u}(x_1)) \bar{u}_{x_1 x_1 x_1} \\ b^{jklm}(x_1) &:= b^{jklm}(\bar{u}(x_1)), \end{aligned} \qquad (1.4)$$

with also

$$\begin{aligned} Q^m &= \mathbf{O}(v^2) + \sum_{jkl} \mathbf{O}(|v||v_{x_j x_k x_l}|), \quad \text{for each } m = 1, \ldots, d, \\ R^1 &= \mathbf{O}(e^{-\bar{\eta}|x_1|}) \Big[\mathbf{O}(|\delta|) \mathbf{O}\big(\sum_{j \neq 1} |\delta_{x_j}| + \sum_{jk \neq 1} |\delta_{x_j x_k}| + \sum_{jkl \neq 1} |\delta_{x_j x_k x_l}|\big) \\ &\quad + \mathbf{O}(|\delta \delta_t|) + \mathbf{O}\big(\sum_{j \neq 1} |\delta_{x_j}|\big)\, \mathbf{O}\big(\sum_{k \neq 1} |\delta_{x_k}| + \sum_{kl \neq 1} |\delta_{x_k x_l}|\big) \Big], \\ R^m &= \mathbf{O}(e^{-\bar{\eta}|x_1|}) \Big[\mathbf{O}(|\delta|) \mathbf{O}\big(\sum_{j \neq 1} |\delta_{x_j}| + \sum_{jk \neq 1} |\delta_{x_j x_k}| + \sum_{jkl \neq 1} |\delta_{x_j x_k x_l}|\big) \\ &\quad + \mathbf{O}\big(\sum_{j \neq 1} |\delta_{x_j}|\big) \mathbf{O}\big(\sum_{kl \neq 1} |\delta_{x_k x_l}|\big) \Big], \quad m \neq 1, \\ S^m &= \mathbf{O}(e^{-\bar{\eta}|x_1|}) \Big[\mathbf{O}(|\delta|) \mathbf{O}(|v|) + \mathbf{O}(|\delta|) \mathbf{O}\big(\sum_{jkl} |v_{x_j x_k x_l}|\big) \\ &\quad + \mathbf{O}(|v|) \mathbf{O}\big(\sum_{jkl} |\delta| + |\delta_{x_j}| + |\delta_{x_j x_k}| + |\delta_{x_j x_k x_l}|\big) \Big], \quad \text{for each } m = 1, \ldots, d. \end{aligned} \qquad (1.5)$$

We remark that the four critical terms will be

$$\sum_{jkl} \mathbf{O}(|v||v_{x_j x_k x_l}|), \qquad \mathbf{O}(e^{-\bar{\eta}|x_1|})\mathbf{O}(|\delta|)\mathbf{O}(\sum_{j\neq 1} |\delta_{x_j}|)$$
$$\mathbf{O}(e^{-\bar{\eta}|x_1|}|\delta\delta_t|), \qquad \mathbf{O}(e^{-\bar{\eta}|x_1|})\mathbf{O}(|\delta|)\mathbf{O}(\sum_{jkl} |v_{x_j x_k x_l}|),$$

which we will show dominate those remaining (see Section 2 for a derivation of (1.3)). According to (H0)–(H2), we can make the following conclusions regarding the coefficients of the linear part of (1.3):

(C0) (regularity) $a_j(x_1) \in C^1(\mathbb{R})$, $b^{jklm}(x_1) \in C^2(\mathbb{R})$, $b^{1111}(x_1) \geqq b_0 > 0$,
(C0') (asymptotic decay) For some $\alpha > 0$, there holds

$$\left|\frac{\partial^k}{\partial x_1^k}(a_j(x_1) - a_j^{\pm})\right| = \mathbf{O}(e^{-\alpha|x_1|}), \quad k = 0, 1,$$
$$\left|\frac{\partial^k}{\partial x_1^k}(b^{jklm}(x_1) - b_{\pm}^{jklm})\right| = \mathbf{O}(e^{-\alpha|x_1|}), \quad k = 0, 1, 2,$$

(C1) (non-sonicity) either $a_1^+ < 0 < a_1^-$ (Lax case) or $\text{sgn}(a_1^+ a_1^-) = 1$ (undercompressive case),
(C2) $\sum_{jklm} b^{jklm}(x_1)\xi_j\xi_k\xi_l\xi_m \geqq \theta|\xi|^4$ for all $\xi \in \mathbb{R}^d$ and some $\theta > 0$.
Here,

$$a_j^{\pm} := \lim_{x_1\to\pm\infty} a_j(x_1); \quad \text{and} \quad b_{\pm}^{jklm} := \lim_{x_1\to\pm\infty} b^{jklm}(x_1).$$

Upon integration of (1.3) we arrive at the integral equation

$$\begin{aligned} v(t,x) = &\int_{\mathbb{R}^d} G(t,x;y)v_0(y)dy \\ &+ \int_0^t \int_{\mathbb{R}^d} G(t-s,x;y) \\ &\times\Big[(\partial_s - L_y)(\bar{u}_{y_1}\delta) + \sum_{m=1}^d (Q_{y_m}^m + R_{y_m}^m + S_{y_m}^m)\Big]dyds, \end{aligned} \tag{1.6}$$

where $G(t,x;y)$ represents the Green's function for the linear part of (1.3):

$$G_t + \sum_{j=1}^d (a_j(x_1)G)_{x_j} = -\sum_{jklm}(b^{jklm}(x_1)G_{x_j x_k x_l})_{x_m}; \quad G(0,x;y) = \delta_y(x). \tag{1.7}$$

The idea behind the pointwise Green's function approach to stability is to obtain estimates on $G(t,x;y)$ sufficiently sharp so that an iteration on (1.6) can be closed. (See, for example, [9, 10, 12] for complete nonlinear analyses in similar situations.) Such estimates on $G(t,x;y)$ have been obtained in [6], wherein the authors observe that the coefficients of the operator L depend only on the distinguished variable x_1 and proceed by taking a Fourier transform of (1.7) in the transverse direction, transforming $(x_2, x_3, \ldots, x_d)$ into $(\xi_1, \xi_2, \ldots, \xi_{d-1})$. In what follows, we denote the transformed linear operator L_ξ.

Our stability theorems will assume spectral conditions on the Evans function, $D(\lambda, \xi)$, associated with the eigenvalue problem, $L_\xi \phi = \lambda \phi$. Briefly, the Evans function serves as a characteristic function for the operator L_ξ. More precisely, away from the essential spectrum, zeros of the Evans function correspond in location and multiplicity with eigenvalues of L_ξ, an observation that has been made precise in [1] in the case—pertaining to reaction–diffusion equations—of isolated eigenvalues, and in [5, 13] in the case—pertaining to conservation laws—of nonstandard "effective" eigenvalues embedded in the essential spectrum. For a development and analysis of the Evans function in the current setting, the reader is referred to [6].

As in [6], we will analyze the Evans function with respect to a radial coordinate ρ, defined through

$$(\lambda, \xi) = (\rho\lambda_0, \rho\xi_0), \quad \text{where } |(\lambda_0, \xi_0)| = 1. \tag{1.8}$$

Clearly, $\rho = |(\lambda, \xi)| = \sqrt{|\lambda|^2 + |\xi|^2}$. In particular, we will analyze

$$D_{\lambda_0, \xi_0}(\rho) := D(\rho\lambda_0, \rho\xi_0), \tag{1.9}$$

and the *reduced Evans functions*

$$\bar{\Delta}(\lambda_0, \xi_0) := \lim_{\rho \to 0} \rho^{-1} D_{\lambda_0, \xi_0}(\rho).$$

In terms of these definitions, our stability conditions take the form $(\mathcal{D}_s)$ (the subscript s follows the notation of [6], for which the *sufficient* condition for stability $(\mathcal{D}_s)$ was distinguished from the *necessary* condition $(\mathcal{D}_n)$).

$(\mathcal{D}_s)$ *Sufficient conditions for linear (and therefore nonlinear) stability.*

Condition (1).

$$\begin{aligned} D(\lambda, \xi) &\neq 0, \quad \{(\lambda, \xi) : \xi \in \mathbb{R}^{d-1}, \mathrm{Re}\lambda \geqq 0, (\lambda, \xi) \neq (0, 0)\}, \\ \bar{\Delta}(\lambda_0, \xi_0) &\neq 0, \quad \{(\lambda_0, \xi_0) : \xi_0 \in \mathbb{R}^{d-1}, \mathrm{Re}\lambda_0 > 0\}, \end{aligned}$$

Condition (2). There is a neighborhood V of zero in (complex) ξ-space so that L_ξ has a unique L^2 eigenvalue, $\lambda_*(\xi)$, defined through $D(\lambda_*(\xi), \xi) = 0$, $\lambda_*(0) = 0$, satisfying

$$\lambda_*(\xi) = -i\frac{[\tilde{f}]}{[u]} \cdot \xi - \lambda_2^{jk} \xi_j \xi_k + i\lambda_3^{jkl} \xi_j \xi_k \xi_l - \lambda_4^{jklm} \xi_j \xi_k \xi_l \xi_m + \mathbf{O}(|\xi|^5),$$

where summation is assumed over repeated indices, and we use the notation $[u] = u_+ - u_-$, and

$$[\tilde{f}] = (f^2(u_+) - f^2(u_-), f^3(u_+) - f^3(u_-), \ldots, f^d(u_+) - f^d(u_-)).$$

We assume that one of the following holds (Condition (2a) or Condition (2b)):

Condition (2a).

$$\lambda_2^{jk}\xi_j\xi_k \geqq \lambda_2^0|\xi|^2, \quad \xi \in \mathbb{R}^{d-1}, \lambda_2^0 > 0,$$

Condition (2b).

$$\lambda_2^{jk} = \lambda_3^{jkl} = 0, \quad \text{all } j, k, l$$

and

$$\lambda_4^{jklm}\xi_j\xi_k\xi_l\xi_m \geqq \lambda_4^0|\xi|^4, \quad \xi \in \mathbb{R}^{d-1}, \lambda_4^0 > 0.$$

Condition (3). For $\rho \geqq \rho_0 > 0$, there exist constants $c_1 > 0$ and $C_2 > 0$ so that the spectrum of L_ξ lies entirely to the left of a contour defined through the relation

$$\text{Re}\,\lambda = -c_1(|\text{Re}\,\xi|^4 - C_2|\text{Im}\,\xi|^4 + |\text{Im}\,\lambda|).$$

We will refer to the contour defined by this relation as Γ_{bound}.

Remark on thin film equations. Undercompressive viscous shock waves arising in the thin films equation (1.2) (multidimensional case) have been shown, through numerical calculations to satisfy Condition (2a) (see [3], especially the discussion around Fig. 6). In the case of compressive waves arising in (1.2) (and generalizations; see (3.1) of [3]), the authors of [3] established the exact representation

$$\lambda_2^0 = \int_{-\infty}^{+\infty} \frac{f(u_-) - f(\bar{u})}{u_+ - u_-} dx.$$

Remark on Conditions (2a), (2b) & (3). In the case of incoming characteristics, signals propagate into the shock layer and then convect and diffuse along the shock layer with rates depending on Conditions (2a) and (2b). In the case of Condition (2a), the signal propagates along the shock layer similarly to the solution to a second-order convection–diffusion equation. In the case of Condition (2b), the signal propagates along the shock layer similarly to a fourth-order convection–regularity equation. Condition (3) ensures that the small t behavior in the shock layer is fourth order.

Remark on Condition (1). The condition $\bar{\Delta}(\lambda_0, \xi_0) \neq 0$ is a *transversality* condition. In the case of compressive waves,

$$\bar{\Delta}(\lambda_0, \xi_0) = \gamma(i\xi_0 \cdot [\tilde{f}] + \lambda_0[u]),$$

from which $\bar{\Delta}(\lambda_0, \xi_0) \neq 0 \Rightarrow \gamma \neq 0$, and additionally $\bar{\Delta}(\lambda_0, \xi_0)$ is only 0 for λ_0 purely complex.

Before stating our main theorem, we must specify our selection mechanism for the local tracking function $\delta(t, \tilde{x})$. In either the compressive, or the undercompressive case, the distinguished eigenvalues $\lambda_*(\xi)$ reduce the time asymptotic decay rate of the Green's function $G(t, x; y)$. (In the case $d = 1$, the eigenvalue at $\lambda = 0$

determines that the Green's function does *not* decay in time.) In our selection of $\delta(t, \tilde{x})$, we separate our Green's function as,

$$\begin{aligned} G(t, x; y) &= \bar{u}_{x_1}(x_1)\tilde{e}(t, \tilde{x}; y) + \tilde{G}(t, x; y), \\ \partial_{y_k} G(t, x; y) &= \bar{u}_{x_1}(x_1)\tilde{e}_k(t, \tilde{x}; y) + \tilde{G}_k(t, x; y), \end{aligned} \tag{1.10}$$

where $\bar{u}_{x_1}(x_1)\tilde{e}(t, \tilde{x}; y)$ decays at the reduced rate determined by $\lambda_*(\xi)$, with $\tilde{G}(t, x; y)$ the more rapidly decaying remainder, and the expressions $\tilde{e}_k(t, \tilde{x}; y)$ and $\tilde{G}_k(t, \tilde{x}; y)$ need not correspond precisely with y_k-derivatives of $\tilde{e}$ and $\tilde{G}$. In terms of $\tilde{e}(t, \tilde{x}; y)$, we define our shift $\delta(t, \tilde{x})$ through the integral relation:

$$\delta(t, \tilde{x}) = -\int_{\mathbb{R}^d} \tilde{e}(t, \tilde{x}; y)v_0(y)dy + \int_0^t \int_{\mathbb{R}^d} \sum_{m=1}^{d} \tilde{e}_m(t-s, \tilde{x}; y)N^m(y, s)dyds,$$

for which, we will show in Section 2, there holds

$$v(t, x) = \int_{\mathbb{R}^d} \tilde{G}(t, x; y)v_0(y)dy - \int_0^t \int_{\mathbb{R}^d} \sum_{m=1}^{d} \tilde{G}_m(t-s, x; y)N^m(s, y)dyds,$$

where, for notational brevity, we take

$$N^m = Q^m + R^m + S^m.$$

We will establish our main results in terms of the *transverse norm*

$$\|v(t, x)\|_{L^p_{\tilde{x}}(t, x_1)} := \left(\int_{\mathbb{R}^{d-1}} |v(t, x)|^p d\tilde{x}\right)^{\frac{1}{p}}, \tag{1.11}$$

and the decay functions

$$\begin{aligned} \theta(t, x_1) &:= (1+t)^{-\frac{d}{4}+\frac{d-1}{4p}} \exp\left(-\frac{(x_1 - a_1^+ t)^{\frac{4}{3}}}{Lt^{\frac{1}{3}}}\right) \\ \psi_1(t, x_1) &:= (1+t)^{-\frac{d-1}{4}\left(1-\frac{1}{p}\right)}[(1+t)^{-\frac{1}{4}} \wedge (1+|x_1 - a_1^+ t|)^{-r}] \\ \psi_2(t, x_1) &:= (1+|x_1|+t)^{-\frac{d}{4}+\frac{d-1}{4p}} (1+|x_1 - a_1^+ t|)^{-\frac{d}{4}} I_{\{|x_1| \leqq |a_1^+|t\}} \\ \psi_3(t, x_1) &:= (1+|x_1|+t)^{-\frac{d-1}{4}\left(2-\frac{1}{p}\right)} (1+|x_1|)^{-\frac{1}{2}} I_{\{|x_1| \leqq |a_1^+|t\}} \\ \psi_4(t, x_1) &:= (1+t)^{-\frac{d-1}{4}\left(1-\frac{1}{p}\right)} (1+|x_1|+t)^{-r} \\ \psi_5(t, x_1) &:= (1+t)^{-\frac{d}{4}+\frac{d-1}{4p}} \exp(-\eta|x_1|). \end{aligned}$$

We remark that the terms with a_1^+ will only appear in the undercompressive case, for which $a_1^+ > 0$ is taken as the outgoing characteristic.

We are now in a position to state the main results of our analysis.

Theorem 1.1. *Let* $\bar{u}(x_1)$ *be a planar wave solution for* (1.1)*, and suppose assumptions* (H0)–(H2) *hold, as well as spectral criteria* ($\mathcal{D}_s$)*, with Condition* (2a)*. Then, for Hölder continuous initial perturbations* $v_0(x)$ *satisfying* $\|v_0\|_{L^\infty} \leqq \zeta_0$ *and*

$$\int_{\mathbb{R}^{d-1}} |v_0(x)|d\tilde{x} \leqq \zeta_0(1+|x_1|)^{-r}, \quad r > \frac{d}{4} + \frac{1}{2} \tag{1.12}$$

for ζ_0 *sufficiently small, we conclude:*

(I) Compressive case. *For $\partial_u f^1(u_-) > 0 > \partial_u f^1(u_+)$, and for $d \geqq 2$ and $|\alpha| \leqq 1$,*

$$\|u(t,x) - \bar{u}(x_1 - \delta(t,\tilde{x}))\|_{L^p_{\tilde{x}}} \leqq C\zeta_0\Big[\psi_4(t,x_1) + (1+t)^{-\frac{d}{4}+\frac{d-1}{4p}}\psi_5(t,x_1)\Big],$$

$$\|\partial_{\tilde{x}}^{\alpha}\delta(t,\tilde{x})\|_{L^p_{\tilde{x}}} \leqq C\zeta_0(1+t)^{-\frac{d-1}{2}\left(1-\frac{1}{p}\right)-\frac{|\alpha|}{2}};$$

(II) Undercompressive case. *For $\partial_u f^1(u_-), \partial_u f^1(u_+) \gtreqless 0$, and for $d \geqq 3$ and $|\alpha| \leqq 1$,*

$$\begin{aligned}
&\|u(t,x) - \bar{u}(x_1 - \delta(t,\tilde{x}))\|_{L^p_{\tilde{x}}} \\
&\quad \leqq C\zeta_0\Big[\theta(t,x_1) + \psi_1(t,x_1) + (1+|x_1 - a_1^+ t|)^{\frac{d}{4}}\psi_2(t,x_1) \\
&\qquad + (1+|x_1|+1)^{-\frac{d-1}{4}(2-\frac{1}{p})-\frac{1}{4}}\psi_3(t,x_1) + (1+t)^{-\frac{d}{4}+\frac{d-1}{4p}}\psi_5(t,x_1)\Big],
\end{aligned}$$

$$\|\partial_{\tilde{x}}^{\alpha}\delta(t,\tilde{x})\|_{L^p_{\tilde{x}}} \leqq C\zeta_0(1+t)^{-\frac{d-1}{2}\left(1-\frac{1}{p}\right)-\frac{|\alpha|}{2}}.$$

Remark on condition $\mathbf{r > \frac{d}{4} + \frac{1}{2}}$. While our general condition $r > 1$ is critical to the argument, the condition $r > \frac{d}{4} + \frac{1}{2}$ has been taken for technical reasons (that will be pointed out in the proofs) to simplify the arguments. We note that in the case $d = 2$, $r > 1$ still suffices.

Remark on the dimensions of validity. We observe that the restriction of our result in the undercompressive case to dimensions $d \geqq 3$ is because of the outgoing characteristic, not behavior, in the shock layer. In particular, the application of our method to linearization about a constant state, which also necessarily has an outgoing field, is also restricted to the cases $d \geqq 3$. More generally, we point out that fourth-order regularization is less stabilizing than is second-order regularization, at least in so much as the rate of approach to the wave is reduced. Such reduction in rate serves, especially, to complicate the step from linear stability to nonlinear stability. It is for this step, that the higher dimensions are required.

According to Theorem 1.1, we obtain the following corollary on nonlinear L^p stability.

Corollary 1.1. (L^p stability.) *Under the assumptions of Theorem 1.1, the planar wave $\bar{u}(x_1)$ is stable in L^p, for $p \geqq 1$ in the compressive case and $p > 1$ in the undercompressive case, with decay rates as follows:*

(I) Compressive waves. *For $\partial_u f^1(u_-) > 0 > \partial_u f^1(u_+)$, and for $d \geqq 2$,*

$$\begin{aligned}
&\|u(t,x) - \bar{u}(x_1 - \delta(t,\tilde{x}))\|_{L^p(\mathbb{R}^d)} \\
&\quad \leqq C\zeta_0\Big[(1+t)^{-\frac{d-1}{4}\left(1-\frac{1}{p}\right)-r+\frac{1}{p}} + (1+t)^{-\frac{d}{2}+\frac{d-1}{2p}}\Big],
\end{aligned}$$

(II) Undercompressive waves. *For $\partial_u f^1(u_-), \partial_u f^1(u_+) \gtrless 0$, and for $d \geqq 3$,*

$$\|u(t,x) - \bar{u}(x_1 - \delta(t,\tilde{x}))\|_{L^p(\mathbb{R}^d)} \leqq C\zeta_0(1+t)^{-\frac{d}{4}\left(1-\frac{1}{p}\right)}.$$

We have a similar, though weaker, result under the higher-order Condition (2b).

Theorem 1.2. *Let $\bar{u}(x_1)$ be a planar wave solution for* (1.1)*, and suppose assumptions* (H0)–(H2) *hold, as well as spectral criteria* ($\mathcal{D}_s$)*, with Condition* (2b)*, and that the regularity coefficients b^{jklm} in* (1.1) *are all constant. Then, for initial perturbations $v_0(x)$ satisfying $\|v_0\|_{L^\infty} \leqq \zeta_0$ and*

$$\int_{\mathbb{R}^{d-1}} |v_0(x)| d\tilde{x} \leqq \zeta_0(1+|x_1|)^{-r}, \quad r > 1, \tag{1.13}$$

for ζ_0 sufficiently small, we conclude:

(I) Compressive waves. *For $\partial_u f^1(u_-) > 0 > \partial_u f^1(u_+)$, and for $d \geqq 3$ and $|\alpha| \leqq 1$,*

$$\|u(t,x) - \bar{u}(x_1 - \delta(t,\tilde{x}))\|_{L^p_{\tilde{x}}} \leqq C\zeta_0\Big[\psi_4(t,x_1) + \psi_5(t,x_1)\Big],$$
$$\|\partial_{\tilde{x}}^\alpha \delta(t,\tilde{x})\|_{L^p_{\tilde{x}}} \leqq C\zeta_0(1+t)^{-\frac{d-1}{4}\left(1-\frac{1}{p}\right)-\frac{|\alpha|}{4}};$$

(II) Undercompressive waves. *For $\partial_u f^1(u_-), \partial_u f^1(u_+) \gtrless 0$, and for $d \geqq 5$,*

$$\|u(t,x) - \bar{u}(x_1 - \delta(t,\tilde{x}))\|_{L^p_{\tilde{x}}} \leqq C\zeta_0\Big[\theta_\pm(t,x_1) + \psi_1(t,x_1) + \psi_2(t,x_1)$$
$$+\psi_3(t,x_1) + \psi_5(t,x_1)\Big],$$
$$\|\delta(t,\tilde{x})\|_{L^p_{\tilde{x}}} \leqq C\zeta_0(1+t)^{-\frac{d-1}{4}\left(1-\frac{1}{p}\right)-\frac{|\alpha|}{4}}.$$

Moreover, under the additional restriction of Hölder continuity on the initial perturbation, $v_0 \in C^{0+\gamma}$, $\gamma > 0$, we can conclude exactly the same estimates for non-constant b^{jklm} satisfying only (H0)–(H2).

According to Theorem 1.2, we obtain the following corollary on nonlinear L^p stability.

Corollary 1.2. (L^p stability.) *Under the assumptions of Theorem* 1.2*, the planar wave $\bar{u}(x_1)$ is stable in L^p, for $p \geqq 1$ in the compressive case and $p > 1$ in the undercompressive case, with decay rates as follows:*

(I) Compressive waves. *For $\partial_u f^1(u_-) > 0 > \partial_u f^1(u_+)$, and for $d \geqq 3$,*

$$\|u(t,x) - \bar{u}(x_1 - \delta(t,\tilde{x}))\|_{L^p(\mathbb{R}^d)}$$
$$\leqq C\zeta_0\Big[(1+t)^{-\frac{d-1}{4}\left(1-\frac{1}{p}\right)-r+\frac{1}{p}} + (1+t)^{-\frac{d}{4}+\frac{d-1}{4p}}\Big];$$

(II) Undercompressive waves. *For* $\partial_u f^1(u_-), \partial_u f^1(u_+) \gtrless 0$*, and for* $d \geqq 5$*,*

$$\|u(t,x) - \bar{u}(x_1 - \delta(t,\tilde{x}))\|_{L^p(\mathbb{R}^d)} \leqq C\zeta_0(1+t)^{-\frac{d}{4}\left(1-\frac{1}{p}\right)}.$$

Remarks on Corollary 1.2. We note that with $r > 1$, our condition on the initial perturbation $v_0(x)$ (1.13) is only slightly stronger than boundedness by ζ_0 in L^1. The assumption in [9] (Theorem 1.2) is $\|x_1 v_0\|_{L^1} \leqq \zeta_0$, which would correspond in the pointwise framework with $r > 2$. In the case $r > 2$ (in fact for $r \geqq 5/4$) our estimate in the compressive case is

$$\|u(t,x) - \bar{u}(x_1 - \delta(t,\tilde{x}))\|_{L^p(\mathbb{R}^d)} \leqq C(1+t)^{-\frac{d-1}{4}\left(1-\frac{1}{p}\right)-\frac{1}{4}},$$

which corresponds with Theorem 1.2 of [9] (for which 4 is replaced everywhere by 2).

