Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

SPECTRAL ANALYSIS FOR TRANSITION FRONT SOLUTIONS IN CAHN-HILLIARD SYSTEMS

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(Communicated by the associate editor name)

Abstract. We consider the spectrum associated with the linear operator obtained when a Cahn–Hilliard system on $\mathbb{R}$ is linearized about a transition wave solution. In many cases it's possible to show that the only non-negative eigenvalue is $\lambda = 0$, and so stability depends entirely on the nature of this neutral eigenvalue. In such cases, we identify a stability condition based on an appropriate Evans function, and we verify this condition under strong structural conditions on our equations. More generally, we discuss and implement a straightforward numerical check of our condition, valid under mild structural conditions.

1. **Introduction.** We consider the spectrum associated with transition wave solutions $\bar{u}(x)$, $\bar{u}(\pm\infty) = u_\pm$, $u_- \neq u_+$, for Cahn–Hilliard systems on $\mathbb{R}$,

$$u_t = \Big(M(u)(-\Gamma u_{xx} + f(u))_x\Big)_x, \tag{1}$$

where $u, f \in \mathbb{R}^m$, m is an integer greater than or equal to 2 ($m+1$ phases are possible) and $M, \Gamma \in \mathbb{R}^{m\times m}$. We will give a brief discussion of the history and physicality of this equation below, and also review reasonable choices for f, M, and Γ, but we first record here, for convenient reference, a group of technical assumptions that will be made throughout the paper.

(H0) (Assumptions on Γ) Γ denotes a constant, symmetric, positive definite matrix.

(H1) (Assumptions on f) $f \in C^3(\mathbb{R}^m)$, and f has at least two zeros on $\mathbb{R}^m$. For convenience we denote this set

$$\mathcal{M} := \{u \in \mathbb{R}^m : f(u) = 0\}. \tag{2}$$

(H2) (Transition wave existence and structure) There exists a transition front solution to (1) $\bar{u}(x)$, so that

$$-\Gamma\bar{u}_{xx} + f(\bar{u}) = 0, \tag{3}$$

2000 *Mathematics Subject Classification.* Primary: 35B35, 35P05; Secondary: 35Q99.
Key words and phrases. Cahn-Hilliard systems, transition fronts, stability, Evans function.
Both authors were supported in part by NSF grant DMS-0906370.

with $\bar{u}(\pm\infty) = u_\pm$, $u_\pm \in \mathcal{M}$. When (3) is written as a first order autonomous ODE system $\bar{u}$ arises as a transverse connection either from the m-dimensional unstable linearized subspace for u_-, denoted $\mathcal{U}^-$, to the m-dimensional stable linearized subspace for u_+, denoted $\mathcal{S}^+$, or (by isotropy) vice versa. (We recall that since our ambient manifold is $\mathbb{R}^{2m}$, the intersection of $\mathcal{U}^-$ and $\mathcal{S}^+$ is referred to as transverse if at each point of intersection the tangent spaces associated with $\mathcal{U}^-$ and $\mathcal{S}^+$ generate $\mathbb{R}^{2m}$. In particular, in this setting a transverse connection is one in which the the intersection of these two manifolds has dimension 1; i.e., our solution manifold will comprise shifts of $\bar{u}$.)

(H3) (Assumptions on M and Γ) $M \in C^2(\mathbb{R}^m)$; M is uniformly positive definite along the wave; i.e., there exists $\theta > 0$ so that for all $\xi \in \mathbb{R}^m$ and all $x \in \mathbb{R}$ we have

$$\xi^{tr} M(\bar{u}(x))\xi \geq \theta|\xi|^2.$$

(H4) (Endstate Assumptions) Setting $B_\pm := f'(u_\pm)$ and $M_\pm := M(u_\pm)$, we assume $B_\pm$ and $M_\pm$ are both symmetric and positive definite. (Of course, $M_\pm$ is already positive definite from (H3).) In addition, we assume that for each of the matrices $M_\pm B_\pm$ and $\Gamma^{-1}B_\pm$, the spectrum is distinct except possibly for repeated eigenvalues that have an associated eigenspace with dimension equal to eigenvalue multiplicity. In the case of repeated eigenvalues, we assume additionally that the solutions μ of