2. The integral equations

In this section, we develop the framework for our nonlinear iteration on $v(t,x)$ and $\delta(t,\tilde{x})$. Given a standing wave solution $\bar{u}(x_1)$ to (1.1), we define our perturbation variable through the relation

$$v(t,x) = u(t,x) - \bar{u}(x_1 - \delta(t,\tilde{x})).$$

Upon substitution of $u(t,x)$ into (1.1), we find

$$\begin{aligned}\partial_t(\bar{u}(x_1-\delta)+v) &+ \sum_{j=1}^{d} f^j(\bar{u}(x_1-\delta)+v)_{x_j} \\ &= -\sum_{jklm}[b^{jklm}(\bar{u}(x_1-\delta)+v)\partial_{x_jx_kx_l}(\bar{u}(x_1-\delta)+v)]_{x_m}.\end{aligned}$$

Taylor expanding for small v, we find

$$\begin{aligned}&\partial_t\bar{u}(x_1-\delta) + v_t + \sum_{j=1}^{d}\Big(f^j(\bar{u}(x_1-\delta)) + \partial_u f^j(\bar{u}(x_1-\delta))v + \mathbf{O}(v^2)\Big)_{x_j} \\ &= -\sum_{jklm}\Big[\Big(b^{jklm}(\bar{u}(x_1-\delta)) + \partial_u b^{jklm}(\bar{u}(x_1-\delta))v + \mathbf{O}(v^2)\Big) \\ &\quad\times\Big(\partial_{x_jx_kx_l}\bar{u}(x_1-\delta) + \partial_{x_jx_kx_l}v\Big)\Big]_{x_m}.\end{aligned}$$

Taylor expanding now $\bar{u}(x_1+\delta)$ for small δ, and then Taylor expanding around $\bar{u}(x_1)$, we arrive at the three relations:

1.
$$\sum_{j=1}^{d}(\partial_u f^j(\bar{u}(x_1-\delta))v)_{x_j} = \sum_{j=1}^{d}(\partial_u f^j(\bar{u}(x_1))v)_{x_j} + \sum_{j=1}^{d}\Big(\mathbf{O}(e^{-\bar{\eta}|x_1|}|\delta||v|)\Big)_{x_j},$$

2.
$$\sum_{jklm}(b^{jklm}(\bar{u}(x_1-\delta))\partial_{x_jx_kx_l}v)_{x_m} = \sum_{jklm}(b^{jklm}(\bar{u}(x_1))\partial_{x_jx_kx_l}v)_{x_m} + \sum_{jklm}\Big(\mathbf{O}(e^{-\bar{\eta}|x_1|}|\delta||v_{x_jx_kx_l}|)\Big)_{x_m},$$

3.
$$\begin{aligned}\sum_{jklm}&\Big(\partial_u b^{jklm}(\bar{u}(x_1-\delta))v\partial_{x_jx_kx_l}\bar{u}(x_1-\delta)\Big)_{x_m}\\ &= \sum_m(\partial_u b^{111m}(\bar{u}(x_1))\bar{u}_{x_1x_1x_1}v)_{x_m}\\ &\quad+\sum_m\Big(\mathbf{O}(e^{-\bar{\eta}|x_1|})\mathbf{O}(|v|)\mathbf{O}(\sum_{jkl}|\delta|+|\delta_{x_j}|+|\delta_{x_jx_k}|+|\delta_{x_jx_kx_l}|)\Big)_{x_m}.\end{aligned}$$

For the remaining terms, we have the following proposition.

Proposition 2.1. *Suppose $\bar{u}(x_1)$ is a standing wave solution to* (1.1)*, and let* (H0) *hold. Then,*

$$\begin{aligned}\partial_t\bar{u}(x_1-\delta(t,\tilde{x}))&+\sum_{j=1}^{d}\partial_{x_j}f^j(\bar{u}(x_1-\delta(t,\tilde{x})))\\ &+\sum_{jklm}\partial_{x_m}\Big(b^{jklm}(\bar{u}(x_1-\delta(t,\tilde{x})))\partial^3_{x_jx_kx_l}\bar{u}(x_1-\delta(t,\tilde{x}))\Big)\\ &=-(\partial_t-L)(\bar{u}_{x_1}\delta)-\sum_{m=1}^{d}R^m_{x_m},\end{aligned}$$

where R is as in (1.5).

Proof. Denoting $\bar{u}'=\partial_{x_1}\bar{u}$ and Taylor expanding, we have the three relations

$$\begin{aligned}\bar{u}(x_1-\delta)_t &= -\bar{u}'(x_1)\delta_t+\Big(\mathbf{O}(e^{-\bar{\eta}|x_1|}|\delta\delta_t|)\Big)_{x_1},\\ \partial_{x_1}f^1(\bar{u}(x_1-\delta)) &= \partial_u f^1(\bar{u}(x_1-\delta(t,\tilde{x})))\bar{u}'(x_1-\delta(t,\tilde{x})),\\ \sum_{j\neq1}f^j(\bar{u}(x_1-\delta))_{x_j} &= -\sum_{j\neq1}\partial_u f^j(\bar{u}(x_1))(\bar{u}'(x_1)\delta)_{x_j}\\ &\quad+\sum_{j\neq1}\Big(\mathbf{O}(e^{-\bar{\eta}|x_1|}|\delta\delta_{x_j}|)\Big)_{x_1},\end{aligned}$$

and finally,

$$\begin{aligned}
&\sum_{jklm} \Big(b^{jklm}(\bar{u}(x_1-\delta))\partial_{x_j x_k x_l}\bar{u}(x_1-\delta)\Big)_{x_m} \\
&\quad = \Big(b^{1111}(\bar{u}(x_1-\delta))\bar{u}'''(x_1-\delta)\Big)_{x_m} \\
&\qquad - \sum_{jklm} \Big(b^{jklm}(\bar{u}(x_1))\partial_{x_j x_k x_l}(\bar{u}'\delta)\Big)_{x_m} \\
&\qquad - \sum_{m\neq 1} \partial_u b^{111m}(\bar{u}(x_1))\bar{u}'''(\bar{u}_{x_1}\delta)_{x_m} \\
&\qquad + b^{1111}(\bar{u}(x_1))\bar{u}^{(5)}\delta \\
&\qquad + \left(\mathbf{O}\left(\sum_{jk\neq 1} e^{-\bar{\eta}|x_1|}(|\delta\delta_{x_j}| + |\delta_{x_j}\delta_{x_k}|)\right)\right)_{x_1} \\
&\qquad + \sum_m \left(\mathbf{O}(e^{-\bar{\eta}|x_1|}|\delta|)\mathbf{O}\left(\sum_{jkl\neq 1} |\delta_{x_j x_k}| + |\delta_{x_j x_k x_l}|\right)\right)_{x_m} \\
&\qquad + \sum_m \left(\mathbf{O}(e^{-\bar{\eta}|x_1|}|\delta_{x_j}|)\mathbf{O}\left(\sum_{jkl\neq 1} |\delta_{x_k x_l}|\right)\right)_{x_m}.
\end{aligned}$$

Proposition 2.1 follows from combining these relations. □

The linearization (1.3) follows through direct substitution from Proposition 2.1 and the relations directly preceding it. Following [9], we next establish a convenient representation for the integral

$$\int_0^t \int_{\mathbb{R}^d} G(t-s,x;y)(\partial_s - L_y)(\bar{u}_{y_1}(y_1)\delta(t,\tilde{y}))dyds.$$

Proposition 2.2. *For any $f(t,x)$ such that for each t, $f(t,\cdot)\in L^p(\mathbb{R}^d)\cap C^4(\mathbb{R}^d)$, with also $f(\cdot,x)\in C^1(\mathbb{R})$ we have:*

$$\int_0^t \int_{\mathbb{R}^d} G(t-s,x;y)(\partial_s - L_y)f(s,y)dyds = f(t,x) - \int_{\mathbb{R}^d} G(t,x;y)f(0,y)dy.$$

Proof. We first observe that for $w(t,x)$ defined through

$$w(t,x) := \int_0^t \int_{\mathbb{R}^d} G(t-s,x;y)(\partial_s - L_y)f(s,y)dyds,$$

we must have

$$\begin{aligned}
w_t - L_x w &= f_t - L_x f, \\
w(0,x) &= 0.
\end{aligned}$$

We then set, $v = w - f$, so that

$$v_t = Lv,$$
$$v(0, x) = -f(0, x),$$

with solution

$$v(t, x) = -\int_{\mathbb{R}^d} G(t, x; y) f(0, y) dy.$$

Substituting back for w completes the proof. □

According to Proposition 2.2, our integral equation (1.6) becomes

$$\begin{aligned} v(t, x) = &\int_{\mathbb{R}^d} G(t, x; y)[v_0(y) - \delta(0, \tilde{y})\bar{u}_{y_1}(y_1)]dy + \bar{u}_{x_1}(x_1)\delta(t, \tilde{x}) \\ &- \int_0^t \int_{\mathbb{R}^d} \sum_m G_{y_m}(t - s, x; y) N^m(s, y) dy ds, \end{aligned}$$

where the N^m is as in (1.11), and we have integrated by parts once. We now choose $\delta(t, \tilde{x})$ to annihilate the part of $G(t, x; y)$ that corresponds with mass accumulating in the shock layer (measured by the excited terms) designated in (1.10) as $\tilde{e}(t, \tilde{x}; y)$ and (for the y_k derivative) $\tilde{e}_k(t, \tilde{x}; y)$. We take

$$\begin{aligned} \delta(t, \tilde{x}) = &-\int_{\mathbb{R}^d} \tilde{e}(t, \tilde{x}; y)[v_0(y) - \delta(0, \tilde{y})\bar{u}_{y_1}(y_1)]dy \\ &+ \int_0^t \int_{\mathbb{R}^d} \sum_m \tilde{e}_m(t - s, x; y) N^m(s, y) dy ds. \end{aligned}$$

According to the Green's function estimates of [6], $\tilde{e}(t, \tilde{x}; y)$, for $y_1 \gtrless 0$, takes the form

$$\tilde{e}(t, \tilde{x}; y) = E(t, \tilde{x}; \tilde{y}) I_{\{|y_1| \leqq |a_1^{\pm}|t\}},$$

where E varies by case. In the compressive case, and under Condition (2b) of $(\mathcal{D}_s)$, we have,

$$\tilde{e}(t, \tilde{x}; y) = \mathbf{O}(t^{-\frac{d-1}{4}}) \exp\left(-\frac{|\tilde{x} - \tilde{y} - \tilde{a}_{\text{eff}}^{\pm} t|^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right) I_{\{|y_1| \leqq |a_1^{\pm}|t\}},$$

where the *effective* transmission rate $\tilde{a}_{\text{eff}}^{\pm}$ is described in Section 3. We see that our linear integral in the definition of $\delta(t, \tilde{x})$ satisfies

$$\begin{aligned} \Big| &\int_{-|a_1^-|t}^{+|a_1^+|t} \int_{\mathbb{R}^{d-1}} \mathbf{O}(t^{-\frac{d-1}{4}}) \exp\left(-\frac{|\tilde{x} - \tilde{y} - \tilde{a}_{\text{eff}}^{\pm} t|^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right) \\ &\times \left[v_0(y) - \delta(0, \tilde{y})\bar{u}_{y_1}(y_1)\right] d\tilde{y} dy \Big| \\ &\leqq C \int_{-|a_1^-|t}^{+|a_1^+|t} \sup_{\tilde{y}} |v_0(y) - \delta(0, \tilde{y})\bar{u}_{y_1}(y_1)|(y_1) dy_1, \end{aligned}$$

which tends to 0 as $t \to 0$. Similarly, we can conclude in each case that $\delta(0,\tilde{x}) = 0$, from which we have our integral equation for δ,

$$D^{\alpha}\delta(t,\tilde{x}) = -\int_{\mathbb{R}^d} D^{\alpha}\tilde{e}(t,\tilde{x};y)v_0(y)dy + \int_0^t \int_{\mathbb{R}^d} \sum_m D^{\alpha}\tilde{e}_m(t-s,\tilde{x};y)N^m(s,y)dyds, \qquad (2.1)$$

where $|\alpha| \leqq 3$ is a standard multi-index. We contrast this approach with the analysis of [9], in which the shift is initialized with a non-zero value determined by the initial perturbation. With this (implicit) choice of $\delta(t,\tilde{x})$, we arrive at the integral equation for $v(t,x)$,

$$v(t,x) = \int_{\mathbb{R}^d} \tilde{G}(t,x;y)v_0(y)dy - \int_0^t \int_{\mathbb{R}^d} \sum_m \tilde{G}_m(t-s,x;y)N^m(s,y)dyds, \qquad (2.2)$$

with $\tilde{G}(t,x;y)$ as in (1.10).

3. The compressive case

In the compressive case ($a_1^- > 0 > a_1^+$), we have the following result from [6].

Lemma 3.1. *Under assumptions (H0)–(H2) and ($\mathcal{D}_s$), and for the compressive case, the following estimates on solutions $G(t,x;y)$ to the Green's function equation* (1.7) *are obtained. For some constants M and η, and for d-dimensional multi-index α, with $|\alpha| \leqq 3$, $\alpha_1 \leqq 1$,*

(i) $y_1, x_1 \leqq 0$

$$\begin{aligned}\partial_y^{\alpha}G(t,x;y) = {}& \mathbf{O}(t^{-\frac{d+|\alpha|}{4}})\exp\left(-\frac{|x-y-a_{-}t|^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right) + \bar{u}_{x_1}(x_1)\partial_y^{\alpha}e(t,\tilde{x},y)\\ &+\mathbf{O}(\exp(-\eta|x_1|))R(t,\tilde{x},y;d+|\alpha|)\\ &+\mathbf{O}(\exp(-\eta|x_1|))\mathbf{O}(\exp(-\eta(|\tilde{x}-\tilde{y}|+t)))I_{\{|y_1|\leqq|a_1^-|t\}};\end{aligned}$$

(ii) $x_1 \leqq 0 \leqq y_1$

$$\begin{aligned}\partial_y^{\alpha}G(t,x;y) = {}& \mathbf{O}\left(t^{-\frac{d+|\alpha|}{4}}\right)\mathbf{O}(\exp(-\eta|x_1|))\exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}^{+}t|^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right)\\ &\times\exp\left(-\frac{(y_1+a_1^{+}t)^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right)\\ &+\mathbf{O}(\exp(-\eta|x_1|))R(t,\tilde{x},y;d+|\alpha|)\\ &+\mathbf{O}(\exp(-\eta|x_1|))\mathbf{O}(\exp(-\eta(|\tilde{x}-\tilde{y}|+t)))I_{\{|y_1|\leqq|a_1^{+}|t\}}\\ &+\bar{u}_{x_1}(x_1)\partial_y^{\alpha}e(t,\tilde{x},y),\end{aligned}$$

where, for $y_1 \gtrless 0$*,*

$$\partial_y^\alpha e(t, \tilde{x}, y) = \mathbf{O}\left(t^{-\frac{d-1+|\alpha|}{4}}\right) \exp\left(-\frac{|\tilde{x} - \tilde{y} - \tilde{a}^\pm t|^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right) \times \exp\left(-\frac{(y_1 + a_1^\pm t)^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right) + R(t, \tilde{x}, y; d - 1 + |\alpha|),$$

with

$$\tilde{a}^\pm_{\text{eff}}(t, y_1) := \left(1 + \frac{y_1}{a_1^\pm t}\right)\tilde{a}_{\text{ave}} - \frac{y_1}{a_1^\pm t}\tilde{a}^\pm,$$

and for the case of Condition (2a)*, we have* (*again for* $y_1 \gtrless 0$)

$$R(t, \tilde{x}, y; \kappa) = \mathbf{O}(t^{-\frac{\kappa}{4}} \wedge |y_1 \mp a_1^\pm t|^{-\frac{\kappa}{2}}) \times \left[\exp\left(-\frac{|\tilde{x} - \tilde{y} - \tilde{a}^\pm_{\text{eff}} t|^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right) + \exp\left(-\frac{|\tilde{x} - \tilde{y} - \tilde{a}^\pm_{\text{eff}} t|^2}{M|y_1 \mp a_1^\pm t|}\right)\right] I_{\{|y_1| \leqq |a_1^\pm| t\}}, \tag{3.1}$$

while in the case of Condition (2b)*, we have*

$$R(t, \tilde{x}, y; \kappa) = \mathbf{O}(t^{-\frac{\kappa}{4}}) \exp\left(-\frac{|\tilde{x} - \tilde{y} - \tilde{a}^\pm_{\text{eff}} t|^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right) I_{\{|y_1| \leqq |a_1^\pm| t\}}.$$

In the Lax case, estimates for $x_1 \geqq 0$ *are symmetric. Moreover, owing to the non-sonicity condition* (H1)*, the first time derivative of* $e(t, \tilde{x}; y)$ *yields the same estimate as the first space derivatives.*

Remark on Lemma 3.1. For a general discussion of Lemma 3.1 and its proof, see [6]. We remark here that $\tilde{e}(t, \tilde{x}; y)$ corresponds to the second summand in the estimate on $e(t, x; y)$, $R(t, \tilde{x}, y; d - 1)$. In the case $x_1, y_1 \leqq 0$, we refer to the first term in $\tilde{G}$ as the *scattering* term, the third as the *remainder* term, and the contribution from $e(t, x; y)$ as the *excited* term. The final estimate in each term $\mathbf{O}(e^{-\eta|x_1|})\mathbf{O}(e^{-\eta(|\tilde{x}-\tilde{y}|+t)})I_{\{|y_1| \leqq |a_1^+| t\}}$ yields exponential decay in time and has no effect on our iteration.

Using the estimates of Lemma 3.1, we can conclude the following norm estimates on the linear integrals in our iteration.

Lemma 3.2 (Linear integral estimates)**.** *For* $G(t, x; y)$ *as in Lemma 3.1, we have the following estimates for* $|\alpha| \leqq 3$*:*

(i) *Under Condition* (2a), *and for* $v_0(y)$ *satisfying* (1.12),

$$\|\int_{\mathbb{R}^d} \tilde{G}(t,x;y)v_0(y)dy\|_{L^p_{\tilde{x}}} \leqq C\zeta_0(\psi_4(t,x_1) + (1+t)^{-\frac{d}{4}+\frac{d-1}{4p}}\psi_5(t,x_1)),$$

$$\|\int_{\mathbb{R}^d} \partial^\alpha_{\tilde{x}}\tilde{e}(t,\tilde{x};y)v_0(y)dy\|_{L^p_{\tilde{x}}} \leqq C\zeta_0(1+t)^{-\frac{d-1}{2}\left(1-\frac{1}{p}\right)} \begin{cases}(1+t)^{-\frac{|\alpha|}{2}}, & |\alpha| = 0, 1,\\ (1+t)^{-\frac{1}{2}}, & |\alpha| \geqq 2.\end{cases}$$

$$\|\partial_t \int_{\mathbb{R}^d} \tilde{e}(t,\tilde{x};y)v_0(y)dy\|_{L^p_{\tilde{x}}} \leqq C\zeta_0(1+t)^{-\frac{d-1}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}}$$

(ii) *Under Condition* (2b), *and for* $v_0(y)$ *satisfying* (1.13),

$$\|\int_{\mathbb{R}^d} \tilde{G}(t,x;y)v_0(y)dy\|_{L^p_{\tilde{x}}} \leqq C\zeta_0(\psi_4(t,x_1) + \psi_5(t,x_1)),$$

$$\|\int_{\mathbb{R}^d} \partial^\alpha_{\tilde{x}}\tilde{e}(t,\tilde{x};y)v_0(y)dy\|_{L^p_{\tilde{x}}} \leqq C\zeta_0(1+t)^{-\frac{d-1}{4}\left(1-\frac{1}{p}\right)-\frac{|\alpha|}{4}},$$

$$\|\partial_t \int_{\mathbb{R}^d} \tilde{e}(t,\tilde{x};y)v_0(y)dy\|_{L^p_{\tilde{x}}} \leqq C\zeta_0(1+t)^{-\frac{d-1}{4}\left(1-\frac{1}{p}\right)-\frac{1}{4}},$$

Lemma 3.3 (Nonlinear interaction estimates)**.** *For* $G(t,x;y)$ *as in Lemma 3.1, we have the following estimates:*

(i) *Under Condition* (2a) *and for* $d \geqq 2$,

$$\|\int_0^t \int_{\mathbb{R}^d} \sum_k \tilde{G}_{y_k}(t-s,x;y)\Psi(s,y)dyds\|_{L^p_{\tilde{x}}},$$
$$\leqq C(\psi_4(t,x_1) + (1+t)^{-\frac{d}{4}+\frac{d-1}{4p}}\psi_5(t,x_1)),$$
$$\|\int_0^t \int_{\mathbb{R}^d} \sum_k D^\alpha_{\tilde{x}}\tilde{e}_k(t-s,\tilde{x};y)\Psi(s,y)dyds\|_{L^p_{\tilde{x}}}$$
$$\leqq \begin{cases} C(1+t)^{-\frac{d-1}{2}\left(1-\frac{1}{p}\right)-\frac{|\alpha|}{2}}, & |\alpha| \leqq 1,\\ C(1+t)^{-\frac{d-1}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}}, & |\alpha| \geqq 2,\end{cases}$$
$$\|\partial_t \int_0^t \int_{\mathbb{R}^d} \sum_k \tilde{e}_k(t-s,\tilde{x};y)\Psi(s,y)dyds\|_{L^p_{\tilde{x}}}$$
$$\leqq C(1+t)^{-\frac{d-1}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}},$$

for any $\Psi(s,y)$, *satisfying*

$$\|\Psi(s,y)\|_{L^p_{\tilde{y}}} \leqq Cs^{-\frac{3}{4}}(1+s)^{\frac{3}{4}}\Big[(1+s)^{-\frac{d-1}{4}\left(2-\frac{1}{p}\right)}(1+|y_1|+s)^{-2r}$$
$$+(1+s)^{-\frac{d}{2}-\frac{d-1}{2}(1-\frac{1}{p})}e^{-\bar{\eta}|y_1|}\Big],$$

where $\bar{\eta} > \eta$, η *as in* ψ_5.

(ii) *Under Condition* (2b) *and for* $d \geqq 3$,

$$\begin{aligned}
&\| \int_0^t \int_{\mathbb{R}^d} \sum_k \tilde{G}_{y_k}(t-s,x;y)\Psi(s,y)dyds\|_{L^p_{\tilde{x}}} \\
&\quad \leqq C(\psi_4(t,x_1)+\psi_5(t,x_1)) \\
&\| \int_0^t \int_{\mathbb{R}^d} \sum_k D^\alpha_{\tilde{x}}\tilde{e}_k(t-s,\tilde{x};y)\Psi(s,y)dyds\|_{L^p_{\tilde{x}}} \\
&\quad \leqq \begin{cases} C(1+t)^{-\frac{d-1}{4}\left(1-\frac{1}{p}\right)-\frac{|\alpha|}{4}}, & |\alpha| \leqq 2, \\ C(1+t)^{-\frac{d-1}{4}\left(1-\frac{1}{p}\right)-\frac{|\alpha|}{4}}\log(2+t), & |\alpha| = 3, \end{cases} \\
&\|\partial_t \int_0^t \int_{\mathbb{R}^d} \sum_k \tilde{e}_k(t-s,\tilde{x};y)\Psi(s,y)dyds\|_{L^p_{\tilde{x}}} \\
&\quad \leqq C(1+t)^{-\frac{d-1}{4}(1-\frac{1}{p})-\frac{1}{4}},
\end{aligned} \tag{3.2}$$

for any $\Psi(s,y)$ *satisfying*

$$\begin{aligned}
\|\Psi(s,y)\|_{L^p_{\tilde{y}}} \leqq Cs^{-\frac{3}{4}}(1+s)^{\frac{3}{4}}\Big[&(1+s)^{-\frac{d-1}{4}\left(2-\frac{1}{p}\right)}(1+|y_1|+s)^{-2r} \\
&+(1+s)^{-\frac{d}{4}-\frac{d-1}{4}\left(1-\frac{1}{p}\right)}e^{-\bar{\eta}|y_1|}\Big],
\end{aligned}$$

where $\bar{\eta} > \eta$, η *as in* ψ_5.

Given the estimates of Lemmas 3.2 and 3.3, the proofs of Theorem 1.1 (Part I) and Theorem 1.2 (Part I) are almost identical. We thus proceed only in the (slightly shorter) case of Theorem 1.2 (Part I).

Proof of Theorem 1.2 (Part I). In order to accomodate the case of non-constant regularity, we require the following lemma on small time behavior

Lemma 3.4. *Under the assumptions of Lemma 3.1, and under the additional restriction of Holder continuity on the initial perturbation,* $v_0 \in C^{0+\gamma}(x)$, $\gamma > 0$, *the integral equations* (2.1) *and* (2.2) *yield a unique local solution* $v \in C^{0+\frac{\gamma}{4}}(t) \cap C^{0+\gamma}(\mathrm{x})$, $\delta \in C^{1+\frac{\gamma}{4}}(t) \cap C^{3+\gamma}(\tilde{x})$, *extending so long as* $|v|_{C^{0+\gamma}}$ *remains bounded. Moreover, on this time interval*

$$\sup_{x_1\in\mathbb{R}} \|v\|_{L^p_{\tilde{x}}}(\psi_4+\psi_5)^{-1}(t,x_1)$$

remains continuous, so long as it, and $|\delta_t(t+1,\tilde{x})|$, *are uniformly bounded, and for* $\tau > 0$ *sufficiently small and* $t \geqq \tau$,

$$\sup_{x_1\in\mathbb{R}} \|D^\alpha v\|_{L^p_{\tilde{x}}}(\psi_4+\psi_5)^{-1}(t,x_1) \leqq C\tau^{-\frac{|\alpha|}{4}} \sup_{x_1\in\mathbb{R}} \|v\|_{L^p_{\tilde{x}}}(\psi_4+\psi_5)^{-1}(t-\tau,x),$$

where α *is a standard multi-index* $|\alpha| \leqq 3$.

Remark on Lemma 3.4. The key observation of Lemma 3.4 is that we can estimate derivatives of v in terms of the value of v at slightly previous times. Similar estimates hold in the other three cases. A proof of Lemma 3.4 is given in Section 5.

We proceed now by defining the iteration variable

$$\zeta(t) := \sup_{\substack{y_1 \in \mathbb{R},\, 0 \leqq s \leqq t \\ 1 \leqq p \leqq \infty}} \Big[\|v(s, y_1, \cdot)\|_{L^p_{\tilde{y}}} (\psi_4(s, y_1) + \psi_5(s, y_1))^{-1} + \sum_{|\alpha| \leqq 2} \|D^\alpha \delta(s, \cdot)\|_{L^p_{\tilde{y}}} (1+s)^{\frac{d-1}{4}\left(1-\frac{1}{p}\right)+\frac{|\alpha|}{4}} + \sum_{|\alpha|=3} \|D^\alpha \delta(s, \cdot)\|_{L^p_{\tilde{y}}} (1+s)^{\frac{d-1}{4}\left(1-\frac{1}{p}\right)+\frac{3}{4}} \log(e+t) + \|\partial_s \delta(s, \cdot)\|_{L^p_{\tilde{y}}} (1+s)^{\frac{d-1}{4}\left(1-\frac{1}{p}\right)+\frac{1}{4}} \Big].$$

Claim 3.1. Suppose there exists some constant C such that

$$\zeta(t) \leqq C(\zeta_0 + \zeta(t)^2),$$

where ζ_0 is as in Theorem 1.1. Then, for ζ_0 sufficiently small, $\zeta(t) < 2C\zeta_0$.

Proof of Claim 3.1. Regarding $\zeta(0)$, we first observe that since $\|v(0,\cdot)\|_{L^1} \leqq \zeta_0$, $\|v(0,\cdot)\|_{L^\infty} \leqq \zeta_0$, and

$$\|v(0, y_1, \cdot)\|_{L^1_{\tilde{y}}} \leqq \zeta_0 (1+|y_1|)^{-r}, \quad r > 1,$$

we can conclude from (2.1) and (2.2) that there exists some constant C_1 such that $\zeta(0) \leqq C_1\zeta_0$. We can now choose ζ_0 sufficiently small so that $\zeta_0 < \frac{1}{4C}$ and (by choosing $\zeta_0 < \frac{1}{C_1}$) $\zeta(0)^2 < \zeta_0$. In this way, we ensure $\zeta(0) < 2C\zeta_0$, where C is as in the statement of our claim. By continuity of $\zeta(t)$, there exists some $\tau > 0$, possibly small, so that for $0 \leqq t \leqq \tau$, $\zeta(t) < 2C\zeta_0$. Arguing by continuous induction, we suppose now that T is the first time for which $\zeta(T) = 2C\zeta_0$. If no such T exists, the claim is established. If T does exist, we compute directly,

$$\zeta(T) \leqq C(\zeta_0 + \zeta(T)^2) = C(\zeta_0 + 4C^2\zeta_0^2) < 2C\zeta_0,$$

a contradition. □

It remains to show that

$$\|v(s, y_1, \cdot)\|_{L^p_{\tilde{y}}} (\psi_4(s, y_1) + \psi_5(s, y_1))^{-1},$$

and

$$\|\delta(s, \cdot)\|_{L^p_{\tilde{y}}} (1+s)^{\frac{d-1}{4}\left(1-\frac{1}{p}\right)},$$

are both bounded by $C(\zeta_0 + \zeta(t)^2)$ for some $C > 0$, all $0 \leqq s \leqq t$, so long as ζ remains sufficiently small.

Combining (3.3) with Lemma 3.4, we find that for $t \leqq 1$, $|\alpha| \leqq 3$, (noting that as $t \to 0$ in Lemma 3.4, t must approach τ from above, forcing, τ to 0),

$$\begin{aligned} &\|D^\alpha v(t,x)\|_{L^p_{\tilde{x}}}(\psi_4(t,x_1)+\psi_5(t,x_1))^{-1} \\ &\quad \leqq Ct^{-\frac{|\alpha|}{4}}\|v(0,x)\|_{L^p_{\tilde{x}}}(\psi_4(0,x_1)+\psi_5(0,x_1))^{-1} \leqq Ct^{-\frac{|\alpha|}{4}}\zeta(0), \end{aligned}$$

so that (observing that $\zeta(0) \leqq \zeta(t)$),

$$\|D^\alpha v(t,x)\|_{L^p_{\tilde{x}}} \leqq Ct^{-\frac{|\alpha|}{4}}\zeta(t)(\psi_4(t,x_1)+\psi_5(t,x_1)).$$

Similarly, for $t \geqq 1$, and some fixed $0 < \tau_0 < 1$,

$$\begin{aligned} &\|D^\alpha v(t,x)\|_{L^p_{\tilde{x}}}(\psi_4(t,x_1)+\psi_5(t,x_1))^{-1} \\ &\leqq C\tau_0^{-\frac{|\alpha|}{4}}\|v(t-\tau_0,x)\|_{L^p_{\tilde{x}}}(\psi_4(t-\tau_0,x_1)+\psi_5(t-\tau_0,x_1))^{-1} \\ &\leqq C_1\zeta(t-\tau_0), \end{aligned}$$

so that (observing that $\zeta(t-\tau_0) \leqq \zeta(t)$),

$$\|D^\alpha v(t,x)\|_{L^p_{\tilde{x}}} \leqq C_1\zeta(t)(\psi_4(t,x_1)+\psi_5(t,x_1)).$$

We conclude the derivative estimate

$$\|D^\alpha v(t,x)\|_{L^p_{\tilde{x}}} \leqq Ct^{-\frac{|\alpha|}{4}}(1+t)^{\frac{|\alpha|}{4}}\zeta(t)(\psi_4(t,x_1)+\psi_5(t,x_1)), \quad |\alpha| \leqq 3.$$

We have then from our definition of N^m, and our definition of $\zeta(t)$,

$$\sum_m \|N^m(s,y)\|_{L^p_{\tilde{y}}} \leqq \zeta(t)^2\|\Psi(s,y)\|_{L^p_{\tilde{y}}}$$

for some $\Psi(s,y)$ as in Lemma 3.3. (though the justification of this last assertion requires considerable calculation, the analysis of each particular term is straightforward).

Now combining our integral equations (2.2) and (2.1) with the estimates of Lemmas 3.2 , we obtain,

$$\begin{aligned} \|v(t,x)\|_{L^p_{\tilde{x}}} \leqq\ & \|\int_{\mathbb{R}^d}\tilde{G}(t,x;y)v_0(y)dy\|_{L^p_{\tilde{x}}} \\ & +\|\int_0^t\int_{\mathbb{R}^d}\sum_m \tilde{G}_{y_m}(t-s,x;y)N^m(s,y)dyds\|_{L^p_{\tilde{x}}} \\ \leqq\ & C\zeta_0(\psi_4(t,x_1)+\psi_5(t,x_1))+C\zeta(t)^2(\psi_4(t,x_1)+\psi_5(t,x_1)) \\ \leqq\ & C(\zeta_0+\zeta(t)^2)(\psi_4(t,x_1)+\psi_5(t,x_1)). \end{aligned}$$

Proceeding similarly, we determine,

$$\begin{aligned} \|D^\alpha\delta(t,\cdot)\|_{L^p_{\tilde{y}}} &\leqq C(\zeta_0+\zeta(t)^2)(1+t)^{\frac{d-1}{4}\left(1-\frac{1}{p}\right)-\frac{|\alpha|}{4}}, \quad |\alpha| \leqq 2, \\ \|D^\alpha\delta(t,\cdot)\|_{L^p_{\tilde{y}}} &\leqq C(\zeta_0+\zeta(t)^2)(1+t)^{\frac{d-1}{4}\left(1-\frac{1}{p}\right)-\frac{3}{4}}\log(2+t), \quad |\alpha| = 3, \\ \|\partial_t\delta(t,\cdot)\|_{L^p_{\tilde{y}}} &\leqq C(\zeta_0+\zeta(t)^2)(1+t)^{\frac{d-1}{4}\left(1-\frac{1}{p}\right)-\frac{1}{4}}, \end{aligned}$$

from which we have

$$\zeta(t) \leqq C(\zeta_0 + \zeta(t)^2),$$

whereby the Lax case of Theorem 1.1 follows directly from Claim 3.1. □

4. The undercompressive case

In the undercompressive case, taken without loss of generality as $a_1^-, a_1^+ > 0$, we have the following lemma from [6].

Lemma 4.1. *Under assumptions (H0)–(H2) and ($\mathcal{D}_s$), and for the undercompressive case, we have the following estimates on solutions $G(t, x; y)$ to the Green's function equation* (1.7). *For some constants M and η and for d-dimensional multi-index α, with $|\alpha| \leqq 3$, $\alpha_1 \leqq 1$*

(i) $y_1, x_1 \leqq 0$

$$\begin{aligned}\partial_y^\alpha G(t, x; y) = \mathbf{O}\left(t^{-\frac{d+|\alpha|}{4}}\right) \exp\left(-\frac{|x - y - a_- t|^{\frac{4}{3}}}{M t^{\frac{1}{3}}}\right)\\
+\bar{u}_{x_1}(x_1)\partial_y^\alpha e(t, \tilde{x}, y)\\
+\mathbf{O}(t^{-\frac{d}{4}})\mathbf{O}(\exp(-\eta|x_1|))\\
\times\left(\mathbf{O}(t^{-\frac{|\alpha|}{4}}) + \alpha_1\mathbf{O}(\exp(-\eta|y_1|))\mathbf{O}(t^{-\frac{|\alpha|-\alpha_1}{4}})\right)\\
\times \exp\left(-\frac{|\tilde{x} - \tilde{y} - \tilde{a}^- t|^{\frac{4}{3}}}{M t^{\frac{1}{3}}}\right) \exp\left(-\frac{(y_1 + a_1^- t)^{\frac{4}{3}}}{M t^{\frac{1}{3}}}\right)\\
+ \mathbf{O}(\exp(-\eta|x_1|)) R_\alpha^-(t, \tilde{x}, y; d)\\
+\mathbf{O}(\exp(-\eta|x_1|))\mathbf{O}(\exp(-\eta(|\tilde{x} - \tilde{y}| + t))) I_{\{|y_1| \leqq |a_1^-|t\}}.\end{aligned}$$

(ii) $x_1 \leqq 0 \leqq y_1$

$$\begin{aligned}&\partial_y^\alpha G(t, x, y)\\
&= \bar{u}_{x_1}(x_1)\partial_y^\alpha e(t, \tilde{x}, y)\\
&\quad+\mathbf{O}(t^{-\frac{d+|\alpha|-\alpha_1}{4}})\mathbf{O}(\exp(-\eta|x_1|))\\
&\quad\times\mathbf{O}(\exp(-\eta|y_1|)) \exp\left(-\frac{|\tilde{x} - \tilde{y} - \tilde{a}^+ t|^{\frac{4}{3}}}{M t^{\frac{1}{3}}}\right) \exp\left(-\frac{(y_1 - a_1^+ t)^{\frac{4}{3}}}{M t^{\frac{1}{3}}}\right)\\
&\quad+ \mathbf{O}(\exp(-\eta|x_1|)) R^+(t, \tilde{x}, y; d + |\alpha| - \alpha_1)\\
&\quad+\mathbf{O}(\exp(-\eta|x_1|))\mathbf{O}(\exp(-\eta(|\tilde{x} - \tilde{y}| + t))) I_{\{|y_1| \leqq |a_1^+|t\}}.\end{aligned}$$

(iii) $y_1 \leqq 0 \leqq x_1$

$$\begin{aligned}\partial_y^\alpha G(t,x;y) = \mathbf{O}(t^{-\frac{d+|\alpha|}{4}}) \exp\left(-\frac{(x_1 - \frac{a_1^+}{a_1^-} y_1 - a_1^+ t)^{\frac{4}{3}}}{Mt^{1/3}}\right) \\ \times \exp\left(-\frac{|\tilde{x} - \tilde{y} - (\frac{\tilde{a}^+ - \tilde{a}^-}{a_1^- t} y_1 + \tilde{a}^+) t|^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right) \\ + \bar{u}_{x_1}(x_1) \partial_y^\alpha e(t,\tilde{x},y) \\ + \mathbf{O}(\exp(-\eta|x_1|)) R_\alpha^-(t,\tilde{x},y;d) \\ + \mathbf{O}(\exp(-\eta|x_1|)) \\ \times \mathbf{O}(\exp(-\eta(|\tilde{x}-\tilde{y}|+t))) I_{\{|y_1| \leqq |a_1^-|t\}}.\end{aligned}$$

(iv) $x_1, y_1 \geqq 0$

$$\begin{aligned}\partial_y^\alpha G(t,x;y) = \mathbf{O}(t^{-\frac{d+|\alpha|}{4}}) \exp\left(-\frac{|x - y - a_+ t|^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right) \\ + \bar{u}_{x_1}(x_1) \partial_y^\alpha e(t,\tilde{x},y) \\ + \mathbf{O}(\exp(-\eta|x_1|)) R^+(t,\tilde{x},y;d+|\alpha|-\alpha_1) \\ + \mathbf{O}(\exp(-\eta|x_1|)) \mathbf{O}(\exp(-\eta(|\tilde{x}-\tilde{y}|+t))) I_{\{|y_1| \leqq |a_1^+|t\}},\end{aligned}$$

where for $y_1 < 0$,

$$\begin{aligned}\partial_y^\alpha e(t,\tilde{x};y) = \mathbf{O}\left(t^{-\frac{d-1}{4}}\right)\left[\mathbf{O}\left(t^{-\frac{|\alpha|}{4}}\right) + \alpha_1 \mathbf{O}(\exp(-\eta|y_1|)) \mathbf{O}\left(t^{-\frac{|\alpha|-\alpha_1}{4}}\right)\right] \\ \times \exp\left(-\frac{|\tilde{x} - \tilde{y} - \tilde{a}^- t|^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right) \exp\left(-\frac{(y_1 + a_1^- t)^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right) \\ + R_\alpha^-(t,\tilde{x};y,d-1),\end{aligned}$$

while for $y_1 > 0$ *we have,*

$$\begin{aligned}\partial_y^\alpha e(t,\tilde{x};y) = \mathbf{O}\left(t^{-\frac{d-1}{4}}\right) \mathbf{O}(\exp(-\eta|y_1|)) \\ \times \mathbf{O}\left(t^{-\frac{|\alpha|-\alpha_1}{4}}\right) \exp\left(-\frac{|\tilde{x} - \tilde{y} - \tilde{a}^+ t|^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right) \\ \times \exp\left(-\frac{(y_1 - a_1^+ t)^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right) \\ + R^+(t,\tilde{x};y,d-1).\end{aligned}$$

In the case of Condition (2a)*, we have*

$$\begin{aligned}
R_\alpha^-(t,\tilde{x},y;\kappa) &= \mathbf{O}\left(t^{-\frac{\kappa}{4}}\wedge|y_1+a_1^-t|^{-\frac{\kappa}{2}}\right)\left(\mathbf{O}\left(t^{-\frac{|\alpha|}{4}}\wedge|y_1+a_1^-t|^{-\frac{|\alpha|}{2}}\right)\right.\\
&\quad \left.+\alpha_1\mathbf{O}(e^{-\eta|y_1|})\mathbf{O}\left(t^{-\frac{|\alpha|-\alpha_1}{4}}\wedge|y_1+a_1^-t|^{-\frac{|\alpha|-\alpha_1}{2}}\right)\right)\\
&\quad \times\left[\exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{\text{eff}}^-t|^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right)\right.\\
&\quad \left.+\exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{\text{eff}}^-t|^{2}}{M|y_1+a_1^-t|}\right)\right]I_{\{|y_1|\leqq|a_1^-|t\}}
\end{aligned}$$

$$\begin{aligned}
R^+(t,\tilde{x},y;\kappa) &= \mathbf{O}(t^{-\frac{\kappa}{4}}\wedge|y_1-a_1^+t|^{-\frac{\kappa}{2}})\\
&\quad \times\mathbf{O}(\exp(-\eta|y_1|))\\
&\quad \times\left[\exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{\text{eff}}^+t|^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right)\right.\\
&\quad \left.+\exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{\text{eff}}^+t|^{2}}{M|y_1-a_1^+t|}\right)\right]I_{\{|y_1|\leqq|a_1^+|t\}},
\end{aligned}$$

while in the case of Condition (2b)*, we have*

$$\begin{aligned}
R_\alpha^-(t,\tilde{x};y,\kappa) &= \mathbf{O}\left(t^{-\frac{\kappa}{4}}\right)\left(\mathbf{O}\left(t^{-\frac{|\alpha|}{4}}\right)+\alpha_1\mathbf{O}\left(\exp(-\eta|y_1|)\right)\right.\\
&\quad \left.\times\mathbf{O}\left(t^{-\frac{|\alpha|-\alpha_1}{4}}\right)\right)\\
&\quad \times\exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{\text{eff}}^-t|^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right)I_{\{|y_1|\leqq|a_1^-|t\}}
\end{aligned}$$

$$\begin{aligned}
R^+(t,\tilde{x};y,\kappa) &= \mathbf{O}(t^{-\frac{\kappa}{4}})\mathbf{O}(\exp(-\eta|x_1|))\\
&\quad \times\mathbf{O}(\exp(-\eta|y_1|))\exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{\text{eff}}^+t|^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right)\\
&\quad \times I_{\{|y_1|\leqq|a_1^+|t\}}.
\end{aligned}$$

Moreover, owing to non-sonicity condition (H1)*, the first time derivative of* $e(t,\tilde{x};y)$ *yields the same estimate as a single derivative with respect to any transverse coordinate.*

Remarks on Lemma 4.1. There are two critical differences between the estimates of the undercompressive case and those of the compressive case. First, the undercompressive case has a *transmission* term (Case (iii)), which describes a signal moving through the shock layer. Secondly, y_1 derivatives on the excited term $e(t,\tilde{x};y)$ in the undercompressive case *do not* decay with an additional rate in time as do their counterparts in the compressive case.

Lemma 4.2 (Linear integral estimates)**.** *For $G(t, x; y)$ as in Lemma 4.1, we have the following estimates:*

(i) *Under Condition* (2a) *and for $v_0(y)$ as in* (1.12)*,*

$$\|\int_{\mathbb{R}^d} \tilde{G}(t, x; y)v_0(y)dy\|_{L^p_{\tilde{x}}} \leqq C\zeta_0 \begin{cases} \psi_4(t, x_1) + (1+t)^{-\frac{d}{4}+\frac{d-1}{4p}}\psi_5(t, x_1), & x_1 \leqq 0, \\ \theta(t, x_1) + \psi_1(t, x_1) + (1+t)^{-\frac{d}{4}+\frac{d-1}{4p}}\psi_5(t, x_1), & x_1 \geqq 0, \end{cases}$$

$$\|\int_{\mathbb{R}^d} \partial^\alpha_{\tilde{x}}\tilde{e}(t, \tilde{x}; y)v_0(y)dy\|_{L^p_{\tilde{x}}} \leqq C\zeta_0(1+t)^{-\frac{d-1}{2}\left(1-\frac{1}{p}\right)} \begin{cases} (1+t)^{-\frac{|\alpha|}{2}}, & |\alpha| \leqq 1, \\ (1+t)^{-\frac{1}{2}}, & |\alpha| \geqq 2. \end{cases}$$

$$\|\partial_t \int_{\mathbb{R}^d} \tilde{e}(t, \tilde{x}; y)v_0(y)dy\|_{L^p_{\tilde{x}}} \leqq C\zeta_0(1+t)^{-\frac{d-1}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}}.$$

(ii) *Under Condition* (2b) *and for $v_0(y)$ as in* (1.13)

$$\|\int_{\mathbb{R}^d} \tilde{G}(t, x; y)v_0(y)dy\|_{L^p_{\tilde{x}}} \leqq C\zeta_0 \begin{cases} \psi_4(t, x_1) + \psi_5(t, x_1), & x_1 \leqq 0 \\ \theta(t, x_1) + \psi_1(t, x_1) + \psi_5(t, x_1), & x_1 \geqq 0 \end{cases}$$

$$\|\int_{\mathbb{R}^d} \partial^\alpha_{\tilde{x}}\tilde{e}(t, \tilde{x}; y)v_0(y)dy\|_{L^p_{\tilde{x}}} \leqq C\zeta_0(1+t)^{-\frac{d-1}{4}\left(1-\frac{1}{p}\right)-\frac{|\alpha|}{4}}$$

$$\|\partial_t \int_{\mathbb{R}^d} \tilde{e}(t, \tilde{x}; y)v_0(y)dy\|_{L^p_{\tilde{x}}} \leqq C\zeta_0(1+t)^{-\frac{d-1}{4}\left(1-\frac{1}{p}\right)-\frac{1}{4}}.$$

Lemma 4.3 (Nonlinear interaction estimates)**.** *For $G(t, x; y)$ as in Lemma 4.1, we have the following estimates:*

(i) *Under Condition* (2a) *and for $d \geqq 3$,*
For $x_1 \leqq 0$,

$$\|\int_0^t \int_{\mathbb{R}^d} \sum_k \tilde{G}_{y_k}(t-s, x; y)\Psi(s, y)dyds\|_{L^p_{\tilde{x}}} \leqq C[\psi_4(t, x_1) + (1+t)^{-\frac{d}{4}+\frac{d-1}{4p}}\psi_5(t, x_1)].$$

For $x_1 \geqq 0$,

$$\|\int_0^t \int_{\mathbb{R}^d} \sum_k \tilde{G}_{y_k}(t-s,x;y)\Psi(s,y)dyds\|_{L^p_{\tilde{x}}}$$
$$\leqq C\Big[\theta(t,x_1)+\psi_1(t,x_1)$$
$$+(1+t)^{-\frac{d}{4}+\frac{d-1}{4p}}\psi_5(t,x_1)+(1+|x_1-a_1^+t|)^{-\frac{d}{4}}\psi_2$$
$$+(1+|x_1|+t)^{-\frac{d-1}{4}\left(2-\frac{1}{p}\right)-\frac{1}{4}}\psi_3\Big].$$