$$\det\Big(-\mu^4 M_\pm \Gamma + \mu^2 M_\pm B_\pm - \lambda I\Big) = 0$$

can be strictly divided into two cases: if $\mu(0) \neq 0$ then $\mu(\lambda)$ is analytic in λ for $|\lambda|$ sufficiently small, while if $\mu(0) = 0$ $\mu(\lambda)$ can be written as $\mu(\lambda) = \sqrt{\lambda}h(\lambda)$, where h is analytic in λ for $|\lambda|$ sufficiently small.

Regarding (H1) we observe that for Cahn-Hilliard systems we can often write f as the gradient of an appropriate bulk free energy density F (i.e. $f(u) = F'(u)$), where F has $m+1$ local minima on $\mathbb{R}^m$. In this way, it's natural for f to have precisely $m+1$ zeros. Since F would appear in (1) with both a u and an x derivative, we can subtract from it any affine function without changing (1). It is often convenient to subtract a supporting hyperplane from F so that F is also 0 on $\mathcal{M}$.

Regarding (H4), we first observe that the symmetry condition on $f'(u_\pm)$ is natural since $F''(u)$ is a Hessian matrix. Also, we note that we can ensure that our system satisfies the determinant condition by taking arbitrarily small perturbations of the matrices M and Γ. Since we expect stability to be insensitive to such perturbations, we view this assumption as purely for technical convenience. In particular, our estimates of Lemma 4.1 would take a more complicated form if we removed them.

When the Cahn–Hilliard system (1) is linearized about a standing wave solution $\bar{u}(x)$, as described in (H2), the resulting linear equation is

$$v_t = \Big(M(x)(-\Gamma v_{xx} + B(x)v)_x\Big)_x, \tag{4}$$

where (with a slight abuse of notation) $M(x) := M(\bar{u}(x))$ and $B(x) := f'(\bar{u}(x))$. Assumptions (H0)–(H3) imply the following:

(C1) $B \in C^2(\mathbb{R})$; there exists a constant $\alpha_B > 0$ so that

$$\partial_x^j (B(x) - B_\pm) = \mathbf{O}(e^{-\alpha_B|x|}), \quad x \to \pm\infty,$$

for $j = 0, 1, 2$; $B_\pm$ are both symmetric and positive definite matrices.

(C2) $M \in C^2(\mathbb{R})$; there exists a constant $\alpha_M > 0$ so that

$$\partial_x^j (M(x) - M_\pm) = \mathbf{O}(e^{-\alpha_M |x|}), \quad x \to \pm\infty,$$

for $j = 0, 1, 2$; $M(x)$ is uniformly positive definite on $\mathbb{R}$; and $M_\pm$ are symmetric. We will set $\alpha = \min\{\alpha_B, \alpha_M\}$.

We note, in particular, that if (3) is written as a first order system

$$\begin{pmatrix} \bar{u} \\ \bar{u}' \end{pmatrix}' = \begin{pmatrix} 0 & I \\ \Gamma^{-1} f(\bar{u}) & 0 \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{u}' \end{pmatrix},$$

then by (H1) the endstates $u_\pm$ correspond with hyperbolic equilibrium points

$$\begin{pmatrix} u_\pm \\ 0 \end{pmatrix}$$

This guarantees that $\bar{u}(x)$ approaches its endstates at exponential rate, and (C1)-(C2) follow by the continuity assumed on f and M.