In addition, we have the tracking estimates,

$$\|\int_0^t \int_{\mathbb{R}^d} \sum_k \partial_{\tilde{x}}^\alpha \tilde{e}_k(t-s,\tilde{x};y)\Psi(s,y)dyds\|_{L^p_{\tilde{x}}}$$
$$\leqq \begin{cases} (1+t)^{-\frac{d-1}{2}\left(1-\frac{1}{p}\right)-\frac{|\alpha|}{2}}, & |\alpha|\leqq 1,\\ (1+t)^{-\frac{d-1}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}}, & |\alpha|\geqq 2,\end{cases}$$

$$\|\partial_t \int_0^t \int_{\mathbb{R}^d} \sum_k \tilde{e}_k(t-s,\tilde{x};y)\Psi(s,y)dyds\|_{L^p_{\tilde{x}}}$$
$$\leqq (1+t)^{-\frac{d-1}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}},$$

where

$$\Psi(s,y)=\Psi_-(s,y)I_{\{y_1\leqq 0\}}+\Psi_+(s,y)I_{\{y_1\geqq 0\}}$$

for any $\Psi_-(s,y)$ and $\Psi_+(s,y)$ satisfying

$$\|\Psi_-(s,y)\|_{L^p_{\tilde{y}}}\leqq Cs^{-\frac{3}{4}}(1+s)^{\frac{3}{4}}\Big[(1+s)^{-\frac{d-1}{4}(2-\frac{1}{p})}(1+|y_1|+s)^{-2r}$$
$$+(1+s)^{-\frac{d}{2}-\frac{d-1}{2}(1-\frac{1}{p})}e^{-\bar{\eta}|y_1|}\Big],$$

and

$$\|\Psi_+(s,y)\|_{L^p_{\tilde{y}}}$$
$$\leqq Cs^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d-1}{4}\left(2-\frac{1}{p}\right)}\Big[(1+s)^{-\frac{1}{2}}\exp\left(-\frac{(y_1-a_1^+s)^{\frac{4}{3}}}{Ms^{\frac{1}{3}}}\right)$$
$$+(1+s)^{-\frac{1}{2}}\wedge(1+|y_1-a_1^+s|)^{-2r}\Big]$$
$$+Cs^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d}{2}-\frac{d-1}{2}\left(1-\frac{1}{p}\right)}\exp\left(-\bar{\eta}|y_1|\right)$$
$$+C(1+|y_1|+s)^{-\frac{d}{2}+\frac{d-1}{4p}}(1+|y_1-a_1^+s|)^{-d}I_{\{|y_1|\leqq|a_1^+|s\}}$$
$$+C(1+|y_1|+s)^{-\frac{d-1}{2}\left(4-\frac{1}{p}\right)-\frac{1}{2}}(1+|y_1|)^{-1}I_{\{|y_1|\leqq a_1^+s\}},$$

where $\wedge$ represents min*, and $\bar{\eta}>\eta$, η as in ψ_5.*

(ii) *Under Condition* (2b) *and for* $d \geqq 5$*,*

$$\|\int_0^t \int_{\mathbb{R}^d} \sum_k \tilde{G}_{y_k}(t-s,x;y)\Psi(s,y)dyds\|_{L^p_{\tilde{x}}}$$

$$\leqq C\begin{cases}\psi_4(t,x_1)+\psi_5(t,x_1), & x_1 \leqq 0,\\ \theta(t,x_1)+\sum_{k=1}^3 \psi_k(t,x_1)+\psi_5(t,x_1), & x_1 \geqq 0,\end{cases}$$

$$\|\int_0^t \int_{\mathbb{R}^d} \sum_k \partial_{\tilde{x}}^\alpha \tilde{e}_k(t-s,\tilde{x};y)\Psi(s,y)dyds\|_{L^p_{\tilde{x}}}$$

$$\leqq \begin{cases}(1+t)^{-\frac{d-1}{4}(1-\frac{1}{p})-\frac{|\alpha|}{4}}, & |\alpha| \leqq 2,\\ (1+t)^{-\frac{d-1}{4}(1-\frac{1}{p})-\frac{|\alpha|}{4}}\ln(e+t), & |\alpha| = 3,\end{cases}$$

$$\|\partial_t \int_0^t \int_{\mathbb{R}^d} \sum_k \tilde{e}_k(t-s,\tilde{x};y)\Psi(s,y)dyds\|_{L^p_{\tilde{x}}}$$

$$\leqq (1+t)^{-\frac{d-1}{4}\left(1-\frac{1}{p}\right)-\frac{1}{4}},$$

where

$$\Psi(s,y) = \Psi_-(s,y)I_{\{y_1 \leqq 0\}} + \Psi_+(s,y)I_{\{y_1 \geqq 0\}}$$

for any $\Psi_-(s,y)$*, and* $\Psi_+(s,y)$ *satisfying*

$$\|\Psi_-(s,y)\|_{L^p_{\tilde{y}}} \leqq Cs^{-\frac{3}{4}}(1+s)^{\frac{3}{4}}\Big[(1+s)^{-\frac{d-1}{4}\left(2-\frac{1}{p}\right)}(1+|y_1|+s)^{-2r} + (1+s)^{-\frac{d}{4}-\frac{d-1}{4}\left(1-\frac{1}{p}\right)}\exp(-\bar{\eta}|y_1|)\Big],$$

and

$$\|\Psi_+(s,y)\|_{L^p_{\tilde{y}}}$$

$$\leqq Cs^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d-1}{4}(2-\frac{1}{p})}\Big[(1+s)^{-\frac{1}{2}}\exp\left(-\frac{(y_1-a_1^+s)^{\frac{4}{3}}}{Ms^{\frac{1}{3}}}\right)$$

$$+(1+s)^{-\frac{1}{2}} \wedge (1+|y_1-a_1^+s|)^{-2r}\Big]$$

$$+Cs^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d}{4}-\frac{d-1}{4}\left(1-\frac{1}{p}\right)}\exp(-\bar{\eta}|y_1|)$$

$$+C(1+|y_1|+s)^{-\frac{d}{2}+\frac{d-1}{4p}}(1+|y_1-a_1^+s|)^{-\frac{d}{2}}I_{\{|y_1|\leqq|a_1^+|s\}}$$

$$+C(1+|y_1|+s)^{-\frac{d-1}{4}\left(4-\frac{1}{p}\right)}(1+|y_1|)^{-1}I_{\{|y_1|\leqq a_1^+s\}},$$

where $\wedge$ *represents* min*, and* $\bar{\eta} > \eta$*,* η *as in* ψ_5*.*

Remark on Proofs of Theorem 1.1 (Part II) and Theorem 1.2 (Part II). The proofs of Theorem 1.1 (Part II) and Theorem 1.2 (Part II) follow from Lemmas 4.2 and 4.3 and the proof of Theorem 1.2 (Part I) above.

5. Proofs of estimate lemmas

In this section we close our arguments in the case of Condition (2b) from $(\mathcal{D}_s)$ by establishing Lemmas 3.2, 3.3 and 3.4 (for the compressive case) and Lemmas 4.2 and 4.3 (for the undercompressive case). The estimates in the case of Condition (2a) are similar, and hence we omit them here. In each case, the estimates will proceed from a careful balance between decay of the kernel $\tilde{G}$ and decay of the initial perturbation $v_0(y)$ or the interaction nonlinearity Ψ.

For a general review of our terminology for the *scattering* case, the *remainder* case, and the *excited* case, we refer the reader to our remark immediately following Lemma 3.1. Throughout, we will refer to the first nonlinearity, the second nonlinearity, etc., by which we mean the summands in our estimates on $\|\Psi\|_{L^p_{\tilde{x}}}$ in their order given in Lemmas 3.3 and 4.3.

Proof of Lemma 3.2. *First estimate of Lemma 3.2.* Owing to the symmetry of the Lax case, we need only consider the case $x_1 \leqq 0$, which we divide into the subcases $y_1 \leqq 0$ and $y_1 \geqq 0$. In the case $y_1 \leqq 0$ we have from, Lemma 3.1, three integrals to analyze, We begin with the scattering term:

$$\begin{aligned}
&\| \int_{-\infty}^{0} \int_{\mathbb{R}^{d-1}} \mathbf{O}\left(t^{-\frac{d}{4}}\right) \exp\left(-\frac{|x-y-a_{-}t|^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right) v_0(y) d\tilde{y} dy_1 \|_{L^p_{\tilde{x}}} \\
&\quad \leqq Ct^{-\frac{d}{4}} \int_{-\infty}^{0} \int_{\mathbb{R}^{d-1}} \| \exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{-}t|^{\frac{4}{3}}}{\bar{M}t^{\frac{1}{3}}}\right) \|_{L^p_{\tilde{x}}} \\
&\qquad \times \exp\left(-\frac{(x_1-y_1-a_1^{-}t)^{\frac{4}{3}}}{\bar{M}t^{\frac{1}{3}}}\right) |v_0(y)| d\tilde{y} dy_1 \\
&\quad \leqq Ct^{-\frac{d}{4}} \sup_{\tilde{y}} \| \exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{-}t|^{\frac{4}{3}}}{\bar{M}t^{\frac{1}{3}}}\right) \|_{L^p_{\tilde{x}}} \\
&\qquad \times \int_{-\infty}^{0} \exp\left(-\frac{(x_1-y_1-a_1^{-}t)^{\frac{4}{3}}}{\bar{M}t^{\frac{1}{3}}}\right) \|v_0\|_{L^1_{\tilde{y}}} dy_1 \\
&\quad \leqq Ct^{-\frac{d}{4}+\frac{d-1}{4p}} \int_{-\infty}^{0} \exp\left(-\frac{(x_1-y_1-a_1^{-}t)^{\frac{4}{3}}}{\bar{M}t^{\frac{1}{3}}}\right) (1+|y_1|)^{-r} dy_1
\end{aligned}$$

for some $\bar{M} > M$. In the final integral, we observe that either y_1 is near $x_1 - a_1^{-}t$ $(= -|x_1| - |a_1^{-}t|$; i.e., no cancellation) for which we have decay with rate $(1 + |x_1| + t)^{-r}$, or else y_1 is small compared with $|x_1 - a_1^{-}t|$ and we have exponential decay in both $|x_1|$ and t from the kernel. More precisely, we have the balance estimate:

$$\exp\left(-\frac{(x_1-y_1-a_1^{-}t)^{\frac{4}{3}}}{M't^{\frac{1}{3}}}\right)(1+|y_1|)^{-r} \leqq C(1+|x_1|+t)^{-r},$$

where, by taking $M' > \bar{M}$ we may reserve a part of the kernel to integrate in y_1. Integrating the kernel (which adds growth with rate $t^{1/4}$), we obtain an estimate by

$$Ct^{-\frac{d-1}{4}\left(1-\frac{1}{p}\right)}(1+|x_1|+t)^{-r}.$$

We observe that the seeming blow-up as $t \to 0$ is an artifact of the approach. For small time, we integrate the kernel in every component of y, which yields "growth" rate $t^{+\frac{d}{4}}$, canceling the blow-up exactly. We next consider integration over the remainder term. Proceeding as above, we compute,

$$\begin{aligned}
&\|\int_{-\infty}^{0}\int_{\mathbb{R}^{d-1}} \mathbf{O}\left(t^{-\frac{d}{4}}\right)\mathbf{O}\left(\exp\left(-\eta|x_1|\right)\right)\exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{\text{eff}}^{-}t|^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right)\\
&\quad\times I_{\{|y_1|\leqq|a_1^-|t\}}v_0(y)dy\|_{L^p_{\tilde{x}}}\\
&\quad\leqq Ct^{-\frac{d}{4}+\frac{d-1}{4p}}\exp\left(-\eta|x_1|\right)\int_{-a_1^-t}^{0}(1+|y_1|)^{-r}dy_1\\
&\quad\leqq Ct^{-\frac{d}{4}+\frac{d-1}{4p}}\exp\left(-\eta|x_1|\right).
\end{aligned}$$

Finally, for the excited term, we have

$$\begin{aligned}
&\|\int_{-\infty}^{0}\int_{\mathbb{R}^{d-1}} \mathbf{O}\left(t^{-\frac{d-1}{4}}\right)\mathbf{O}\left(\exp\left(-\eta|x_1|\right)\right)\exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}^{-}t|^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right)\\
&\qquad\qquad\times\exp\left(-\frac{(y_1+a_1^-t)^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right)v_0(y)dy\|_{L^p_{\tilde{x}}}\\
&\leqq Ct^{-\frac{d-1}{4}+\frac{d-1}{4p}}\exp\left(-\eta|x_1|\right)\int_{-\infty}^{0}\exp\left(-\frac{(y_1+a_1^-t)^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right)(1+|y_1|)^{-r}dy_1\\
&\leqq Ct^{-\frac{d-1}{4}\left(1-\frac{1}{p}\right)+\frac{1}{4}}\exp\left(-\eta|x_1|\right)(1+t)^{-r},
\end{aligned}$$

which can be subsumed for $r \geqq \frac{1}{2}$. The cases $x_1 < 0 < y_1$ and $x_1 > 0$ are similar.

Second estimate of Lemma 3.2. For the second integral estimate of Lemma 3.2, we have

$$\begin{aligned}
&\|\int_{-\infty}^{0}\int_{\mathbb{R}^{d-1}} \mathbf{O}\left(t^{-\frac{d-1}{4}-\frac{|\alpha|}{4}}\right)\exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{\text{eff}}^{-}t|^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right)I_{\{|y_1|\leqq|a_1^-|t\}}v_0(y)dy\|_{L^p_{\tilde{x}}}\\
&\leqq Ct^{-\frac{d-1}{4}\left(1-\frac{1}{p}\right)-\frac{|\alpha|}{4}}\int_{-a_1^-t}^{0}(1+|y_1|)^{-r}dy_1 \leqq Ct^{-\frac{d-1}{4}\left(1-\frac{1}{p}\right)-\frac{|\alpha|}{4}},
\end{aligned}$$

where we observe that for $|\alpha| \leqq 4$ the kernel can be integrated as before and then we can use the time dependence of the limits of integration to ensure there is no genuine blow-up as $t \to 0$.

Third estimate of Lemma 3.2. Finally, for the third estimate in Lemma 3.2, we observe the relation:

$$\begin{aligned}&\int_{\mathbb{R}} \tilde{e}(t,\tilde{x};y)v_0(y)dy\\ &\quad= \int_{-a_1^- t}^{0}\int_{\mathbb{R}^{d-1}} \mathbf{O}\left(t^{-\frac{d-1}{4}}\right)\exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{\text{eff}}^- t|^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right)v_0(y)dy,\end{aligned}$$

with the integrand differentiable in t (see the development of [6]), with an estimate

$$\begin{aligned}&\partial_t\left[\mathbf{O}\left(t^{-\frac{d-1}{4}}\right)\exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{\text{eff}}^- t|^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right)\right]\\ &\quad= \mathbf{O}\left(t^{-\frac{d}{4}}\right)\exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{\text{eff}}^- t|^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right).\end{aligned}$$

We then have,

$$\begin{aligned}&\|\partial_t\int_{-a_1^- t}^{0}\int_{\mathbb{R}^{d-1}} \mathbf{O}\left(t^{-\frac{d-1}{4}}\right)\exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{\text{eff}}^- t|^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right)v_0(y)dy\|_{L^p_{\tilde{x}}}\\ &= \|-\int_{\mathbb{R}^{d-1}} \mathbf{O}\left(t^{-\frac{d-1}{4}}\right)\exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{\text{eff}}^- t|^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right)v_0(-a_1^- t,\tilde{y})d\tilde{y}\\ &\quad+\int_{-a_1^- t}^{0}\int_{\mathbb{R}^{d-1}} \mathbf{O}\left(t^{-\frac{d}{4}}\right)\exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{\text{eff}}^- t|^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right)v_0(y)dy\|_{L^p_{\tilde{x}}},\\ &\leqq Ct^{-\frac{d-1}{4}\left(1-\frac{1}{p}\right)-\frac{1}{4}}.\end{aligned}$$

This completes the proof of Lemma 3.2. □

Proof of Lemma 3.3. First we consider integrals of the form

$$\begin{aligned}&\|\int_0^t\int_{\mathbb{R}^d}\sum_k \tilde{G}_{y_k}(t-s,x;y)\Psi(s,y)dyds\|_{L^p_{\tilde{x}}}\\ &\quad\leqq \int_0^t\int_{-\infty}^{+\infty}\|\int_{\mathbb{R}^{d-1}}\sum_k \tilde{G}_{y_k}(t-s,x;y)\Psi(s,y)d\tilde{y}\|_{L^p_{\tilde{x}}}dy_1ds.\end{aligned}$$

We have two approaches for estimating this norm, which we choose depending upon whether we want more $t-s$ decay (typically, for $s\in[0,t/\gamma]$, $\gamma>1$) or more s decay (typically, for $s\in[t/\gamma,t]$). For the first, we have

$$\|\int_{\mathbb{R}^{d-1}}\sum_k \tilde{G}_{y_k}(t-s,x;y)\Psi(s,y)d\tilde{y}\|_{L^p_{\tilde{x}}}\leqq\sup_{\tilde{y}}\|\tilde{G}(t-s,x;y)\|_{L^p_{\tilde{x}}}\|\Psi\|_{L^1_{\tilde{y}}}.$$

For our second approach, we take p and q satisfying $\frac{1}{p} + \frac{1}{q} = 1$ and compute

$$\begin{aligned}
&\| \int_{\mathbb{R}^{d-1}} |\tilde{G}_{y_k}(t-s, x; y)\Psi(s, y)| d\tilde{y} \|_{L^p_{\tilde{x}}} \\
&\quad = \| \int_{\mathbb{R}^{d-1}} |\tilde{G}_{y_k}|^{\frac{1}{p}} |\Psi| |\tilde{G}_{y_k}|^{\frac{1}{q}} d\tilde{y} \|_{L^p_{\tilde{x}}} \\
&\quad \leqq \| \Big(\int_{\mathbb{R}^{d-1}} |\tilde{G}_{y_k}| |\Psi|^p d\tilde{y} \Big)^{\frac{1}{p}} \Big(\int_{\mathbb{R}^{d-1}} |\tilde{G}_{y_k}| d\tilde{y} \Big)^{\frac{1}{q}} \|_{L^p_{\tilde{x}}} \\
&\quad \leqq \sup_{\tilde{y}} \|\tilde{G}_{y_k}\|^{\frac{1}{p}}_{L^1_{\tilde{x}}} \sup_{\tilde{x}} \|\tilde{G}_{y_k}\|^{\frac{1}{q}}_{L^1_{\tilde{y}}} \|\Psi\|_{L^p_{\tilde{y}}}.
\end{aligned}$$

By symmetry of the Lax case, we need only consider $x_1 \leqq 0$, which we divide into the subcases $y_1 \leqq 0$ and $y_1 \geqq 0$. For $y_1 \leqq 0$, we begin with the scattering term, for which we have

$$\begin{aligned}
&\| \int_0^t \int_{-\infty}^0 \int_{\mathbb{R}^{d-1}} \mathbf{O}\left((t-s)^{-\frac{d+1}{4}} \right) \\
&\quad \times \exp\left(-\frac{|x - y - a_-(t-s)|^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}} \right) \Psi(s, y) dy ds \|_{L^p_{\tilde{x}}} \\
&\quad \leqq C_1 \int_0^{t/\gamma} \int_{-\infty}^0 (t-s)^{-\frac{d+1}{4} + \frac{d-1}{4p}} \\
&\qquad \times \exp\left(-\frac{(x_1 - y_1 - a_1^-(t-s))^{\frac{4}{3}}}{\bar{M}(t-s)^{\frac{1}{3}}} \right) \|\Psi\|_{L^1_{\tilde{y}}} dy_1 ds \\
&\qquad + C_2 \int_{t/\gamma}^t \int_{-\infty}^0 (t-s)^{-\frac{1}{2}} \exp\left(-\frac{(x_1 - y_1 - a_1^-(t-s))^{\frac{4}{3}}}{\bar{M}(t-s)^{\frac{1}{3}}} \right) \|\Psi\|_{L^p_{\tilde{y}}} dy_1 ds
\end{aligned} \tag{5.1}$$

for some constant $\gamma > 1$. (typically, γ can be chosen as any fixed value in $(0, 1)$, though in certain cases we will choose it more precisely). We have two terms to consider for $\|\Psi\|_{L^1_{\tilde{y}}}$. We first consider $s \in [0, t/\gamma]$, for which we have

$$\begin{aligned}
&\int_0^{t/\gamma} \int_{-\infty}^0 (t-s)^{-\frac{d+1}{4} + \frac{d-1}{4p}} \exp\left(-\frac{(x_1 - y_1 - a_1^-(t-s))^{\frac{4}{3}}}{\bar{M}(t-s)^{\frac{1}{3}}} \right) \\
&\quad \times s^{-\frac{3}{4}} (1+s)^{\frac{3}{4} - \frac{d-1}{4}} (1 + |y_1| + s)^{-2r} dy_1 ds,
\end{aligned}$$

for which we observe the estimate

$$\exp\left(-\frac{(x_1 - y_1 - a_1^-(t-s))^{\frac{4}{3}}}{\bar{M}(t-s)^{\frac{1}{3}}} \right) (1 + |y_1| + s)^{-2r} \leqq C(1 + |x_1| + t)^{-2r}.$$

Upon integrating the kernel, we obtain the estimate

$$Ct^{-\frac{d}{4}+\frac{d-1}{4p}}(1+t)^{-\frac{d-1}{4}+\frac{3}{4}}(1+|x_1|+t)^{-2r}\int_0^{t/2} s^{-3/4}ds$$
$$\leqq Ct^{-\frac{d-1}{4}\left(1-\frac{1}{p}\right)}(1+t)^{-\frac{d-1}{4}+\frac{3}{4}}(1+|x_1|+t)^{-2r},$$

which for $r \geqq 1/2$ is bounded by $\psi_4(t,x_1)$ for all $d \geqq 2$. For the second term in $\|\Psi\|_{L^1_{\tilde{y}}}$, we have

$$\int_0^{t/\gamma}\int_{-\infty}^0 (t-s)^{-\frac{d+1}{4}+\frac{d-1}{4p}} \exp\left(-\frac{(x_1-y_1-a_1^-(t-s))^{\frac{4}{3}}}{\bar{M}(t-s)^{\frac{1}{3}}}\right)s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d}{4}}$$
$$\times \exp(-\bar{\eta}|y_1|)dy_1 ds,$$

for which we observe the inequality

$$\exp\left(-\frac{(x_1-y_1-a_1^-(t-s))^{\frac{4}{3}}}{\bar{M}(t-s)^{\frac{1}{3}}}\right)\exp(-(\bar{\eta}-\varepsilon)|y_1|)$$
$$\leqq C\exp\left(-\eta_1|x_1-a_1^-(t-s)|\right)$$
$$= C\exp(-\eta_1|x_1|)\exp\left(-\eta_1 a_1^-(t-s)\right), \tag{5.2}$$

in which $\eta_1 > 0$ and we have reserved a term $e^{-\varepsilon|y_1|}$ to integrate in y_1. Clearly, on this interval of s, we have exponential decay in both $|x_1|$ and t, which can be absorbed by our claimed estimate. For the integrals over $s \in [t/\gamma, t]$, we first observe that for the nonlinearity $s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d-1}{4}\left(2-\frac{1}{p}\right)}(1+|y_1|+s)^{-2r}$ the argument remains almost identical. For the second nonlinearity, we proceed by estimate (5.2), observing that for γ chosen sufficiently close to 1, η_1 can be taken larger than η. We arrive at an estimate by

$$\int_{t/\gamma}^{t}\int_{-\infty}^0 (t-s)^{-\frac{1}{2}}\exp\left(-\frac{(x_1-y_1-a_1^-(t-s))^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)$$
$$\times s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d}{4}-\frac{d-1}{4}\left(1-\frac{1}{p}\right)}\exp(-\bar{\eta}|y_1|)dy_1 ds$$
$$\leqq Ct^{-\frac{3}{4}}(1+t)^{\frac{3}{4}-\frac{d}{4}-\frac{d-1}{4}\left(1-\frac{1}{p}\right)}\exp(-\eta|x_1|)$$
$$\int_{t/\gamma}^{t}(t-s)^{-\frac{1}{2}}\exp\left(-\eta_1 a_1^-(t-s)\right)ds$$
$$\leqq Ct^{-\frac{3}{4}}(1+t)^{\frac{3}{4}-\frac{d}{4}-\frac{d-1}{4}\left(1-\frac{1}{p}\right)}\exp(-\eta|x_1|),$$

which can be bounded by ψ_5. Again, the seeming blow-up as $t \to 0$ can be eliminated by integrating the kernel to eliminate $(t-s)^{-\frac{1}{4}}$ from the final integration over s. In this way, for small time, we integrate $(t-s)^{-\frac{1}{4}}$ to $t^{+\frac{3}{4}}$, which cancels exactly with the derivative blow-up $t^{-\frac{3}{4}}$.