The eigenvalue problem associated with (4) has the form

$$L\phi := \Big(M(x)(-\Gamma\phi'' + B(x)\phi)' \Big)' = \lambda\phi. \tag{5}$$

In many cases it's possible to verify that the only non-negative eigenvalue for this equation is $\lambda = 0$ (see, for example, [1, 2, 31] and our discussion in Section 3), and so stability depends entirely on the nature of this neutral eigenvalue. Our main goal in this paper is to develop and verify an appropriate stability condition for this leading eigenvalue. We construct this condition in terms of an appropriate Evans function, which can be defined in terms of the asymptotically growing/decaying solutions of (5). As we show in Lemma 4.1, for $|\lambda| > 0$ sufficiently small, and $\text{Arg} z \neq \pi$ (i.e., excluding negative real numbers), there are $2m$ linearly independent solutions of (5) that decay as $x \to -\infty$ and $2m$ linearly independent solutions of (5) that decay as $x \to +\infty$. Moreover, these functions can be constructed so that they are analytic in $\rho = \sqrt{\lambda}$. If we denote these functions $\{\phi_j^\pm(x;\rho)\}_{j=1}^{2m}$ and set $\Phi_j^\pm = (\phi_j^\pm, \phi_j^{\pm\prime}, \phi_j^{\pm\prime\prime}, \phi_j^{\pm\prime\prime\prime})^{\text{tr}}$, the Evans function can be expressed as

$$D_a(\rho) = \det(\Phi_1^+, \dots, \Phi_{2m}^+, \Phi_1^-, \dots, \Phi_{2m}^-)\Big|_{x=0}. \tag{6}$$

We will show in Section 5 that under our conditions (H0)-(H4) the first m derivatives of $D_a(\rho)$ all vanish at $\rho = 0$. In a companion paper, currently in preparation [23], we establish that nonlinear asymptotic stability of $\bar{u}(x)$ is implied by the following condition on the order $m+1$ derivative of D_a.

Condition 1. *The set $\sigma(L)\backslash\{0\}$ lies entirely on the negative real axis, and*

$$\frac{d^{m+1}}{d\rho^{m+1}} D_a(\rho)\Big|_{\rho=0} \neq 0.$$

We will discuss particular cases in which Condition 1 holds at the end of this introduction, after our discussion of physicality.

Our analysis is particularly motivated by the study of spinodal decomposition, a phenomenon in which the rapid cooling of a homogeneously mixed alloy with $m+1$ components causes separation to occur, resolving the mixture into regions of different crystalline structure, separated by steep transition layers, in which one or more component concentrations rise above their high-temperature concentrations while one or more fall below their high-temperature concentrations. In this context, the vector u typically contains concentrations for m components of the alloy, and the final component concentration is obtained from conservation of mass

$$\sum_{j=1}^{m+1} u_j = 1. \tag{7}$$

Each component of u is a conserved quantity, so if we denote by J_j the (vector) flux associated with concentration u_j we have

$$u_{j_t} + \nabla \cdot J_j = 0; \quad j = 1, \dots, m. \tag{8}$$

The molecular transfer during spinodal decomposition corresponds with motion from configurations in which small fluctuations in concentration correspond with large fluctuations in system internal energy to configurations in which small fluctuations in concentration correspond with small fluctuations in system internal energy. In order to capture this behavior the J_j are typically chosen to have the form (see [8], p. 12)

$$J_j = -\sum_{i=1}^{m} M_{ji}(u) \nabla \frac{\delta E}{\delta u_i}, \tag{9}$$

where $M_{ji}(u)$ denotes (scalar) molecular mobility, E denotes a total free energy functional for the alloy, and $\frac{\delta E}{\delta u_i}$ denotes the kernel of the variational derivative of E with respect to u_i. The Cahn–Hilliard system arises from these considerations and a form of the free energy functional suggested by Cahn and Hilliard in 1958 for the case of binary alloys ($m = 1$) [7] and generalized by de Fontaine to multicomponent alloys in 1967 [8]. For the case of a bounded domain $U \subset \mathbb{R}^n$, de Fontaine's functional can be written as (see [8], p. 10)

$$E(u) = \int_U F(u) + \frac{1}{2} Du : (\Gamma(u) Du) dx, \tag{10}$$

where $F(u)$ denotes the bulk free energy density for the alloy with uniform composition u, $\Gamma(u)$ is a gauge of interfacial energy (so, in particular, the term involving Γ describes energy associated with a transition of composition), Du denotes the $m \times n$ Jacobian of u, and the notation $A : B$ refers to matrix inner product

$$A : B := \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} b_{ij}. \tag{11}$$

We note for comparison with our references that Eyre uses the same energy in [12], though he replaces $Du : (\Gamma(u) Du)$ with the equivalent expression $\text{tr}(Du^{tr} \Gamma(u) Du)$. If Γ is taken to be constant (which certainly need not be the case physically) then

$$\frac{\delta E}{\delta u_i} = F_{u_i}(u) - (\Gamma \Delta u)_i, \tag{12}$$

and we obtain the Cahn–Hilliard system on $\mathbb{R}^n$

$$u_{j_t} = \nabla \cdot \Big\{ \sum_{i=1}^{m} M_{ji}(u) \nabla \Big((-\Gamma \Delta u)_i + F_{u_i}(u) \Big) \Big\}, \quad j = 1, 2, \dots, m. \tag{13}$$

This corresponds, for example, with equation (24) in [8] and equation (4) in [27], except that the author in [8] adds an inhomogeneous term and the authors in [27] take Γ as the identity. (Also, in contrast to the case here, the authors in [27] are considering degenerate mobility, but that does not show up in the notation.) We note that for $n = 1$ (13) is a special case of our equation (1), obtained by taking

f to be the Jacobian (with respect to u) of F. (For the case of a binary alloy (13) first appeared in Cahn's 1961 paper [6], and de Fontaine suggests the Cahn–Hilliard equation would more correctly be designated the Cahn equation. Hilliard, de Fontaine's advisor, apparently referred to this equation as "the last unnumbered equation after Eq. (18) in Cahn's 1961 paper" [11].)

Equation (13) has been the subject of considerable study [8, 9, 10, 12, 15, 27, 28], though certainly the case $m = 1$ is much better understood than the case currently under investigation.

Alternatively, we can derive a form of (1) by regarding the total internal energy as a map on all $m+1$ component concentrations and introducing a Lagrange multiplier to impose the total mass constraint (7). In this approach we will assume the flow is governed by a functional $\mathcal{L}$, equivalent with total internal energy along (1), which includes a Lagrange multiplier as a constraint term. Following the analysis of Boyer and Lapuerta in [5], we obtain, for $j = 1, 2, \dots, m$, the system

$$\begin{aligned} u_{j_t} &= \Big(\frac{\tilde{M}(u)}{\gamma_j}\Big(- \gamma_j u_{j_{xx}} + \gamma_0 \sum_{i=1}^{m+1} \frac{1}{\gamma_i}(F_{u_j}(u) - F_{u_i}(u))\Big)_x\Big)_x, \\ u_{m+1} &= 1 - \sum_{j=1}^{m} u_j, \end{aligned} \tag{14}$$

where in this case $\tilde{M}$ and F are regarded as functions of $m + 1$ variables, and we note that Boyer and Lapuerta focused on the case $m = 2$ (ternary alloys). We remark for clarity of comparison that system (14) is taken from equation (8) of [5], given in the case $m = 2$, with

$$\tilde{M} = \frac{3\epsilon}{4} M_0; \quad \gamma_j = \Sigma_j; \quad j = 1, 2, 3; \quad \gamma_0 = \frac{16\Sigma_T}{3\epsilon^2},$$

$$\Sigma_T = \frac{3}{\frac{1}{\Sigma_1} + \frac{1}{\Sigma_2} + \frac{1}{\Sigma_3}},$$

where the expressions on the right hand sides are in the notation of [5]. This is the special case of our (1) obtained by taking $M(u)$ diagonal, with entries $\tilde{M}/\gamma_j$, Γ diagonal with entries $\{\gamma_j\}_{j=1}^m$, and

$$f_j(u) = \Gamma_0 \sum_{i=1}^{m+1} \frac{1}{\gamma_i}\Big(F_{u_j}(u, 1 - \sum_{k=1}^{m} u_k) - F_{u_i}(u, 1 - \sum_{k=1}^{m} u_k)\Big).$$

Remark 1. As the analysis of systems of form (14) differs somewhat from the analysis of systems of form (13), we will find it convenient to have a terminological distinction between the cases. While we certainly regard both cases as Cahn-Hilliard systems, we will, for brevity, sub-categorize equations (14) as *Boyer-Lapuerta* systems and equations

$$u_t = \Big(M(u)(-\Gamma u_{xx} + F'(u))_x\Big)_x, \tag{15}$$

(slightly more general than (13)) as *gradient* systems.