We next consider the remainder term in the Green's function, for which we have

$$\begin{aligned} \| \int_0^t \int_{-\infty}^0 \int_{\mathbb{R}^{d-1}} &\mathbf{O}\left((t-s)^{-\frac{d+1}{4}}\right) \exp(-\eta|x_1|) \exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{\text{eff}}^-(t-s)|^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right) \\ &\times I_{\{|y_1|\leqq|a_1^-|(t-s)\}} \Psi(s,y) dy ds \|_{L^p_{\tilde{x}}} \\ &\leqq C_1 \int_0^{t/\gamma} \int_{-a_1^-(t-s)}^0 (t-s)^{-\frac{d+1}{4}+\frac{d-1}{4p}} \exp(-\eta|x_1|) \|\Psi\|_{L^1_{\tilde{y}}} dy_1 ds \qquad (5.3) \\ &+ C_2 \int_{t/\gamma}^t \int_{-a_1^-(t-s)}^0 (t-s)^{-\frac{1}{2}} \exp(-\eta|x_1|) \|\Psi\|_{L^p_{\tilde{y}}} dy_1 ds. \end{aligned}$$

Proceeding as above, we consider each summand of $\|\Psi\|_{L^1_{\tilde{y}}}$ in turn, beginning with the first, for which we have (for $r > 1$ and $s \in [0, t/\gamma]$):

$$\begin{aligned} \int_0^{t/\gamma} \int_{-a_1^-(t-s)}^0 &(t-s)^{-\frac{d+1}{4}+\frac{d-1}{4p}} e^{-\eta|x_1|} \\ &\times s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d-1}{4}}(1+|y_1|+s)^{-2r} dy_1 ds \\ &\leqq Ct^{-\frac{d+1}{4}+\frac{d-1}{4p}} \exp(-\eta|x_1|) \int_0^{t/\gamma} s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d-1}{4}-r} ds \\ &\leqq Ct^{-\frac{d+1}{4}+\frac{d-1}{4p}} \exp(-\eta|x_1|), \end{aligned}$$

which is bounded by ψ_5. For the second summand of $\|\Psi\|_{L^1_{\tilde{y}}}$, we have, for $d \geqq 3$:

$$\begin{aligned} \int_0^{t/\gamma} \int_{-a_1^-(t-s)}^0 &(t-s)^{-\frac{d+1}{4}+\frac{d-1}{4p}} \exp(-\eta|x_1|) s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d}{4}} \exp(-\bar{\eta}|y_1|) dy_1 ds \\ &\leqq Ct^{-\frac{d+1}{4}+\frac{d-1}{4p}} \exp(-\eta|x_1|) \int_0^{t/\gamma} s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d}{4}} ds \\ &\leqq Ct^{-\frac{d}{4}+\frac{d-1}{4p}} \exp(-\eta|x_1|). \end{aligned}$$

We remark at this point, the significance of the interaction between the remainder term and the second nonlinearity, restricting our analysis to the case $d \geqq 3$. In the case $d = 1$ there is no term corresponding to this remainder, thus the analysis can be completed; but, the analysis fails in the case $d = 2$. We compare this with the case of second-order regularity, for which the analogous time integration becomes:

$$Ct^{-\frac{d+1}{2}+\frac{d-1}{2p}} \int_0^{t/\gamma} (1+s)^{-\frac{d}{2}} ds \leqq C_1 t^{-\frac{d+1}{2}+\frac{d-1}{2p}} \log(2+t),$$

which is subsumed by the linear estimate in that case $t^{-\frac{d-1}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}}$ (see [9] (Theorem 2.1)).

For the integral over $s \in [t/\gamma, t]$, analysis of the first nonlinearity is again straightforward. For the second, we have

$$\int_{t/\gamma}^{t} \int_{-\infty}^{0} (t-s)^{-\frac{1}{2}} \exp(-\eta|x_1|) s^{-\frac{3}{4}} (1+s)^{\frac{3}{4}-\frac{d}{4}-\frac{d-1}{4}\left(1-\frac{1}{p}\right)} \exp(-\bar{\eta}|y_1|) dy_1 ds$$
$$\leqq C t^{-\frac{3}{4}} (1+t)^{\frac{3}{4}-\frac{d}{4}-\frac{d-1}{4}\left(1-\frac{1}{p}\right)} \exp(-\eta|x_1|) \int_{t/\gamma}^{t} (t-s)^{-\frac{1}{2}} ds$$
$$\leqq C t^{-\frac{1}{4}} (1+t)^{\frac{3}{4}-\frac{d}{4}-\frac{d-1}{4}\left(1-\frac{1}{p}\right)} \exp(-\eta|x_1|),$$

which for $d \geqq 3$ can be bounded by ψ_5.

Finally, for the excited term, we have

$$\| \int_{0}^{t} \int_{-\infty}^{0} \int_{\mathbb{R}^{d-1}} \mathbf{O}\left((t-s)^{-\frac{d}{4}}\right) \exp(-\eta|x_1|) \exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{\text{eff}}^{-}(t-s)|^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)$$
$$\times \exp\left(-\frac{|y_1+a_1^{-}(t-s)|^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right) \Psi(s,y) dy ds \|_{L^p_{\tilde{x}}}$$
$$\leqq C_1 \int_{0}^{t/\gamma} \int_{-a_1^{-}(t-s)}^{0} (t-s)^{-\frac{d}{4}+\frac{d-1}{4p}} \exp(-\eta|x_1|)$$
$$\times \exp\left(-\frac{|y_1+a_1^{-}(t-s)|^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right) \|\Psi\|_{L^1_{\tilde{y}}} dy_1 ds$$
$$+ C_2 \int_{t/\gamma}^{t} \int_{-a_1^{-}(t-s)}^{0} (t-s)^{-\frac{1}{4}} \exp(-\eta|x_1|)$$
$$\times \exp\left(-\frac{|y_1+a_1^{-}(t-s)|^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right) \|\Psi\|_{L^p_{\tilde{y}}} dy_1 ds. \tag{5.4}$$

For $s \in [0, t/\gamma]$ and the first nonlinearity, we obtain

$$\int_{0}^{t/\gamma} \int_{-a_1^{-}(t-s)}^{0} (t-s)^{-\frac{d}{4}+\frac{d-1}{4p}} \exp(-\eta|x_1|) \exp\left(-\frac{|y_1+a_1^{-}(t-s)|^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)$$
$$\times s^{-\frac{3}{4}} (1+s)^{\frac{3}{4}-\frac{d-1}{4}} (1+|y_1|+s)^{-2r} dy_1 ds,$$

for which we observe the inequality

$$\exp\left(-\frac{|y_1+a_1^{-}(t-s)|^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)(1+|y_1|+s)^{-2r}$$
$$\leqq C(1+|y_1|+t)^{-2r} \exp\left(-\frac{|y_1+a_1^{-}(t-s)|^{\frac{4}{3}}}{M'(t-s)^{\frac{1}{3}}}\right),$$

through which we obtain an estimate by

$$C t^{-\frac{d}{4}+\frac{d-1}{4p}} \exp(-\eta|x_1|) \int_{0}^{t/\gamma} s^{-\frac{3}{4}} (1+t)^{-\frac{d-1}{4}-r} ds$$
$$\leqq C t^{-\frac{d}{4}+\frac{d-1}{4p}} (1+t)^{-\frac{d-1}{4}+1-r} \exp(-\eta|x_1|),$$

which can be bounded by ψ_5. For the second nonlinearity, we have

$$\int_0^{t/\gamma}\int_{-a_1^-(t-s)}^0 (t-s)^{-\frac{d}{4}+\frac{d-1}{4p}} \exp(-\eta|x_1|)\exp\left(-\frac{|y_1+a_1^-(t-s)|^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)$$
$$\times s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d}{4}}\exp(-\bar{\eta}|y_1|)dy_1ds,$$

for which we observe the inequality

$$\exp\left(-\frac{|y_1+a_1^-(t-s)|^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)\exp(-\bar{\eta}|y_1|) \leqq C\exp(-\eta_1|y_1|)\exp^{-\eta_2(t-s)} \tag{5.5}$$

for some constants $\eta_1, \eta_2 > 0$. By the integrability of this last expression in both y_1 and s, we conclude an estimate of form ψ_5. For $s \in [t/\gamma, t]$ and the first nonlinearity, we proceed almost exactly as in the case $s \in [0, t/\gamma]$. For the second nonlinearity, an estimate better than ψ_5 follows from the integrability of (5.5) in both y_1 and s.

Tracking estimates. We now turn to the second estimate of Lemma 3.3, for which we have

$$\begin{aligned}
&\|\int_0^t\int_{-\infty}^0\int_{\mathbb{R}^{d-1}} \mathbf{O}\left((t-s)^{-\frac{d}{4}-\frac{|\alpha|}{4}}\right)\exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{\text{eff}}^-(t-s)|^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)\\
&\quad\times I_{\{|y_1|\leqq|a_1^-|(t-s)\}}\Psi(s,y)dyds\|_{L^p_{\tilde{x}}}\\
&\quad\leqq C_1\int_0^{t/\gamma}\int_{-a_1^-(t-s)}^0 (t-s)^{-\frac{d}{4}+\frac{d-1}{4p}-\frac{|\alpha|}{4}}\|\Psi\|_{L^1_{\tilde{y}}}dy_1ds\\
&\quad+C_2\int_{t/\gamma}^t\int_{-a_1^-(t-s)}^0 (t-s)^{-\frac{1+|\alpha|}{4}}\|\Psi\|_{L^p_{\tilde{y}}}dy_1ds.
\end{aligned} \tag{5.6}$$

For $s \in [0, t/\gamma]$, and for the first nonlinearity, we have

$$\begin{aligned}
&\int_0^{t/\gamma}\int_{-a_1^-(t-s)}^0 (t-s)^{-\frac{d}{4}+\frac{d-1}{4p}-\frac{|\alpha|}{4}}s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d-1}{4}}(1+|y_1|+s)^{-2r}dy_1ds\\
&\quad\leqq Ct^{-\frac{d}{4}+\frac{d-1}{4p}-\frac{|\alpha|}{4}}\int_0^{t/\gamma}s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d-1}{4}-r}ds\\
&\quad\leqq Ct^{-\frac{d}{4}+\frac{d-1}{4p}-\frac{|\alpha|}{4}},
\end{aligned}$$

which is better than the claimed estimate. Once again we observe that the apparent blow-up as $t \to 0$ can be eliminated through an alternative integration of the kernel, and through the observation that our limits of integration vanish as $t \to 0$. For the second nonlinearity, we have

$$\begin{aligned}
&\int_0^{t/\gamma}\int_{-a_1^-(t-s)}^0 (t-s)^{-\frac{d}{4}+\frac{d-1}{4p}-\frac{|\alpha|}{4}}s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d}{4}}e^{-\bar{\eta}|y_1|}dy_1ds\\
&\quad\leqq Ct^{-\frac{d}{4}+\frac{d-1}{4p}-\frac{|\alpha|}{4}}\int_0^{t/\gamma}s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d}{4}}ds\\
&\quad\leqq Ct^{-\frac{d-1}{4}(1-\frac{1}{p})-\frac{|\alpha|}{4}}
\end{aligned}$$

for $d \geqq 3$. For $s \in [t/\gamma, t]$ and the first nonlinearity, we have

$$\int_{t/\gamma}^{t}\int_{-a_1^-(t-s)}^{0}(t-s)^{-\frac{1+|\alpha|}{4}}s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d-1}{4}\left(2-\frac{1}{p}\right)}(1+|y_1|+s)^{-2r}dy_1ds$$
$$\leqq C(1+t)^{-\frac{d-1}{4}\left(2-\frac{1}{p}\right)-2r+1}\int_{t/\gamma}^{t}(t-s)^{-\frac{1+|\alpha|}{4}}s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}}ds$$
$$\leqq Ct^{-\frac{d-1}{4}\left(1-\frac{1}{p}\right)-\frac{|\alpha|}{4}}$$

for $|\alpha| \leqq 3$.

We observe, here, that for t small (e.g., $t < 1$), we can proceed alternatively, taking advantage of the length of the y_1 integration, to write

$$\int_{t/\gamma}^{t}\int_{-a_1^-(t-s)}^{0}(t-s)^{-\frac{1+|\alpha|}{4}}s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d-1}{4}\left(2-\frac{1}{p}\right)}(1+|y_1|+s)^{-2r}dy_1ds$$
$$\leqq C\int_{t/\gamma}^{t}(t-s)^{1-\frac{1+|\alpha|}{4}}s^{-\frac{3}{4}}ds,$$

which is $\mathbf{O}(1)$ as $t \to 0$ for $|\alpha| \leqq 4$. For the second nonlinearity, we have

$$\int_{t/\gamma}^{t}\int_{-a_1^-(t-s)}^{0}(t-s)^{-\frac{1+|\alpha|}{4}}s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d}{4}-\frac{d-1}{4}(1-\frac{1}{p})}\exp(-\bar{\eta}|y_1|)dy_1ds$$
$$\leqq C(1+t)^{-\frac{d}{4}-\frac{d-1}{4}(1-\frac{1}{p})}\int_{t/\gamma}^{t}(t-s)^{-\frac{1+|\alpha|}{4}}s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}}ds$$

for $d \geqq 3$, which is bounded by the claimed estimate, and is the term where the $\log t$ arises for $|\alpha| = 3$. (We observe that derivative estimates for $|\alpha| > 1$ are not critical to the analysis, because these will always be dominated in the iteration by lower-order terms).

Finally, we turn to the time derivative estimate (the third estimate of Lemma 3.3). We have

$$\partial_t\int_0^t\int_{\mathbb{R}^d}\sum_k \tilde{e}_k(t-s,\tilde{x};y)\Psi(s,y)dyds$$
$$=\partial_t\int_0^t\int_{-a_1^-(t-s)}^{0}\int_{\mathbb{R}^{d-1}}\mathbf{O}\left((t-s)^{-\frac{d}{4}}\right)$$
$$\times\exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{\text{eff}}^-(t-s)|^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)\Psi(s,y)d\tilde{y}dy_1ds$$
$$=\int_0^t\int_{-a_1^-(t-s)}^{0}\int_{\mathbb{R}^{d-1}}\mathbf{O}\left((t-s)^{-\frac{d+1}{4}}\right)$$
$$\times\exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{\text{eff}}^-(t-s)|^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)\Psi(s,y)d\tilde{y}dy_1ds$$

$$+\int_0^t \int_{\mathbb{R}^{d-1}} \mathbf{O}\left((t-s)^{-\frac{d}{4}}\right)$$
$$\times \exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{\text{eff}}^-(t-s)|^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)\Psi(s,-a_1^-(t-s),\tilde{y})d\tilde{y}ds. \tag{5.7}$$

For the first of these last two integrals we proceed exactly as in the remainder case above, obtaining an estimate bounded by $(1+t)^{-\frac{d}{4}+\frac{d-1}{4p}}$, as required. For the second, we have

$$\begin{aligned}
&\|\int_0^t \int_{\mathbb{R}^{d-1}} \mathbf{O}\left((t-s)^{-\frac{d}{4}}\right)\exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{\text{eff}}^-(t-s)|^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)\\
&\quad \Psi(s,-a_1^-(t-s),\tilde{y})d\tilde{y}ds\|_{L^p_{\tilde{x}}}\\
&\quad \leqq C_1\int_0^{t/\gamma}(t-s)^{-\frac{d}{4}+\frac{d-1}{4p}}\|\Psi(s,-a_1^-(t-s),\cdot)\|_{L^1_{\tilde{y}}}ds\\
&\quad\quad +C_2\int_{t/\gamma}^t (t-s)^{-\frac{1}{4}}\|\Psi(s,-a_1^-(t-s),\cdot)\|_{L^p_{\tilde{y}}}ds.
\end{aligned} \tag{5.8}$$

For the first nonlinearity, evaluated at $y_1=-a_1^-(t-s)$, we obtain

$$\begin{aligned}
&s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d-1}{4}\left(2-\frac{1}{p}\right)}(1+|a_1^-(t-s)|+s)^{-2r}\\
&\quad \leqq Cs^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d-1}{4}\left(2-\frac{1}{p}\right)}(1+t)^{-2r},
\end{aligned}$$

for which the claimed estimate is clear for $r>1$. For the second nonlinearity, evaluated at $y_1=-a_1^-(t-s)$, we obtain

$$s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d}{4}-\frac{d-1}{4}\left(1-\frac{1}{p}\right)}\exp\left(-\bar{\eta}|a_1^-(t-s)|\right),$$

which gives exponential decay in t on $s\in[0,t/\gamma]$, and through integration of $e^{-\bar{\eta}|a_1^-(t-s)|}$ gives, precisely, the claimed decay for $s\in[t/\gamma,t]$. □

Proof of Lemma 3.4. Under assumptions (H0)–(H2), we have the following standard result on local existence.

Lemma 5.1. *For $u(0,x)\in C^\gamma(x)$, $\gamma>0$, (i.e., Holder continuous), and for $\tau>0$ sufficiently small, there exists a unique solution to (1.1) for $0\leqq t\leqq\tau$ satisfying $u(t,x)\in C^{\frac{\gamma}{4}}(t)\cap C^\gamma(x)$, extending so long as $|u|_{C^{0+\gamma}}$ remains bounded, and satisfying uniform bounds,*

$$|u|_{C^{0+\gamma}(x)}\leqq C,\quad |u|_{C^k(x)}\leqq C\left(\frac{t}{1+t}\right)^{-\frac{k}{4}+\gamma},\quad k=1,2,3,4. \tag{5.9}$$

Remark on the proof of Lemma 5.1. Lemma 5.1 can be established in the present setting of fourth-order regularity in a manner similar to the analysis by GARRONI & MENALDI in the case of second-order regularity (see [4] (Chapter V)). The method of proof consists in writing (1.1) in the linear form,

$$u_t + \sum_{j=1}^{d} (A(t,x)u)_x = -\sum_{jklm} (B^{jklm}(t,x)u_{x_j x_k x_l})_{x_m},$$

where

$$A(t,x) = \int_0^1 \partial_u f^j(\gamma u(t,x))d\gamma$$
$$B^{jklm}(t,x) = b^{jklm}(u(t,x)),$$

and integrating over Green's functions. (See Lemma A.1 in the appendix, and the remarks following it.)

Our perturbation takes the form

$$v(t,x) = u(t,x) - \bar{u}(x_1 - \delta(t,\tilde{x})),$$

and consequently it, and its derivatives, are governed by the behavior of $u(t,x)$, $\bar{u}(x_1)$, and $\delta(t,\tilde{x})$. In this way, the estimates (5.9), and the regular behavior of $\bar{u}(x_1)$, are sufficient for closing a small-time iteration on $D_{\tilde{x}}^{\alpha}\delta$ of (2.1) (in a manner similar to the large-time iterations that we are considering in detail), from which we can conclude $\delta(t,\tilde{x}) \in C^{1+\frac{\gamma}{4}}(t) \cap C^{3+\gamma}(x)$. The claimed regularity on our perturbation $v(t,x)$ follows, then, immediately from the regularity of $u(t,x)$, $\bar{u}(x)$, and $\delta(t,\tilde{x})$. In fact, we have

$$|v|_{C^{0+\gamma}(x)} \leqq C, \quad |v|_{C^k(x)} \leqq C\left(\frac{t}{1+t}\right)^{-\frac{k}{4}+\gamma}, k = 1,2,3,4. \tag{5.10}$$

Fixing now t_0 and τ, for τ sufficiently small, we observe that for $t_0 - \tau \leqq t \leqq t_0$, we can develop our perturbation $v(t,x)$ as an iteration beginning at time $t_0 - \tau$ (i.e., initialized by $v = v(t_0 - \tau, y)$), so that for $|\alpha| \leqq 2$,

$$D_x^{\alpha} v(t,x) = \int_{\mathbb{R}^d} D_x^{\alpha} \tilde{G}(t - (t_0 - \tau), x; y) v(t_0 - \tau, y) dy$$
$$- \int_{t_0-\tau}^{t} \int_{\mathbb{R}^d} \sum_m D_x^{\alpha} \tilde{G}_{y_m}(t - s, x; y) N^m(s,y) dy ds, \tag{5.11}$$

and for $|\alpha| = 3$ we choose the alternative representation:

$$D_x^{\alpha} v(t,x) = \int_{\mathbb{R}^d} D_x^{\alpha} \tilde{G}(t - (t_0 - \tau), x; y) v(t_0 - \tau, y) dy$$
$$+ \int_{t_0-\tau}^{t} \int_{\mathbb{R}^d} \sum_m D_x^{\alpha} \tilde{G}(t - s, x; y) (N^m(s,y))_{y_m} dy ds. \tag{5.12}$$

We observe from (5.10) and (1.5) that for the critical case $|\alpha| = 3$

$$
\begin{aligned}
|D_x^\alpha v(t,x)| \leqq \int_{\mathbb{R}^d} & |D_x^\alpha \tilde{G}(t-(t_0-\tau),x;y)||v(t_0-\tau,y)|dy \\
&+ \int_{t_0-\tau}^{t}\int_{\mathbb{R}^d} |D_x^\alpha \tilde{G}_{y_m}(t-s,x;y)|\Big[|v|(s-(t_0-\tau))^{-1+\gamma} \\
&+ \sum_j |v_{x_j}|(s-(t_0-\tau))^{-\frac{3}{4}+\gamma} + \sum_{jk} |v_{x_j x_k}|(s-(t_0-\tau))^{-\frac{1}{2}+\gamma} \\
&+ \sum_{jkl} |v_{x_j x_k x_l}|(s-(t_0-\tau))^{-\frac{1}{4}+\gamma}\Big] dyds,
\end{aligned}
$$

where we have used the observation that our small- and large-time iterations determine that $D^\alpha \delta \in L^\infty$. Iterating (5.11) and (5.12), we determine

$$
\|D_x^\alpha v\|_{L^p_{\tilde{x}}} \leqq C(t-(t_0-\tau))^{-\frac{\alpha}{4}} \sup_{x_1\in\mathbb{R}} \|v(t_0-\tau,x_1,\cdot)\|_{L^p_{\tilde{x}}},
$$

from which the main estimate of Lemma 3.4 follows, see [13] (Lemmas 11.5 and 11.6) for similar arguments. Finally, we remark that in the last iteration we observe a balance in which for s near $t_0-\tau$ (the lower endpoint of our nonlinear iteration), we have $(t-(t_0-\tau))^{-\frac{|\alpha|}{4}}$ behavior in the nonlinear integral from $G(t-s)$, while for s near t we have slightly better behavior from the nonlinearity. □

Proof of Lemma 4.2 *First estimate of Lemma 4.2.* For $x_1 \leqq 0$, Lemma 4.2 can be established through the calculations involved in the proof of Lemma 3.2. For the case $x_1 \geqq 0$, we must consider new integrals involving signals being carried away from the shock layer. In the case $y_1 \leqq 0 \leqq x_1$, and for the scattering kernel, we have

$$
\begin{aligned}
&\Bigg\| \int_{-\infty}^0 \int_{\mathbb{R}^{d-1}} \mathbf{O}\left(t^{-\frac{d}{4}}\right) \exp\left(-\frac{\left(x_1 - \frac{a_1^+}{a_1^-}y_1 - a_1^+ t\right)^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right) \\
&\quad\times \exp\left(-\frac{|\tilde{x}-\tilde{y}-\left(\frac{\tilde{a}^+-\tilde{a}^-}{a_1^- t}y_1 + \tilde{a}^+\right)t|^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right) v_0(y)d\tilde{y}dy_1 \Bigg\|_{L^p_{\tilde{x}}} \\
&\leqq Ct^{-\frac{d}{4}+\frac{d-1}{4p}} \int_{-\infty}^0 \exp\left(-\frac{\left(x_1 - \frac{a_1^+}{a_1^-}y_1 - a_1^+ t\right)^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right)(1+|y_1|)^{-r} dy_1.
\end{aligned}
$$

In this case, either $\frac{a_1^+}{a_1^-}y_1$ is away from $x_1 - a_1^+ t$, for which we have kernel decay of form

$$Ct^{-\frac{d}{4}+\frac{d-1}{4p}} \exp\left(-\frac{\left(x_1 - a_1^+ t\right)^{\frac{4}{3}}}{Lt^{\frac{1}{3}}}\right)$$

for $L > M$, or $\frac{a_1^+}{a_1^-}y_1$ is near $x_1 - a_1^+ t$, for which we have decay of form

$$Ct^{-\frac{d-1}{4p}\left(1-\frac{1}{p}\right)}(1 + |x_1 - a_1^+ t|)^{-r}.$$

In this final estimate, we can alternatively integrate $v_0(y)$ in y_1 to conclude the estimate

$$Ct^{-\frac{d-1}{4p}\left(1-\frac{1}{p}\right)-\frac{1}{4}}.$$

The case of an outgoing signal for $y_1 \geqq 0$ can be analyzed similarly, and the same estimates arise.

Since derivatives of $\tilde{e}(t, \tilde{x}; y)$ with respect to t and $\tilde{x}$ are the same in the undercompressive case as the compressive case, the second and third estimates of Lemma 4.2 follow from the second and third estimates of Lemma 3.2. $\square$

Proof of Lemma 4.3. In the undercompressive case $x_1, y_1 \leqq 0$, analysis of the scattering and excited kernels proceeds as in the compressive case, and we need only consider integration against the new, slowly decaying, *remainder* kernel of form,

$$\mathbf{O}\left(t^{-\frac{d}{4}}\right)\mathbf{O}(\exp(-\eta(|x_1| + |y_1|))) \exp\left(-\frac{|\tilde{x} - \tilde{y} - \tilde{a}_{\text{eff}}^- t|^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right) I_{\{|y_1| \leqq |a_1^-|t\}}$$

for which

$$\begin{aligned}
&\| \int_0^t \int_{-a_1^-(t-s)}^0 \int_{\mathbb{R}^{d-1}} (t-s)^{-\frac{d}{4}} \exp(-\eta|x_1|) \exp(-\eta|y_1|) \\
&\quad \times \exp\left(-\frac{|\tilde{x} - \tilde{y} - \tilde{a}_{\text{eff}}^- t|^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right) \Psi_-(s, y) dy ds \|_{L^p_{\tilde{x}}} \\
&\leqq C \exp(-\eta|x_1|) \int_0^{t/\gamma} \int_{-a_1^-(t-s)}^0 (t-s)^{-\frac{d}{4}+\frac{d-1}{4p}} \\
&\quad \times \exp(-\eta|y_1|) \|\Psi_-(s, y)\|_{L^1_{\tilde{x}}} dy_1 ds \\
&\quad + C \exp(-\eta|x_1|) \int_{t/\gamma}^t \int_{-a_1^-(t-s)}^0 (t-s)^{-\frac{1}{4}} \\
&\quad \times \exp(-\eta|y_1|) \|\Psi_-(s, y)\|_{L^p_{\tilde{x}}} dy_1 ds.
\end{aligned}$$

For the first nonlinearity, we proceed as in previous cases to determine a solution bound by $C_1 e^{-\eta|x_1|} t^{-\frac{d}{4}+\frac{d-1}{4p}}$. For the second nonlinearity, and for $s \in [0, t/\gamma]$, we have

$$\begin{aligned}
&\exp(-\eta|x_1|) \int_0^{t/\gamma} \int_{-a_1^-(t-s)}^{0} (t-s)^{-\frac{d}{4}+\frac{d-1}{4p}} s^{-3/4} (1+s)^{\frac{3}{4}-\frac{d}{4}} \exp(-\bar{\eta}|y_1|) dy_1 ds \\
&\quad \leqq C \exp(-\eta|x_1|) t^{-\frac{d}{4}+\frac{d-1}{4p}} \int_0^{t/\gamma} s^{-\frac{3}{4}} (1+s)^{\frac{3}{4}-\frac{d}{4}} ds \\
&\quad \leqq C_1 \exp(-\eta|x_1|) t^{-\frac{d}{4}+\frac{d-1}{4p}}, \qquad (5.13)
\end{aligned}$$

where the last inequality is true for $d \geqq 5$. We remark that for $d = 4$, a logarithmic term arises from the time integration. In the compressive waves analysis detailed in [9], the authors accomodated such a term by accepting time decay reduced by a sufficiently small exponent σ. In our case, such an augmentation fails, because the reduced decay worsens with at each iteration. The fundamental difference is that the shift in [9] is chosen through purely linear considerations, and its decay is unaffected by the rate of decay arising through the integrations. For $s \in [t/\gamma, t]$ and for the second nonlinearity, we obtain

$$\begin{aligned}
&\exp(-\eta|x_1|) \int_{t/\gamma}^{t} \int_{-a_1^-(t-s)}^{0} (t-s)^{-\frac{d}{4}+\frac{d-1}{4}} \exp(-\eta|y_1|) \\
&\qquad\qquad \times s^{-\frac{3}{4}} (1+s)^{\frac{3}{4}-\frac{d}{4}-\frac{d-1}{4}(1-\frac{1}{p})} \exp(-\bar{\eta}|y_1|) dy_1 ds \\
&\quad \leqq C \exp(-\eta|x_1|)(1+t)^{-\frac{d}{4}-\frac{d-1}{4}(1-\frac{1}{p})} \int_{t/\gamma}^{t} (t-s)^{-\frac{1}{4}} s^{-\frac{3}{4}} (1+s)^{\frac{3}{4}} ds \\
&\quad \leqq C_1 \exp(-\eta|x_1|)(1+t)^{-\frac{d}{4}-\frac{d-1}{4}(1-\frac{1}{p})+\frac{3}{4}}, \qquad (5.14)
\end{aligned}$$

which satisfies the claimed estimates for $d \geqq 4$.

For undercompressive waves, we must consider separately the case $x_1 \geqq 0$, for which our Green's function estimates describe signals carried away from the shock layer along the outgoing characteristic $a_1^+ > 0$. For $y_1 \leqq 0 \leqq x_1$, the remainder and excited Green's function terms are as in the case $x_1, y_1 \leqq 0$, but we have a genuinely new scattering that describes a signal moving *through* the shock layer. We have

$$\begin{aligned}
&\| \int_0^t \int_{-\infty}^{0} \int_{\mathbb{R}^{d-1}} (t-s)^{-\frac{d+1}{4}} \exp\left(- \frac{\left(x_1 - \frac{a_1^+}{a_1^-} y_1 - a_1^+(t-s) \right)^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}} \right) \\
&\qquad \times \exp\left(- \frac{|\tilde{x} - \tilde{y} - \left(\frac{\tilde{a}^+ - \tilde{a}^-}{a_1^- t} y_1 + \tilde{a}^+ \right)(t-s)|^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}} \right) \Psi_-(s, y) dy ds \|_{L^p_{\tilde{x}}}
\end{aligned}$$

$$\leqq C \int_0^{t/\gamma} \int_{-\infty}^0 (t-s)^{-\frac{d+1}{4}+\frac{d-1}{4p}} \exp\left(-\frac{\left(x_1 - \frac{a_1^+}{a_1^-} y_1 - a_1^+(t-s)\right)^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}} \right)$$

$$\times \|\Psi_-(s,y)\|_{L^1_{\tilde{x}}} dy_1 ds$$

$$+ C \int_{t/\gamma}^t \int_{-\infty}^0 (t-s)^{-\frac{1}{2}} \exp\left(-\frac{\left(x_1 - \frac{a_1^+}{a_1^-} y_1 - a_1^+(t-s)\right)^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}} \right)$$

$$\times \|\Psi_-(s,y)\|_{L^p_{\tilde{x}}} dy_1 ds. \tag{5.15}$$

For $s \in [0, t/\gamma]$, and our first nonlinearity, we have

$$\int_0^{t/\gamma} \int_{-\infty}^0 (t-s)^{-\frac{d+1}{4}+\frac{d-1}{4p}} \exp\left(-\frac{\left(x_1 - \frac{a_1^+}{a_1^-} y_1 - a_1^+(t-s)\right)^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}} \right)$$

$$\times s^{-\frac{3}{4}} (1+s)^{\frac{3}{4}-\frac{d-1}{4}} (1+|y_1|+s)^{-2r} dy_1 ds$$

for which the quantity $-\frac{a_1^+}{a_1^-} y_1 + a_1^+ s$ is always positive. In the event that $x_1 \geqq a_1^+ t$, the exponential kernel is always positive, and we have an estimate by

$$C t^{-\frac{d+1}{4}+\frac{d-1}{4p}} \exp\left(-\frac{(x_1 - a_1^+ t)^{\frac{4}{3}}}{L t^{\frac{1}{3}}} \right) \int_{-\infty}^0 s^{-\frac{3}{4}} (1+s)^{\frac{3}{4}-\frac{d-1}{4}-r} ds$$

$$\leqq C_1 t^{-\frac{d+1}{4}+\frac{d-1}{4p}} \exp\left(-\frac{(x_1 - a_1^+ t)^{\frac{4}{3}}}{L t^{\frac{1}{3}}} \right).$$

In the event that $x_1 \leqq a_1^+ t$, we observe the balance inequality

$$\exp\left(-\frac{\left(x_1 - \frac{a_1^+}{a_1^-} y_1 - a_1^+(t-s)\right)^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}} \right) (1+|y_1|+s)^{-r}$$

$$\leqq C\Big[\exp\left(-\frac{(x_1 - a_1^+ t)^{\frac{4}{3}}}{M'(t-s)^{\frac{1}{3}}} \right) (1+|y_1|+s)^{-r}$$

$$+ \exp\left(-\frac{\left(x_1 - \frac{a_1^+}{a_1^-} y_1 - a_1^+(t-s)\right)^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}} \right) (1+|x_1 - a_1^+ t|)^{-r} \Big], \tag{5.16}$$

where $M' > M$. For the first estimate in (5.16), we have precisely the same analysis as in the case $x_1 \geqq a_1^+ t$, while for the second estimate in (5.16), we compute

$$\begin{aligned}
&\int_0^{t/\gamma}\int_{-\infty}^0 (t-s)^{-\frac{d+1}{4}+\frac{d-1}{4p}} s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d-1}{4}} \\
&\qquad\qquad \times(1+|x_1-a_1^+t|)^{-r}(1+|y_1|+s)^{-r}dy_1ds \\
&\quad \leqq Ct^{-\frac{d+1}{4}+\frac{d-1}{4p}}(1+|x_1-a_1^+t|)^{-r}\int_0^{t/\gamma} s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d-1}{4}+1-r}ds \\
&\quad \leqq Ct^{-\frac{d+1}{4}+\frac{d-1}{4p}+\frac{1}{2}}(1+|x_1-a_1^+t|)^{-r},
\end{aligned}$$

where this last inequality is true for $d \geqq 3$ and can be subsumed into the claimed estimates. (Observe that the alternative minimum estimate can be obtained by integrating $(1+|y_1|+s)^{-2r}$ in both y_1 and s). For this nonlinearity the case $s \in [t/\gamma, t]$ is similar. For the second nonlinearity and for $s \in [0, t/\gamma]$, we have

$$\begin{aligned}
&\int_0^{t/\gamma}\int_{-\infty}^0 (t-s)^{-\frac{d+1}{4}+\frac{d-1}{4p}} \exp\left(-\frac{\left(x_1-\frac{a_1^+}{a_1^-}y_1-a_1^+(t-s)\right)^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right) \\
&\qquad \times s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d}{4}}\exp(-\bar{\eta}|y_1|)dy_1ds.
\end{aligned}$$

In the event that $x_1 \geqq a_1^+ t$, we have no cancellation in the exponential kernel, and we obtain an estimate by

$$\begin{aligned}
&Ct^{-\frac{d+1}{4}+\frac{d-1}{4p}}\exp\left(-\frac{(x_1-a_1^+t)^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right)\int_0^{t/\gamma} s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d}{4}}ds \\
&\qquad \leqq Ct^{-\frac{d}{4}+\frac{d-1}{4p}}\exp\left(-\frac{(x_1-a_1^+t)^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right),
\end{aligned}$$

where in this last inequality we have taken $d \geqq 3$. In the event $x_1 \leqq a_1^+ t$, we observe the balance inequality

$$\begin{aligned}
&\exp\left(-\frac{\left(x_1-\frac{a_1^+}{a_1^-}y_1-a_1^+(t-s)\right)^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)\exp(-\bar{\eta}|y_1|) \\
&\qquad \leqq C\left[\exp\left(-\frac{(x_1-a_1^+(t-s))^{\frac{4}{3}}}{M'(t-s)^{\frac{1}{3}}}\right)\exp(-\bar{\eta}|y_1|)\right.
\end{aligned}$$

$$+\exp\left(-\frac{\left(x_1-\frac{a_1^+}{a_1^-}y_1-a_1^+(t-s)\right)^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)\exp\left(-\eta_1|y_1|\right)$$

$$\times\exp\left(-\eta_2|x_1-a_1^+(t-s)|\right)\Bigg]. \tag{5.17}$$

For the first estimate in (5.17), we have

$$\int_0^{t/\gamma}\int_{-\infty}^0 (t-s)^{-\frac{d+1}{4}+\frac{d-1}{4p}}\exp\left(-\frac{\left(x_1-a_1^+(t-s)\right)^{\frac{4}{3}}}{M'(t-s)^{\frac{1}{3}}}\right)s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d}{4}}$$
$$\times\exp\left(-\bar{\eta}|y_1|\right)dy_1ds,$$

where we now observe the additional balance inequality:

$$\exp\left(-\frac{\left(x_1-a_1^+(t-s)\right)^{\frac{4}{3}}}{M'(t-s)^{\frac{1}{3}}}\right)(1+s)^{-\frac{d}{4}}$$
$$\leqq C\left[\exp\left(-\frac{\left(x_1-a_1^+t\right)^{\frac{4}{3}}}{L(t-s)^{\frac{1}{3}}}\right)(1+s)^{-\frac{d}{4}}\right.$$
$$\left.+\exp\left(-\frac{\left(x_1-a_1^+(t-s)\right)^{\frac{4}{3}}}{M'(t-s)^{\frac{1}{3}}}\right)(1+|x_1-a_1^+t|)^{-\frac{d}{4}}\right]. \tag{5.18}$$

For the first estimate in (5.18), we have an estimate by

$$Ct^{-\frac{d+1}{4}+\frac{d-1}{4p}}\exp\left(-\frac{\left(x_1-a_1^+t\right)^{\frac{4}{3}}}{Lt^{\frac{1}{3}}}\right)\int_0^{t/\gamma}s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d}{4}}ds$$
$$\leqq C_1t^{-\frac{d}{4}+\frac{d-1}{4p}}\exp\left(-\frac{(x_1-a_1^+t)^{\frac{4}{3}}}{Lt^{\frac{1}{3}}}\right),$$

where in this last inequality we have taken $d\geqq 3$. For the second estimate in (5.18), we similarly have an estimate by

$$Ct^{-\frac{d+1}{4}+\frac{d-1}{4p}}(1+|x_1-a_1^+t|)^{-\frac{d}{4}}\int_0^{t/\gamma}\exp\left(-\frac{(x_1-a_1^+(t-s))^{\frac{4}{3}}}{M'(t-s)^{\frac{1}{3}}}\right)s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}}ds$$
$$\leqq C_1t^{-\frac{d}{4}+\frac{d-1}{4p}}(1+|x_1-a_1^+t|)^{-\frac{d}{4}}. \tag{5.19}$$

In this final expression we observe that since we are in the case $x_1\leqq a_1^+t$, t decay additionally yields x_1 decay. The second estimate in (5.17) can be analyzed in a similar fashion to the first. For the case $s \in [t/\gamma, t]$,

$$\int_{t/\gamma}^{t} \int_{-\infty}^{0} (t-s)^{-\frac{1}{2}} \exp\left(-\frac{\left(x_1 - \frac{a_1^+}{a_1^-} y_1 - a_1^+(t-s)\right)^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}} \right)$$
$$\times s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d}{4}-\frac{d-1}{4}\left(1-\frac{1}{p}\right)} \exp(-\bar{\eta}|y_1|) dy_1 ds.$$

In the event that $x_1 \geqq a_1^+ t$, we have no cancellation in the exponential kernel, and we obtain an estimate by

$$C(1+t)^{-\frac{d}{4}-\frac{d-1}{4}\left(1-\frac{1}{p}\right)} \exp\left(-\frac{\left(x_1 - a_1^+ t\right)^{\frac{4}{3}}}{L t^{\frac{1}{3}}} \right)$$
$$\times \int_{t/\gamma}^{t} \exp\left(-\varepsilon \frac{\left(x_1 - a_1^+(t-s)\right)^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}} \right) (t-s)^{-\frac{1}{2}} s^{-\frac{3}{4}} (1+s)^{\frac{3}{4}} ds$$
$$\leqq C(1+t)^{-\frac{d}{4}-\frac{d-1}{4}\left(1-\frac{1}{p}\right)+\frac{1}{4}} \exp\left(-\frac{\left(x_1 - a_1^+ t\right)^{\frac{4}{3}}}{L t^{\frac{1}{3}}} \right),$$

which can be subsumed for $d \geqq 3$. In the event that $x_1 \leqq a_1^+ t$, we proceed from estimate (5.17). For the first estimate in (5.17), we have

$$\int_{t/\gamma}^{t} \int_{-\infty}^{0} (t-s)^{-\frac{1}{2}} \exp\left(-\frac{\left(x_1 - a_1^+(t-s)\right)^{\frac{4}{3}}}{M'(t-s)^{\frac{1}{3}}} \right) s^{-\frac{3}{4}} (1+s)^{\frac{3}{4}-\frac{d}{4}-\frac{d-1}{4}\left(1-\frac{1}{p}\right)}$$
$$\times \exp(-\bar{\eta}|y_1|) dy_1 ds$$
$$\leqq C(1+t)^{-\frac{d}{4}-\frac{d-1}{4}\left(1-\frac{1}{p}\right)} \int_{t/\gamma}^{t} (t-s)^{-\frac{1}{2}}$$
$$\times \exp\left(-\frac{\left(x_1 - a_1^+(t-s)\right)^{\frac{4}{3}}}{M'(t-s)^{\frac{1}{3}}} \right) s^{-\frac{3}{4}} (1+s)^{\frac{3}{4}} ds,$$

for which we observe the balance inequality

$$(t-s)^{-\frac{1}{2}} \exp\left(-\frac{\left(x_1 - a_1^+(t-s)\right)^{\frac{4}{3}}}{M'(t-s)^{\frac{1}{3}}} \right) \leqq C x_1^{-\frac{1}{2}} \exp\left(-\frac{\left(x_1 - a_1^+(t-s)\right)^{\frac{4}{3}}}{M'(t-s)^{\frac{1}{3}}} \right), \tag{5.20}$$

from which we obtain an estimate by

$$C(1+t)^{-\frac{d-1}{4}\left(2-\frac{1}{p}\right)} x_1^{-\frac{1}{2}}. \tag{5.21}$$

The second estimate in (5.17) can be analyzed similarly.

We remark here that in the undercompressive case, two new terms have arisen from the nonlinear analysis, (5.19) and (5.21). These terms must be considered in all cases $y_1 \geqq 0$.

For the undercompressive case $x_1 < 0 < y_1$, a signal begins at y_1 and is convected to the far field at $+\infty$, so that our Green's function decays with exponential rate in both x_1 and y_1. In this way, we obtain an estimate in all subcases by $\psi_5(t, x_1)$.

Critical undercompressive case, $x_1, y_1 \geqq 0$. Finally, we consider the case $x_1, y_1 \geqq 0$. For the scattering term, we have

$$\begin{aligned}
&\| \int_0^t \int_0^\infty \int_{\mathbb{R}^{d-1}} (t-s)^{-\frac{d+1}{4}} \exp\left(-\frac{|x-y-a_+(t-s)|^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right) \Psi_+(s,y)dyds \|_{L^p_{\tilde{x}}} \\
&\leqq C \int_0^{t/\gamma} \int_0^\infty (t-s)^{-\frac{d+1}{4}+\frac{d-1}{4p}} \exp\left(-\frac{\left(x_1-y_1-a_1^+(t-s)\right)^{\frac{4}{3}}}{\bar{M}(t-s)^{\frac{1}{3}}}\right) \\
&\quad \times \|\Psi_+(s,y)\|_{L^1_{\tilde{y}}} dy_1 ds \\
&\quad + C \int_{t/\gamma}^t \int_0^\infty (t-s)^{-\frac{1}{2}} \exp\left(-\frac{\left(x_1-y_1-a_1^+(t-s)\right)^{\frac{4}{3}}}{\bar{M}(t-s)^{\frac{1}{3}}}\right) \\
&\quad \times \|\Psi_+(s,y)\|_{L^p_{\tilde{y}}} dy_1 ds.
\end{aligned} \tag{5.22}$$

In this case, we have five nonlinearities to consider. For the first—taken in order from Lemma 4.3—and for $s \in [0, t/\gamma]$, we have

$$\begin{aligned}
&\int_0^{t/\gamma} \int_{-\infty}^0 (t-s)^{-\frac{d+1}{4}+\frac{d-1}{4p}} \exp\left(-\frac{\left(x_1-y_1-a_1^+(t-s)\right)^{\frac{4}{3}}}{\bar{M}(t-s)^{\frac{1}{3}}}\right) \\
&\qquad \times s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d+1}{4}} \exp\left(-\frac{\left(y_1-a_1^+s\right)^{\frac{4}{3}}}{Ms^{\frac{1}{3}}}\right) dy_1 ds;
\end{aligned}$$

we can write

$$x_1 - y_1 - a_1^+(t-s) = (x - a_1^+ t) - (y_1 - a_1^+ s),$$

from which we obtain

$$\begin{aligned}
&\exp\left(-\frac{\left(x_1-y_1-a_1^+(t-s)\right)^{\frac{4}{3}}}{\bar{M}(t-s)^{\frac{1}{3}}}\right) \exp\left(-\frac{\left(y_1-a_1^+s\right)^{\frac{4}{3}}}{Ms^{\frac{1}{3}}}\right) \\
&\quad \leqq C \exp\left(-\varepsilon \frac{\left(x_1-y_1-a_1^+(t-s)\right)^{\frac{4}{3}}}{\bar{M}(t-s)^{\frac{1}{3}}}\right) \exp\left(-\varepsilon \frac{\left(y_1-a_1^+s\right)^{\frac{4}{3}}}{Ms^{\frac{1}{3}}}\right) \\
&\quad \times \exp\left(-\frac{\left(x_1-a_1^+t\right)^{\frac{4}{3}}}{Lt^{\frac{1}{3}}}\right),
\end{aligned} \tag{5.23}$$

which is slightly less accurate than the analogous expression derived through completing a square in the case of second-order regularity (in particular here, we must take $L > M$), but suffices. Integrating the kernel with divisor $Ms^{\frac{1}{3}}$, we compute

$$\begin{aligned}
&\int_0^{t/\gamma} (t-s)^{-\frac{d+1}{4}+\frac{d-1}{4p}} \exp\left(-\frac{\left(x_1 - a_1^+ t\right)^{\frac{4}{3}}}{Lt^{\frac{1}{3}}}\right) s^{-\frac{1}{2}}(1+s)^{\frac{3}{4}-\frac{d+1}{4}} ds \\
&\quad \leqq Ct^{-\frac{d+1}{4}+\frac{d-1}{4p}} \exp\left(-\frac{\left(x_1 - a_1^+ t\right)^{\frac{4}{3}}}{Lt^{\frac{1}{3}}}\right) \int_0^{t/\gamma} s^{-\frac{1}{2}}(1+s)^{\frac{3}{4}-\frac{d+1}{4}} ds \\
&\quad \leqq C_1 t^{-\frac{d}{4}+\frac{d-1}{4p}} \exp\left(-\frac{\left(x_1 - a_1^+ t\right)^{\frac{4}{3}}}{Lt^{\frac{1}{3}}}\right),
\end{aligned}$$

where in the last inequality we have taken $d \geqq 3$. Likewise, for $s \in [t/\gamma, t]$, using again (5.23) and this time integraing in y_1 the kernel with divisor $M(t-s)^{\frac{1}{3}}$, we have

$$\begin{aligned}
&\int_{t/\gamma}^{t} (t-s)^{-\frac{1}{4}} \exp\left(-\frac{\left(x_1 - a_1^+ t\right)^{\frac{4}{3}}}{Lt^{\frac{1}{3}}}\right) s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d-1}{4}(2-\frac{1}{p})-\frac{1}{2}} dy_1 ds \\
&\quad \leqq Ct^{-\frac{3}{4}}(1+t)^{\frac{3}{4}-\frac{d-1}{4}\left(2-\frac{1}{p}\right)-\frac{1}{2}} \exp\left(-\frac{\left(x_1 - a_1^+ t\right)^{\frac{4}{3}}}{Lt^{\frac{1}{3}}}\right) \int_{t/\gamma}^{t} (t-s)^{-\frac{1}{4}} ds \\
&\quad \leqq C_1 (1+t)^{\frac{1}{4}-\frac{d-1}{4}\left(2-\frac{1}{p}\right)} \exp\left(-\frac{(x_1 - a_1^+ t)^{\frac{4}{3}}}{Lt^{\frac{1}{3}}}\right),
\end{aligned}$$

which is sufficient for $d \geqq 3$. For the second nonlinearity, we proceed similarly, beginning with the balance estimate,

$$\begin{aligned}
&\exp\left(-\frac{\left(x_1 - y_1 - a_1^+(t-s)\right)^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)(1+|y_1 - a_1^+ s|)^{-r} \\
&\quad \leqq C\Big[\exp\left(-\varepsilon\frac{\left(x_1 - y_1 - a_1^+(t-s)\right)^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right) \\
&\qquad \times \exp\left(-\frac{\left(x_1 - a_1^+ t\right)^{\frac{4}{3}}}{Lt^{\frac{1}{3}}}\right)(1+|y_1 - a_1^+ s|)^{-r} \\
&\qquad + \exp\left(-\frac{\left(x_1 - y_1 - a_1^+(t-s)\right)^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)(1+|x_1 - a_1^+ t|)^{-r}\Big]. \qquad (5.24)
\end{aligned}$$