Qualitatively, we expect that at high temperatures the bulk free energy density F will decrease as entropy increases (according to the Helmholtz free energy relation $\mathcal{F} = U - TS$, where $\mathcal{F}$ denotes free energy, U denotes internal energy, T denotes system temperature, and S denotes system energy), and so F will have a global minimum in the configuration that maximizes entropy. For example, if our system has $m+1$ components, present in equal amounts, we expect (at high temperature) F to have a global minimum at a concentration vector

$$u = (\frac{1}{m+1}, \frac{1}{m+1}, \dots, \frac{1}{m+1}),$$

and to have global maxima at the $m+1$ low-entropy single-component configurations corresponding with concentration vectors $(1,0,\dots,0)$, $(0,1,0,\dots,0)$ etc., with also $(0,0,\dots,0)$. (To be clear, this discussion is only intuitive, and we are not adding any hypotheses on F.) As temperature decreases (and assuming internal energy remains constant) we have the thermodynamic relation

$$\frac{\partial \mathcal{F}}{\partial T} = -S,$$

and so $\mathcal{F}$ increases (again, as temperature decreases) at a rate proportional to entropy. In this way the free energy increases most rapidly where it was previously minimized and increases most slowly where it was previously maximized. Heuristically, then, we expect that at low temperatures F will have a local maximum where it was previously minimized and that it will have $m+1$ local minima associated (possibly by equivalence) with the $m+1$ global single-component maxima.

More precisely, in [8, 9] de Fontaine attributes the following form of the bulk free energy density for a ternary alloy to Prigogine [29]:

$$F(u_1, u_2, u_3) = \sum_{i \neq j} \omega_{ij} u_i u_j + \kappa T \sum_{i=1}^{3} u_i \ln u_i,$$

where κ denotes Boltzmann's constant, T denotes system temperature, and we haven't yet employed mass conservation to reduce the number of variables. For a system with $m+1$ components it is natural to consider the generalized form

$$F(u) = \frac{1}{2} u \cdot Au + \kappa T \sum_{i=1}^{m+1} l_i u_i \ln u_i, \tag{16}$$

where A is an $(m+1) \times (m+1)$ matrix and the values $\{l_i\}_{i=1}^{m+1}$ are constants associated with the alloy. We recall that the reduction of F from a function of $m+1$ variables to a function of m variables is accomplished by the conservation equation (7).

A form commonly examined due to its simplicity is

$$F(u) = \sum_{i<j} \alpha_{ij} u_i^2 u_j^2. \tag{17}$$

In the case $m=2$ Alikakos et al. have carefully examined bulk free energy functions of the form

$$F(u_1, u_2) = |h(u_1 + iu_2)|^2, \tag{18}$$

where h is analytic on $\mathbb{C}$, and the third component has been eliminated by conservation of mass [1].

Finally, we mention the class of *algebraically consistent* functions suggested by Boyer and Lapuerta. (For the precise definition of algebraic consistency, as Boyer and Lapuerta give it, see Definition 3.1 of [5].) Their functions are given for ternary alloys and have the form

$$\begin{aligned} F(u_1,u_2,u_3) = &\sum_{j<k} \sigma_{jk} u_j^2 u_k^2 + u_1 u_2 u_3 \sum_{j=1}^{3} \gamma_j u_j \\ &+ u_1^2 u_2^2 u_3^2 G(u_1,u_2,u_3) + (u_1+u_2+u_3-1) H(u_1,u_2,u_3), \end{aligned} \tag{19}$$