For $s \in [0, t/\gamma]$ and for the first estimate in (5.24), we integrate $(1+|y_1 - a_1^+ s|)^{-r}$ in y_1 to determine an estimate by

$$\begin{aligned}\int_0^{t/\gamma} &(t-s)^{-\frac{d+1}{4}+\frac{d-1}{4p}} \exp\left(-\frac{(x_1 - a_1^+ t)^{\frac{4}{3}}}{Lt^{\frac{1}{3}}}\right) s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d}{4}} ds \\ &\leqq Ct^{-\frac{d+1}{4}+\frac{d-1}{4p}} \exp\left(-\frac{(x_1 - a_1^+ t)^{\frac{4}{3}}}{Lt^{\frac{1}{3}}}\right) \int_0^{t/\gamma} s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d}{4}} ds \\ &\leqq C_1 t^{-\frac{d}{4}+\frac{d-1}{4p}} \exp\left(-\frac{(x_1 - a_1^+ t)^{\frac{4}{3}}}{Lt^{\frac{1}{3}}}\right),\end{aligned}$$

where in the last inequality we have taken $d \geqq 3$. For the second estimate in (5.24), we integrate similarly $(1+|y_1 - a_1^+ s|)^{-r}$ (only one of these has been lost in (5.24)), to determine an estimate by

$$\begin{aligned}\int_0^{t/\gamma} &(t-s)^{-\frac{d+1}{4}+\frac{d-1}{4p}} (1+|x_1 - a_1^+ t|)^{-r} s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d-1}{4}} ds \\ &\leqq Ct^{-\frac{d+1}{4}+\frac{d-1}{4p}} (1+|x_1 - a_1^+ t|)^{-r} \int_0^{t/\gamma} s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d-1}{4}} ds \\ &\leqq C_1 t^{-\frac{d-1}{4}+\frac{d-1}{4p}} (1+|x_1 - a_1^+ t|)^{-r},\end{aligned}$$

where in this last inequality we have again taken $d \geqq 3$. (We can obtain the alternative t decay by replacing $(1+|x_1 - a_1^+ t|)^{-r}$ with $(1+s)^{-\frac{1}{4}}$). For $s \in [t/\gamma, t]$ and the first estimate in (5.24), we integrate the kernel with divisor $M(t-s)^{\frac{1}{3}}$ (also, replacing $(1+|y_1 - a_1^+ s|)^{-r}$ with its alternative $(1+s)^{-1/4}$), to determine an estimate by

$$\int_{t/\gamma}^{t} (t-s)^{-\frac{1}{4}} \exp\left(-\frac{(x_1 - a_1^+ t)^{\frac{4}{3}}}{Lt^{\frac{1}{3}}}\right) s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d-1}{4}\left(2-\frac{1}{p}\right)-\frac{1}{2}} dy_1 ds,$$

which has been analyzed above, and is sufficient for $d \geqq 3$. For the second estimate in (5.24), we proceed similarly to determine an estimate by

$$\begin{aligned}\int_{t/\gamma}^{t} &(t-s)^{-\frac{1}{4}} (1+|x_1 - a_1^+ t|)^{-r} s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d-1}{4}\left(2-\frac{1}{p}\right)-\frac{1}{4}} dy_1 ds \\ &\leqq Ct^{-\frac{3}{4}}(1+t)^{\frac{3}{4}-\frac{d-1}{4}\left(2-\frac{1}{p}\right)-\frac{1}{4}} (1+|x_1 - a_1^+ t|)^{-r} \int_{t/\gamma}^{t} (t-s)^{-\frac{1}{4}} ds \\ &\leqq C_1 (1+t)^{\frac{1}{2}-\frac{d-1}{4}\left(2-\frac{1}{p}\right)} (1+|x_1 - a_1^+ t|)^{-r},\end{aligned}$$

which is sufficient for $d \geqq 3$.

For the third nonlinearity, we begin with the balance estimate

$$\begin{aligned}
&\exp\left(-\frac{(x_1-y_1-a_1^+(t-s))^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)\exp(-\bar{\eta}|y_1|)\\
&\leqq C\left[\exp\left(-\frac{(x_1-a_1^+(t-s))^{\frac{4}{3}}}{M'(t-s)^{\frac{1}{3}}}\right)\exp(-\bar{\eta}|y_1|)\right.\\
&\quad+\exp\left(-\varepsilon\frac{(x_1-y_1-a_1^+(t-s))^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)\exp\left(-\eta_1|x_1-a_1^+(t-s)|\right)\\
&\quad\left.\times\exp(-\eta_2|y_1|)\right].
\end{aligned}\tag{5.25}$$

For $s\in[0,t/\gamma]$ and the first estimate in (5.25), we have

$$\begin{aligned}
&\int_0^{t/\gamma}\int_{-\infty}^0(t-s)^{-\frac{d+1}{4}+\frac{d-1}{4p}}\exp\left(-\frac{\left(x_1-a_1^+(t-s)\right)^{\frac{4}{3}}}{M'(t-s)^{\frac{1}{3}}}\right)(1+s)^{-\frac{d}{4}}\\
&\quad\times\exp(-\bar{\eta}|y_1|)dy_1ds.
\end{aligned}$$

In the event that $x_1\geqq a_1^+t$, we have no cancellation in the kernel and we determine an estimate by

$$\begin{aligned}
&Ct^{-\frac{d+1}{4}+\frac{d-1}{4p}}\exp\left(-\frac{\left(x_1-a_1^+t\right)^{\frac{4}{3}}}{M't^{\frac{1}{3}}}\right)\int_0^{t/\gamma}(1+s)^{-\frac{d}{4}}ds\\
&\quad\leqq Ct^{-\frac{d}{4}+\frac{d-1}{4p}}\exp\left(-\frac{\left(x_1-a_1^+t\right)^{\frac{4}{3}}}{M't^{\frac{1}{3}}}\right).
\end{aligned}$$

In the event that $x_1\leqq a_1^+t$, we have the estimate

$$\begin{aligned}
&\exp\left(-\frac{\left(x_1-a_1^+(t-s)\right)^{\frac{4}{3}}}{M'(t-s)^{\frac{1}{3}}}\right)(1+s)^{-\frac{d}{4}}\\
&\quad\leqq C\Bigg[\exp\left(-\frac{(x_1-a_1^+t)^{\frac{4}{3}}}{L(t-s)^{\frac{1}{3}}}\right)(1+s)^{-\frac{d}{4}}\\
&\quad\quad+\exp\left(-\frac{\left(x_1-a_1^+(t-s)\right)^{\frac{4}{3}}}{M'(t-s)^{\frac{1}{3}}}\right)(1+|x_1-a_1^+t|)^{-\frac{d}{4}}\Bigg].
\end{aligned}\tag{5.26}$$

For the first estimate in (5.26), we proceed exactly as in the case $x_1\geqq a_1^+t$ and obtain the same estimate. For the second estimate in (5.26) we have an estimate by

$$\begin{aligned}
&Ct^{-\frac{d+1}{4}+\frac{d-1}{4p}}(1+|x_1-a_1^+t|)^{-\frac{d}{4}}\int_0^{t/\gamma}\exp\left(-\frac{\left(x_1-a_1^+(t-s)\right)^{\frac{4}{3}}}{M'(t-s)^{\frac{1}{3}}}\right)ds\\
&\quad\leqq Ct^{-\frac{d}{4}+\frac{d-1}{4p}}(1+|x_1-a_1^+t|)^{-\frac{d}{4}}.
\end{aligned}$$

For $s \in [0, t/\gamma]$ and the second estimate in (5.25), we have

$$\int_0^{t/\gamma} \int_{-\infty}^0 (t-s)^{-\frac{d+1}{4}+\frac{d-1}{4p}} \exp\left(-\eta_1 |x_1 - a_1^+(t-s)|\right) \times \exp^{(-\eta_2 |y_1|)} (1+s)^{-\frac{d}{4}} dy_1 ds,$$

which can be analyzed in a manner similar to that of the previous case. For $s \in [t/\gamma, t]$ and the first estimate in (5.25), we have

$$\int_{t/\gamma}^t \int_{-\infty}^0 (t-s)^{-\frac{1}{2}} \exp\left(-\frac{\left(x_1 - a_1^+(t-s)\right)^{\frac{4}{3}}}{M'(t-s)^{\frac{1}{3}}}\right) \times \exp\left(-\bar{\eta}|y_1|\right) s^{-\frac{3}{4}} (1+s)^{\frac{3}{4}-\frac{d}{4}-\frac{d-1}{4}(1-\frac{1}{p})} dy_1 ds.$$

In the event that $x_1 \geqq a_1^+ t$, we have no cancellation in the kernel and obtain an estimate by

$$C(1+t)^{-\frac{d}{4}-\frac{d-1}{4}\left(1-\frac{1}{p}\right)} \exp\left(-\frac{\left(x_1 - a_1^+ t\right)^{\frac{4}{3}}}{L t^{\frac{1}{3}}}\right) \int_{t/\gamma}^t (t-s)^{-\frac{1}{2}} ds \leqq C_1 (1+t)^{-\frac{d}{4}-\frac{d-1}{4}\left(1-\frac{1}{p}\right)+\frac{1}{2}} \exp\left(-\frac{\left(x_1 - a_1^+ t\right)^{\frac{4}{3}}}{L t^{\frac{1}{3}}}\right),$$

which is sufficient for $d \geqq 3$. In the event that $x_1 \leqq a_1^+ t$, we observe the inequality

$$(t-s)^{-\frac{1}{2}} \exp\left(-\frac{\left(x_1 - a_1^+(t-s)\right)^{\frac{4}{3}}}{M'(t-s)^{\frac{1}{3}}}\right) \leqq C x_1^{-1/2} \exp\left(-\frac{\left(x_1 - a_1^+(t-s)\right)^{\frac{4}{3}}}{M'(t-s)^{\frac{1}{3}}}\right). \tag{5.27}$$

We have, then, an estimate by

$$C(1+t)^{-\frac{d}{4}-\frac{d-1}{4}\left(1-\frac{1}{p}\right)} x_1^{-\frac{1}{2}} \int_{t/\gamma}^t \exp\left(-\frac{\left(x_1 - a_1^+(t-s)\right)^{\frac{4}{3}}}{M'(t-s)^{\frac{1}{3}}}\right)(t-s)^{-\frac{1}{2}} s^{-\frac{3}{4}} (1+s)^{\frac{3}{4}} ds \leqq C(1+t)^{-\frac{d-1}{4}\left(2-\frac{1}{p}\right)} x_1^{-\frac{1}{2}}.$$

For the second estimate in (5.25), we proceed similarly.

For the fourth nonlinearity, we proceed from the inequality

$$
\begin{aligned}
&\exp\left(-\frac{\big(x_1 - y_1 - a_1^+(t-s)\big)^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)(1+|y_1 - a_1^+ s|)^{-\frac{d}{4}} \\
&\quad \leqq C\Big[\exp\left(-\varepsilon\frac{\big(x_1 - y_1 - a_1^+(t-s)\big)^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)\exp\left(-\frac{\big(x_1 - a_1^+ t\big)^{\frac{4}{3}}}{Lt^{\frac{1}{3}}}\right) \\
&\quad\quad \times(1+|y_1 - a_1^+ s|)^{-\frac{d}{4}} \\
&\quad\quad +\exp\left(-\frac{\big(x_1 - y_1 - a_1^+(t-s)\big)^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)(1+|x_1 - a_1^+ t|)^{-\frac{d}{4}}\Big]. \qquad (5.28)
\end{aligned}
$$

For $s \in [0, t/\gamma]$ and the first estimate in (5.28), with $d \geqq 3$, we integrate $(1+|y_1 - a_1^+ s|)^{-\frac{d}{2}}$ in y_1 to obtain an estimate by

$$
\begin{aligned}
&Ct^{-\frac{d+1}{4}+\frac{d-1}{4p}}\exp\left(-\frac{\big(x_1 - a_1^+ t\big)^{\frac{4}{3}}}{Lt^{\frac{1}{3}}}\right)\int_0^{t/\gamma}(1+s)^{-\frac{d}{2}+\frac{d-1}{4p}}ds \\
&\quad \leqq C_1 t^{-\frac{d}{4}+\frac{d-1}{4p}}\exp\left(-\frac{\big(x_1 - a_1^+ t\big)^{\frac{4}{3}}}{Lt^{\frac{1}{3}}}\right).
\end{aligned}
$$

For the second estimate in (5.28), we observe that since $y_1 \leqq a_1^+ s$, we have no cancellation in the kernel for $x_1 \geqq a_1^+ t$, and consequently can proceed exactly as above. For the case $x_1 \leqq a_1^+ t$, we integrate the remaining $(1+|y_1 - a_1^+ s|)^{-\frac{d}{4}}$ in y_1 (on the bounded interval $y_1 \in [0, a_1^+ s]$) to determine an estimate (for $d \geqq 3$) by

$$
\begin{aligned}
&Ct^{-\frac{d+1}{4}+\frac{d-1}{4p}}(1+|x_1 - a_1^+ t|)^{-\frac{d}{4}}\int_0^{t/\gamma}(1+s)^{-\frac{d}{2}+\frac{d-1}{4p}+\frac{1}{4}}dy_1 ds \\
&\quad \leqq C_1 t^{-\frac{d}{4}+\frac{d-1}{4p}}(1+|x_1 - a_1^+ t|)^{-\frac{d}{4}}.
\end{aligned}
$$

For $s \in [t/\gamma, t]$ and the first estimate in (5.28), we integrate the kernel with divisor $M(t-s)^{\frac{1}{3}}$ to determine an estimate by

$$
\begin{aligned}
&C(1+t)^{-\frac{d}{2}+\frac{d-1}{4p}}\exp\left(-\frac{\big(x_1 - a_1^+ t\big)^{\frac{4}{3}}}{Lt^{\frac{1}{3}}}\right)\int_{t/\gamma}^{t}(t-s)^{-\frac{1}{4}}ds \\
&\quad \leqq C_1(1+t)^{-\frac{d}{2}+\frac{d-1}{4p}+\frac{3}{4}}\exp\left(-\frac{\big(x_1 - a_1^+ t\big)^{\frac{4}{3}}}{Lt^{\frac{1}{3}}}\right),
\end{aligned}
$$

which is sufficient for $d \geqq 3$. For the second estimate in (5.28), we observe again the lack of cancellation for $x_1 \geqq a_1^+ t$, and proceed exactly as in the case of the first estimate. For the case $x_1 \leqq a_1^+ t$, we have an estimate by

$$C(1+t)^{-\frac{d}{2}+\frac{d-1}{4p}}(1+|x_1-a_1^+t|)^{-\frac{d}{4}}\int_{t/\gamma}^{t}(t-s)^{-\frac{1}{4}}ds$$
$$\leqq C_1(1+t)^{-\frac{d}{2}+\frac{3}{4}+\frac{d-1}{4p}}(1+|x_1-a_1^+t|)^{-\frac{d}{4}},$$

which is sufficient for $d \geqq 3$.

For the fifth and final nonlinearity, we first observe that for $x_1 \geqq a_1^+ t$, (and $y_1 \leqq a_1^+ s$), we have no cancellation in the kernal decay, and immediately determine an estimate by

$$C_1 t^{-\frac{d+1}{4}+\frac{d-1}{4p}}\exp\left(-\frac{\left(x_1-a_1^+t\right)^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right)\int_0^{t/\gamma}(1+s)^{-\frac{3}{4}(d-1)+\frac{1}{4}}ds$$
$$+C_2(1+t)^{-\frac{d-1}{4}\left(4-\frac{1}{p}\right)}\exp\left(-\frac{\left(x_1-a_1^+t\right)^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right)\int_{t/\gamma}^{t}(t-s)^{-\frac{1}{4}}ds$$
$$\leqq C_3 t^{-\frac{d}{4}+\frac{d-1}{4p}}\exp\left(-\frac{\left(x_1-a_1^+t\right)^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right).$$

For $x_1 \leqq a_1^+ t$, we observe the inequality,

$$\exp\left(-\frac{\left(x_1-y_1-a_1^+(t-s)\right)^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)(1+|y_1|+s)^{-\frac{d-1}{4}\left(4-\frac{1}{p}\right)}(1+|y_1|)^{-1}$$
$$\leqq C\Big[\exp\left(-\varepsilon\frac{\left(x_1-y_1-a_1^+(t-s)\right)^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)\exp\left(-\frac{\left(x_1-a_1^+(t-s)\right)^{\frac{4}{3}}}{M't^{\frac{1}{3}}}\right)$$
$$\times(1+|y_1|+s)^{-\frac{d-1}{4}\left(4-\frac{1}{p}\right)}(1+|y_1|)^{-1}$$
$$+\exp\left(-\frac{\left(x_1-y_1-a_1^+(t-s)\right)^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)(1+|x_1-a_1^+t|+s)^{-\frac{d-1}{4}\left(4-\frac{1}{p}\right)}$$
$$\times(1+|y_1|+|x_1-a_1^+(t-s)|)^{-1}\Big]. \tag{5.29}$$

For the first estimate in (5.29), we integrate $(1+|y_1|)^{-1}$ in y_1 (on the bounded interval $y_1 \in [0, a_1^+ s]$) to determine an estimate by

$$Ct^{-\frac{d+1}{4}+\frac{d-1}{4p}}\int_0^{t/\gamma}\exp\left(-\frac{\left(x_1-a_1^+(t-s)\right)^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)(1+s)^{-\frac{3}{4}(d-1)}\ln(e+s)ds,$$
$$+C(1+t)^{-\frac{d-1}{4}\left(4-\frac{1}{p}\right)}\ln(e+t)\int_{t/\gamma}^{t}(t-s)^{-\frac{1}{2}}\exp\left(-\frac{\left(x_1-a_1^+(t-s)\right)^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)ds.$$

For $s \in [0, t/\gamma]$, we observe the inequality

$$\begin{aligned}
&\exp\left(-\frac{\left(x_1 - a_1^+(t-s)\right)^{\frac{4}{3}}}{M'(t-s)^{\frac{1}{3}}}\right)(1+s)^{-\frac{d}{4}} \\
&\quad \leqq C\Big[\exp\left(-\varepsilon\frac{\left(x_1 - a_1^+(t-s)\right)^{\frac{4}{3}}}{M'(t-s)^{\frac{1}{3}}}\right)\exp\left(-\frac{\left(x_1 - a_1^+ t\right)^{\frac{4}{3}}}{Lt^{\frac{1}{3}}}\right)(1+s)^{-\frac{d}{4}} \\
&\qquad + \exp\left(-\frac{\left(x_1 - a_1^+(t-s)\right)^{\frac{4}{3}}}{M'(t-s)^{\frac{1}{3}}}\right)(1+|x_1 - a_1^+ t|)^{-\frac{d}{4}}\Big]. \qquad (5.30)
\end{aligned}$$

For the first estimate in (5.30), we have an estimate by

$$\begin{aligned}
&Ct^{-\frac{d+1}{4}+\frac{d-1}{4p}}\exp\left(-\frac{\left(x_1 - a_1^+ t\right)^{\frac{4}{3}}}{Lt^{\frac{1}{3}}}\right)\int_0^{t/\gamma}(1+s)^{-\frac{3}{4}(d-1)}\ln(e+s)ds \\
&\quad \leqq C_1 t^{-\frac{d+1}{4}+\frac{d-1}{4p}}\exp\left(-\frac{\left(x_1 - a_1^+ t\right)^{\frac{4}{3}}}{Lt^{\frac{1}{3}}}\right),
\end{aligned}$$

for $d \geqq 3$. For the second estimate in (5.30), we have an estimate by

$$\begin{aligned}
&Ct^{-\frac{d+1}{4}+\frac{d-1}{4p}}(1+|x_1 - a_1^+ t|)^{-\frac{d}{4}} \\
&\quad \int_0^{t/\gamma}(1+s)^{\frac{3}{4}-\frac{d}{2}}\ln(e+s)\exp\left(-\frac{\left(x_1 - a_1^+(t-s)\right)^{\frac{4}{3}}}{M't^{\frac{1}{3}}}\right)ds \\
&\quad \leqq C_1 t^{-\frac{d}{4}+\frac{d-1}{4p}}(1+|x_1 - a_1^+ t|)^{-\frac{d}{4}}.
\end{aligned}$$

For $s \in [t/\gamma, t]$, we observe the inequality

$$(t-s)^{-\frac{1}{2}}\exp\left(-\frac{\left(x_1 - a_1^+(t-s)\right)^{\frac{4}{3}}}{M'(t-s)^{\frac{1}{3}}}\right) \leqq Cx_1^{-\frac{1}{2}}\exp\left(-\frac{\left(x_1 - a_1^+(t-s)\right)^{\frac{4}{3}}}{M'(t-s)^{\frac{1}{3}}}\right).$$

Integrating the kernel, we determine a final estimate by

$$C_1(1+t)^{-\frac{d-1}{4}\left(4-\frac{1}{p}\right)+\frac{1}{2}}\ln(e+t)x_1^{-\frac{1}{2}},$$

which is sufficient for $d \geqq 3$. For the second estimate in (5.29) and for $s \in [0, t/\gamma]$, we integrate $(1+|y_1|)^{-\frac{3}{4}}$ (to avoid the ln) to determine an estimate by

$$Ct^{-\frac{d+1}{4}+\frac{d-1}{4p}} \int_0^{t/\gamma} (1+|x_1 - a_1^+ t| + s)^{-\frac{3}{4}(d-1)+\frac{1}{4}} (1+|x_1 - a_1^+(t-s)|)^{-\frac{1}{4}} ds.$$

$$= Ct^{-\frac{d+1}{4}+\frac{d-1}{4p}} \int_0^{\varepsilon|x_1 - a_1^+ t|} (1+|x_1 - a_1^+ t| + s)^{-\frac{3}{4}(d-1)+\frac{1}{4}}$$
$$\times (1+|x_1 - a_1^+(t-s)|)^{-\frac{1}{4}} ds$$
$$+ Ct^{-\frac{d+1}{4}+\frac{d-1}{4p}} \int_{\varepsilon|x_1 - a_1^+ t|}^{\frac{2}{a_1^+}|x_1 - a_1^+ t|} (1+|x_1 - a_1^+ t| + s)^{-\frac{3}{4}(d-1)+\frac{1}{4}}$$
$$\times (1+|x_1 - a_1^+(t-s)|)^{-\frac{1}{4}} ds$$
$$+ Ct^{-\frac{d+1}{4}+\frac{d-1}{4p}} \int_{\frac{2}{a_1^+}|x_1 - a_1^+ t|}^{t/\gamma} (1+|x_1 - a_1^+ t| + s)^{-\frac{3}{4}(d-1)+\frac{1}{4}}$$
$$\times (1+|x_1 - a_1^+(t-s)|)^{-\frac{1}{4}} ds$$
$$\leqq Ct^{-\frac{d+1}{4}+\frac{d-1}{4p}} (1+|x_1 - a_1^+ t|)^{1-\frac{3}{4}(d-1)},$$

for $\varepsilon > 0$ sufficiently small which is sufficient for $d \geqq 3$. (In the event that $t/\gamma \leqq \frac{2}{a_1^+}|x_1 - a_1^+ t|$, we proceed similarly, altering the second integral and omitting the third.) Finally, for $s \in [t/\gamma, t]$, integrating $(1+|y_1|)^{-\frac{1}{2}}$, we have an estimate by

$$C(1+t)^{-\frac{d-1}{4}\left(4-\frac{1}{p}\right)+\frac{1}{2}} \int_{t/\gamma}^{t} (t-s)^{-\frac{1}{2}} (1+|x_1 - a_1^+(t-s)|)^{-\frac{1}{2}} ds$$
$$\leqq (1+t)^{-\frac{d-1}{4}\left(4-\frac{1}{p}\right)+1} x_1^{-\frac{1}{2}},$$

which is sufficient for $d \geqq 3$.

Remainder estimates. We next consider the remainder term estimates in the case $x_1, y_1 \geqq 0$, for which we have,

$$\| \int_0^t \int_0^{+\infty} \int_{\mathbb{R}^{d-1}} (t-s)^{-\frac{d}{4}} \exp(-\eta|x_1|) \exp(-\eta|y_1|)$$
$$\times \exp\left(-\frac{|\tilde{x} - \tilde{y} - a_{\text{eff}}^+(t-s)|^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}} \right) \Psi_+(s,y) dy ds \|_{L^p_{\tilde{x}}}$$
$$\leqq C \int_0^{t/\gamma} \int_0^{+\infty} (t-s)^{-\frac{d}{4}+\frac{d-1}{4p}} \exp(-\eta|x_1|) \exp(-\eta|y_1|) \|\Psi_+(s,y)\|_{L^1_{\tilde{x}}} dy_1 ds$$
$$+ C \int_{t/\gamma}^{t} \int_0^{+\infty} (t-s)^{-\frac{1}{4}} \exp(-\eta|x_1|) \exp(-\eta|y_1|) \|\Psi_+(s,y)\|_{L^p_{\tilde{x}}} dy_1 ds.$$

For the first nonlinearity, we observe the estimate

$$\exp\left(-\frac{(y_1 - a_1^+ s)^{\frac{4}{3}}}{Ls^{\frac{1}{3}}} \right) \exp(-\eta|y_1|) \leqq C \exp(-\eta_1 s) \exp(-\eta_2 |y_1|)$$

for some $\eta_1 > 0$ and $\eta_2 > 0$. Integrating this last expression in both y_1 and s, we develop an estimate by $Ct^{-\frac{d}{4}+\frac{d-1}{4p}}e^{-\eta|x_1|}$, valid for all $d \geqq 1$. For the second nonlinearity, we proceed similarly from the inequality

$$(1+|y_1-a_1^+s|)^{-2r}\exp(-\eta|y_1|) \leqq C(1+s)^{-2r}\exp(-\eta_1|y_1|),$$

again integrable in both s and y_1. For the third nonlinearity, we proceed exactly as in (5.13) and (5.14) to determine an estimate only sufficient for $d \geqq 5$. For the fourth nonlinearity, we observe the estimate

$$(1+|y_1-a_1^+s|)^{-\frac{d}{2}}\exp(-\eta|y_1|) \leqq C(1+s)^{-\frac{d}{2}}\exp(-\eta_1|y_1|),$$

from which we obtain an estimate by $Ct^{-\frac{d}{4}+\frac{d-1}{4p}}e^{-\eta|x_1|}$ for all $d \geqq 1$. Finally, for the fifth nonlinearity, we have the estimate

$$\begin{aligned} C_1 t^{-\frac{d}{4}+\frac{d-1}{4p}}\exp(-\eta|x_1|)\int_0^{t/\gamma}(1+s)^{-\frac{3}{4}(d-1)}ds \\ +C_2(1+t)^{-\frac{d-1}{4}\left(4-\frac{1}{p}\right)}\exp(-\eta|x_1|)\int_{t/\gamma}^{t}(t-s)^{-\frac{1}{4}}ds \\ \leqq Ct^{-\frac{d}{4}+\frac{d-1}{4p}}\exp(-\eta|x_1|), \end{aligned}$$

where we have taken $d \geqq 3$.

Excited estimates. Finally, for the excited term in the undercompressive case with $y_1 > 0$, we observe that

$$\begin{aligned} \exp(-\eta|y_1|)\exp\left(-\frac{\left(y_1-a_1^+(t-s)\right)^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right) \\ \leqq C\exp(-\eta_1(t-s))\exp(-\eta_2|y_1|), \end{aligned} \tag{5.31}$$

for some $\eta_1 > 0$ and some $\eta_2 > 0$, from which an estimate by $Ct^{-\frac{d}{4}+\frac{d-1}{4p}}e^{-\eta|x_1|}$ follows immediately for all $d \geqq 1$. We observe here that the only difference between the compressive case and the undercompressive case with regard to decay in time is that $(t-s)^{-\frac{d+1}{4}}$ is replaced by $(t-s)^{-\frac{d}{4}}$, the need for either is obviated by (5.31).

Tracking estimates (Second and third estimates of Lemma 4.3.). In the case $y_1 \leqq 0$, and for the second estimate in Lemma 4.3, the nonlinearities for the undercompressive case are identical to those for the compressive case, and the term in $\partial_{\tilde{x}}^{\alpha}\tilde{e}_k(t,\tilde{x};y)$, of form

$$\mathbf{O}(t^{-\frac{d}{4}-\frac{|\alpha|}{4}})\exp\left(-\frac{\left(\tilde{x}-\tilde{y}-\tilde{a}_{\text{eff}}^-t\right)^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}\right)I_{\{|y_1|\leqq|a_1^-|t\}}, \tag{5.32}$$

can be analyzed exactly as before (see (5.6) and the analysis that follows). For the new term in $\partial_{\tilde{x}}^{\alpha}\tilde{e}_k(t,\tilde{x};y)$, we have

$$\begin{aligned}
&\| \int_0^t \int_{-\infty}^0 \int_{\mathbb{R}^{d-1}} \mathbf{O}\left((t-s)^{-\frac{d-1}{4}-\frac{|\alpha|}{4}}\right) \exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{\text{eff}}^{-}(t-s)|^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right) \\
&\quad \times \exp(-\eta|y_1|) I_{\{|y_1|\leqq|a_1^-|(t-s)\}} \Psi_-(s,y)dyds\|_{L^p_{\tilde{x}}} \\
&\quad \leqq C_1 \int_0^{t/\gamma} \int_{-a_1^-(t-s)}^0 (t-s)^{-\frac{d-1}{4}+\frac{d-1}{4p}-\frac{|\alpha|}{4}} \exp(-\eta|y_1|)\|\Psi_-\|_{L^1_{\tilde{y}}} dy_1 ds \\
&\quad + C_2 \int_{t/\gamma}^t \int_{-a_1^-(t-s)}^0 (t-s)^{-\frac{|\alpha|}{4}} \exp(-\eta|y_1|)\|\Psi_-\|_{L^p_{\tilde{y}}} dy_1 ds. \qquad (5.33)
\end{aligned}$$

For the first nonlinearity, we have

$$\begin{aligned}
&\int_0^{t/\gamma} \int_{-a_1^-(t-s)}^0 (t-s)^{-\frac{d-1}{4}+\frac{d-1}{4p}-\frac{|\alpha|}{4}} \exp(-\eta|y_1|) s^{-\frac{3}{4}} \\
&\qquad \times (1+s)^{\frac{3}{4}-\frac{d-1}{4}} (1+|y_1|+s)^{-2r} dy_1 ds \\
&\qquad + \int_{t/\gamma}^t \int_{-a_1^-(t-s)}^0 (t-s)^{-\frac{|\alpha|}{4}} \exp(-\eta|y_1|) s^{-\frac{3}{4}} \\
&\qquad \times (1+s)^{\frac{3}{4}-\frac{d-1}{4}\left(2-\frac{1}{p}\right)} (1+|y_1|+s)^{-2r} dy_1 ds \\
&\quad \leqq C_1 t^{-\frac{d-1}{4}(1-\frac{1}{p})-\frac{|\alpha|}{4}} \int_0^{t/\gamma} s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d-1}{4}-2r} ds \\
&\qquad + C_2(1+t)^{-\frac{d-1}{4}\left(2-\frac{1}{p}\right)-2r} \int_{t/\gamma}^t (t-s)^{-\frac{|\alpha|}{4}} s^{-\frac{3}{4}}(1+s)^{-\frac{3}{4}} ds \\
&\quad \leqq C t^{-\frac{d-1}{4}\left(1-\frac{1}{p}\right)-\frac{|\alpha|}{4}},
\end{aligned}$$

for any $d \geqq 1$. For the second nonlinearity, we have

$$\begin{aligned}
&\int_0^{t/\gamma} \int_{-a_1^-(t-s)}^0 (t-s)^{-\frac{d-1}{4}+\frac{d-1}{4p}-\frac{|\alpha|}{4}} \exp(-\eta|y_1|) s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d}{4}} dy_1 ds \\
&\quad + \int_{t/\gamma}^t \int_{-a_1^-(t-s)}^0 (t-s)^{-\frac{|\alpha|}{4}} \exp(-\eta|y_1|) s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d}{4}-\frac{d-1}{4}\left(1-\frac{1}{p}\right)} dy_1 ds \\
&\leqq C_1 t^{-\frac{d-1}{4}\left(1-\frac{1}{p}\right)-\frac{|\alpha|}{4}} \int_0^{t/\gamma} s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d}{4}} ds \\
&\quad + C_2(1+t)^{-\frac{d}{4}-\frac{d-1}{4}\left(1-\frac{1}{p}\right)} \int_{t/\gamma}^t (t-s)^{-\frac{|\alpha|}{4}} s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}} ds \\
&\leqq C t^{-\frac{d-1}{4}\left(1-\frac{1}{p}\right)-\frac{|\alpha|}{4}},
\end{aligned}$$

where we require $d \geqq 5$ in the first integral and $|\alpha| \leqq 3$ in the second.

In the undercompressive case $y_1 \geqq 0$, our excited kernels all decay at exponential rate in $|y_1|$, and we need only consider estimates

$$\begin{aligned}
&\| \int_0^t \int_0^{+\infty} \int_{\mathbb{R}^{d-1}} \mathbf{O}\left((t-s)^{-\frac{d-1}{4}-\frac{|\alpha|}{4}}\right) \exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{\text{eff}}^{+}(t-s)|^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right) \\
&\quad \times \exp(-\eta|y_1|) I_{\{|y_1| \leqq |a_1^+|(t-s)\}} \Psi_+(s,y) dy ds \|_{L^p_{\tilde{x}}} \\
&\leqq C_1 \int_0^{t/\gamma} \int_0^{a_1^+(t-s)} (t-s)^{-\frac{d-1}{4}+\frac{d-1}{4p}-\frac{|\alpha|}{4}} \exp(-\eta|y_1|) \|\Psi_+\|_{L^1_{\tilde{y}}} dy_1 ds \\
&\quad + C_2 \int_{t/\gamma}^{t} \int_0^{a_1^+(t-s)} (t-s)^{-\frac{|\alpha|}{4}} \exp(-\eta|y_1|) \|\Psi_+\|_{L^p_{\tilde{y}}} dy_1 ds.
\end{aligned}$$

For the first nonlinearity, we have

$$\begin{aligned}
&\int_0^{t/\gamma} \int_0^{a_1^+(t-s)} (t-s)^{-\frac{d-1}{4}+\frac{d-1}{4p}-\frac{|\alpha|}{4}} \exp(-\eta|y_1|) \\
&\quad \times s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d-1}{4}-\frac{1}{2}} \exp\left(-\frac{(y_1-a_1^+ s)^{\frac{4}{3}}}{M s^{\frac{1}{3}}}\right) dy_1 ds \\
&+ \int_{t/\gamma}^{t} \int_0^{a_1^+(t-s)} (t-s)^{-\frac{|\alpha|}{4}} \exp(-\eta|y_1|) \\
&\quad \times s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d-1}{4}(2-\frac{1}{p})-\frac{1}{2}} \exp\left(-\frac{(y_1-a_1^+ s)^{\frac{4}{3}}}{M s^{\frac{1}{3}}}\right) dy_1 ds.
\end{aligned}$$

In this case, we observe the estimate

$$\exp(-\eta|y_1|) \exp\left(-\frac{(y_1-a_1^+ s)^{\frac{4}{3}}}{M s^{\frac{1}{3}}}\right) \leqq C \exp(-\eta_1|y_1|) \exp(-\eta_2 s),$$

from which we immediately obtain an estimate by

$$t^{-\frac{d-1}{4}\left(1-\frac{1}{p}\right)-\frac{|\alpha|}{4}}.$$

Analysis of the second nonlinearity proceeds similarly from the estimate

$$\exp(-\eta|y_1|)(1+|y_1-a_1^+ s|)^{-2r} \leqq C \exp(-\eta_1|y_1|)(1+s)^{-2r}.$$

The third nonlinearity is critical, as it is the only case in which the term $e^{-\eta|y_1|}$, which has replaced $t^{-\frac{1}{4}}$ from the compressive case, is of no additional help. We have

$$\int_0^{t/\gamma}\int_0^{a_1^+(t-s)}(t-s)^{-\frac{d-1}{4}+\frac{d-1}{4p}-\frac{|\alpha|}{4}}\exp(-\eta|y_1|)s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d}{4}}dy_1ds$$
$$+\int_{t/\gamma}^{t}\int_0^{a_1^+(t-s)}(t-s)^{-\frac{|\alpha|}{4}}\exp(-\eta|y_1|)s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d}{4}-\frac{d-1}{4}\left(1-\frac{1}{p}\right)}dy_1ds$$
$$\leqq C_1 t^{-\frac{d-1}{4}\left(1-\frac{1}{p}\right)-\frac{|\alpha|}{4}}\int_0^{t/\gamma}s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}-\frac{d}{4}}ds$$
$$+C_2(1+t)^{-\frac{d}{4}-\frac{d-1}{4}\left(1-\frac{1}{p}\right)}\int_{t/\gamma}^{t}(t-s)^{-\frac{|\alpha|}{4}}s^{-\frac{3}{4}}(1+s)^{\frac{3}{4}}ds$$
$$\leqq Ct^{-\frac{d-1}{4}\left(1-\frac{1}{p}\right)-\frac{|\alpha|}{4}},$$

where in this last inequality we must take $d \geqq 5$. Four the fourth nonlinearity, we have

$$\int_0^{t/\gamma}\int_0^{a_1^+(t-s)}(t-s)^{-\frac{d-1}{4}+\frac{d-1}{4p}-\frac{|\alpha|}{4}}\exp(-\eta|y_1|)$$
$$\times(1+s)^{-\frac{d+1}{4}}(1+|y_1-a_1^+s|)^{-\frac{d}{2}}dy_1ds$$
$$+\int_{t/\gamma}^{t}\int_0^{a_1^+(t-s)}(t-s)^{-\frac{|\alpha|}{4}}\exp(-\eta|y_1|)(1+s)^{-\frac{d}{2}+\frac{d-1}{4p}}$$
$$\times(1+|y_1-a_1^+s|)^{-\frac{d}{2}}dy_1ds$$
$$\leqq C_1 t^{-\frac{d-1}{4}\left(1-\frac{1}{p}\right)-\frac{|\alpha|}{4}}\int_0^{t/\gamma}(1+s)^{-\frac{d+1}{4}}ds$$
$$+C_2(1+t)^{-\frac{d}{2}+\frac{d-1}{4p}}\int_{t/\gamma}^{t}(t-s)^{-\frac{|\alpha|}{4}}ds$$
$$\leqq Ct^{-\frac{d-1}{4}\left(1-\frac{1}{p}\right)-\frac{|\alpha|}{4}},$$

where we have taken $d \geqq 4$. For the final nonlinearity, we have

$$\int_0^{t/\gamma}\int_0^{a_1^+(t-s)}(t-s)^{-\frac{d-1}{4}+\frac{d-1}{4p}-\frac{|\alpha|}{4}}\exp(-\eta|y_1|)(1+s)^{-\frac{3}{4}(d-1)}(1+|y_1|)^{-1}dy_1ds$$
$$+\int_{t/\gamma}^{t}\int_0^{a_1^+(t-s)}(t-s)^{-\frac{|\alpha|}{4}}\exp(-\eta|y_1|)(1+s)^{-\frac{d-1}{4}\left(4-\frac{1}{p}\right)}(1+|y_1|)^{-1}dy_1ds$$
$$\leqq C_1 t^{-\frac{d-1}{4}+\frac{d-1}{4p}-\frac{|\alpha|}{4}}\int_0^{t/\gamma}(1+s)^{-\frac{3}{4}(d-1)}ds$$
$$+C_2(1+t)^{-\frac{d-1}{4}\left(4-\frac{1}{p}\right)}\int_{t/\gamma}^{t}(t-s)^{-\frac{|\alpha|}{4}}ds$$
$$\leqq Ct^{-\frac{d-1}{4}\left(1-\frac{1}{p}\right)-\frac{|\alpha|}{4}},$$

where for the final inequality we require $d \geqq 3$.

For the third estimate of Lemma 4.3 and for $y_1 \leqq 0$, we have a term identical to (5.7), and a term new to the undercompressive case,

$$\begin{aligned}
&\partial_t \int_0^t \int_{-a_1^-(t-s)}^0 \int_{\mathbb{R}^{d-1}} \mathbf{O}\left((t-s)^{-\frac{d-1}{4}}\right) \mathbf{O}(\exp(-\eta|y_1|)) \\
&\quad \times \exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{\text{eff}}^-(t-s)|^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)\Psi_-(s,y)d\tilde{y}dy_1ds \\
&= \int_0^t \int_{-a_1^-(t-s)}^0 \int_{\mathbb{R}^{d-1}} \mathbf{O}\left((t-s)^{-\frac{d}{4}}\right) \mathbf{O}(\exp(-\eta|y_1|)) \\
&\quad \times \exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{\text{eff}}^-(t-s)|^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)\Psi_-(s,y)d\tilde{y}dy_1ds \\
&\quad + \int_0^t \int_{\mathbb{R}^{d-1}} \mathbf{O}\left((t-s)^{-\frac{d-1}{4}}\right) \mathbf{O}(e^{-a_1^-(t-s)}) \\
&\quad \times \exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{\text{eff}}^-(t-s)|^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)\Psi_-(s,-a_1^-(t-s),\tilde{y})d\tilde{y}ds.
\end{aligned} \tag{5.34}$$

For the first integral on the right-hand side of (5.34), we proceed exactly as for the transverse derivative case $y_1 \leqq 0$ with $|\alpha| = 1$. For the second integral on the right-hand side of (5.34), we observe that the exponential decay in $(t-s)$ compensates for the loss of algebraic decay, and we can proceed as in the compressive case. In the case $y_1 \geqq 0$, we have the expression,

$$\begin{aligned}
&\partial_t \int_0^t \int_0^{a_1^+(t-s)} \int_{\mathbb{R}^{d-1}} \mathbf{O}\left((t-s)^{-\frac{d-1}{4}}\right) \mathbf{O}(\exp(-\eta|y_1|)) \\
&\quad \times \exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{\text{eff}}^+(t-s)|^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)\Psi_+(s,y)d\tilde{y}dy_1ds \\
&= \int_0^t \int_0^{a_1^+(t-s)} \int_{\mathbb{R}^{d-1}} \mathbf{O}\left((t-s)^{-\frac{d}{4}}\right) \mathbf{O}(\exp(-\eta|y_1|)) \\
&\quad \times \exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{\text{eff}}^+(t-s)|^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)\Psi_+(s,y)d\tilde{y}dy_1ds \\
&\quad + \int_0^t \int_{\mathbb{R}^{d-1}} \mathbf{O}\left((t-s)^{-\frac{d-1}{4}}\right) \mathbf{O}\left(\exp\left(-a_1^+(t-s)\right)\right) \\
&\quad \times \exp\left(-\frac{|\tilde{x}-\tilde{y}-\tilde{a}_{\text{eff}}^+(t-s)|^{\frac{4}{3}}}{M(t-s)^{\frac{1}{3}}}\right)\Psi_-(s,a_1^+(t-s),\tilde{y})d\tilde{y}ds.
\end{aligned} \tag{5.35}$$

For the first integral on the right-hand side of (5.35), we proceed as for the transverse derivative case $y_1 \geqq 0$ with $|\alpha| = 1$. For the second integral on the right-hand side of (5.35), and for the first undercompressive nonlinearity, we have an estimate by

$$\int_0^{t/\gamma} (t-s)^{-\frac{d-1}{4}+\frac{d-1}{4p}} \exp(-\eta(t-s)) s^{-\frac{3}{4}} (1+s)^{\frac{3}{4}-\frac{d-1}{4}-\frac{1}{2}}$$
$$\times \exp\left(-\frac{(a_1^+ t - 2a_1^+ s)^{\frac{4}{3}}}{Ms^{\frac{1}{3}}}\right) ds$$
$$+ \int_{t/\gamma}^{t} (t-s)^{-\frac{1}{4}} \exp(-\eta(t-s)) s^{-\frac{3}{4}} (1+s)^{\frac{3}{4}-\frac{d-1}{4}\left(2-\frac{1}{p}\right)-\frac{1}{2}}$$
$$\times \exp\left(-\frac{(a_1^+ t - 2a_1^+ s)^{\frac{4}{3}}}{Ms^{\frac{1}{3}}}\right) ds$$
$$\leqq C_1 t^{-\frac{d}{4}+\frac{d-1}{4p}}$$

for $d \geqq 1$, and similarly for the second undercompressive nonlinearity. For the third undercompressive nonlinearity, we have

$$\int_0^{t/\gamma} (t-s)^{-\frac{d-1}{4}+\frac{d-1}{4p}} s^{-\frac{3}{4}} (1+s)^{\frac{3}{4}-\frac{d}{4}} \exp\left(-|a_1^+|(t-s)\right) ds$$
$$+ \int_{t/\gamma}^{t} s^{-\frac{3}{4}} (1+s)^{\frac{3}{4}-\frac{d}{4}-\frac{d-1}{4}\left(1-\frac{1}{p}\right)} \exp\left(-|a_1^+|(t-s)\right) ds$$
$$\leqq C_1 t^{-\frac{d}{4}-\frac{d-1}{4}\left(1-\frac{1}{p}\right)},$$

sufficient for $d \geqq 1$. For the fourth undercompressive nonlinearity, we have an estimate by

$$\int_0^{t/\gamma} (t-s)^{-\frac{d-1}{4}+\frac{d-1}{4p}} e^{-\eta(t-s)} (1+a_1^+(t-s)+s)^{-\frac{d}{2}+\frac{d-1}{4}}$$
$$\times (1+|a_1^+ t - 2a_1^+ s|)^{-\frac{d}{2}} ds$$
$$+ \int_{t/\gamma}^{t} e^{-\eta(t-s)} (1+a_1^+(t-s)+s)^{-\frac{d}{2}+\frac{d-1}{4p}} (1+|a_1^+ t - 2a_1^+ s|)^{-\frac{d}{2}} ds$$
$$\leqq C_1 t^{-\frac{d}{2}+\frac{d-1}{4p}}$$

for $d \geqq 1$.

Finally, for the fifth undercompressive nonlinearity, we have an estimate by

$$\int_0^{t/\gamma} (t-s)^{-\frac{d-1}{4}+\frac{d-1}{4p}} e^{-\eta(t-s)} (1+a_1^+(t-s)+s)^{-\frac{3}{4}(d-1)}$$
$$\times (1+|a_1^+(t-s)|)^{-1} ds$$
$$+ \int_{t/\gamma}^{t} e^{-\eta(t-s)} (1+a_1^+(t-s)+s)^{-\frac{d-1}{4}(4-\frac{1}{p})} (1+|a_1^+(t-s)|)^{-1} ds$$
$$\leqq C_1 t^{-\frac{d}{4}+\frac{d-1}{4p}},$$

for $d \geqq 2$.

This completes the proof of Lemma 4.3. □

Appendix

The following lemma on linear PDE can be verified by the parametrix methods of [4]. See also the proof of Lemma 3.4 of the current analysis, and Proposition 11.3 of [13].

Lemma A.1. *For $x \in \mathbb{R}^d$, $t > 0$, and $j, k, l, m = 1, \ldots, d$, let $A^j(t,x)$, $B^{jklm}(t,x)$, and $C(t,x)$ be uniformly bounded in $L^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ and $C^{0+\frac{\gamma}{4}}(t) \cap C^{0+\gamma}(x)$ for some $\gamma > 0$, with also for some $\theta > 0$*

$$\sum_{jklm} B^{jklm}(t,x)\xi_j\xi_k\xi_l\xi_m \geqq \theta|\xi|^4, \forall \xi \in \mathbb{R}^d,$$

where θ is independent of $x \in \mathbb{R}^d$, and for $t \in [0,T]$, some $T > 0$, depends only on T. Then for $0 \leqq t \leqq \tau \leqq T$, τ sufficiently small, there exists a Green's function $G(t,x;y) \in C^1(t) \cap C^4(x)$ associated with the Cauchy problem for

$$v_t = -C(t,x)v - \sum_j A^j(t,x)v_{x_j} - \sum_{jklm} B^{jklm}(t,x)v_{x_jx_kx_lx_m},$$

satisfying bounds

$$|D_x^\alpha G(t,x;y)| \leqq Ct^{-\frac{d+|\alpha|}{4}} e^{-\frac{|x-y|^{\frac{4}{3}}}{Mt^{\frac{1}{3}}}}, |\alpha| \leqq 4, \tag{5.36}$$

where C and M depend only on the L^∞ bounds of the coefficients and on θ. Moreover, for a linear equation in divergence form

$$v_t = -C(t,x)v - \sum_j (A^j(t,x)v)_{x_j} - \sum_{jklm} (B^{jklm}(t,x)v_{x_jx_kx_l})_{x_m},$$

with the same assumptions in place on $C(t,x)$, $A^j(t,x)$, and $B^{jklm}(t,x)$, there is a Green's function $G(t,x;y) \in C^0(t) \cap C^3(x)$ in the distributional sense satisfying (5.36) for all $|\alpha| \leqq 3$.

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Department of Mathematics
Texas A&M University
College Station
Texas 77843
U.S.A.
e-mail: phoward@math.tamu.edu

and

Department of Mathematics
Texas A&M University
College Station
Texas 77843
U.S.A.
e-mail: cbhu@math.tamu.edu

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