for C^1 functions G and H. Here, σ_{jk} denotes surface tension between components j and k and for $i = 1,2,3$, $i \neq j,k$,

$$\gamma_i = \sigma_{ij} + \sigma_{ik} - \sigma_{jk}.$$

(Aside from a slight change in notation, form (19) is taken directly from [5]. We note here, as the authors do in [5] that since $u_1 + u_2 + u_3 = 1$ we can take $H \equiv 0$ without loss of generality.) In particular, these functions have the property that if any particular component is absent then the resulting expression is the correct bulk free energy for the two remaining components. For example,

$$F(u_1,u_2,0) = \sigma_{12} u_1^2 u_2^2,$$

which with $u_2 = 1 - u_1$ corresponds with the standard double-well form

$$\tilde{F}(u_1) = \sigma_{12} u_1^2 (1-u_1)^2.$$

We conclude our introduction with a discussion of the primary results we cite and establish in the paper. First, though it's difficult to summarize briefly here, we regard the framework we develop in Section 5 for analyzing Condition 1 as the primary contribution of this paper. As we show in Section 6.3 this framework provides a straightforward way in which to numerically check Condition 1.

In analyzing gradient systems, we use the following theorem due to Alikakos and Fusco (see [2] and also our discussion in Section 2.1).

Theorem 1.1 (Existence for gradient systems). *Let (H0) hold and suppose $F \in C^4(\mathbb{R}^m)$ has precisely $m+1$ local minima $\{\xi_j\}_{j=1}^{m+1}$ such that $F''(\xi_j)$ is positive definite for each $j = 1,2,...,m+1$. In addition, suppose u_- and u_+ are elements of the set $\{\xi_j\}_{j=1}^{m+1}$, $u_- \neq u_+$, and*

$$F(tu_- + (1-t)u_+) > 0,$$

for all $t \in (0,1)$. Then there exists a transition front $\bar{u} \in C^5(\mathbb{R})$ so that (3) holds and $\bar{u}(\pm\infty) = u_\pm$.

Moreover, $\bar{u}(x)$ minimizes the energy functional

$$E(u) = \int_{\mathbb{R}} F(u) + \frac{1}{2}(\Gamma u_x, u_x) dx,$$

where $(\cdot,\cdot)$ denotes inner product on $\mathbb{R}^n$ (i.e., dot product).

In Section 6.3 we use the framework of Section 5 to give numerical evidence for Condition 1 in the following case.

Numerical Result 1 (Stability for gradient systems). *For the gradient system (15) with M and Γ both taken as identity, and the choice*

$$F(u_1,u_2) = u_1^2 u_2^2 + u_1^2(1-u_1-u_2)^2 + u_2^2(1-u_1-u_2)^2,$$

we numerically compute $D_a'''(0) \neq 0$ for the transition front $\bar{u}(x)$ depicted in Figure 2.1.

For Boyer-Lapuerta systems we will have a hierarchy of assumptions under which different conclusions can be drawn, and for convenient reference we will summarize here the assumptions and their implications. We also give references to the associated full statements of the conclusions. First, our theoretical framework of Section 5 is valid under our basic assumptions (H0)-(H4), which we assume to hold in all cases. We will designate the assumptions under which we obtain existence of transition front solutions $\bar{u}(x)$ as *(BL1)*; we will designate the assumptions under which we verify that the spectrum associated with $\bar{u}(x)$ lies entirely on the negative real axis (including $\lambda = 0$) as *(BL2)*; and finally we will designate the assumptions under which we establish a relatively straightfoward expression for Condition 1 as *(BL3)*. After stating these conditions we discuss a test case for which we verify that (BL1)-(BL3) all hold, and for which we can analytically verify Condition 1.

(BL1) We assume that F satisfies the relations

$$\begin{aligned} F(u_1, 1-u_1, 0) &= F_{12}(u_1) \\ F(u_1, 0, 1-u_1) &= F_{13}(u_1) \\ F(0, u_2, 1-u_2) &= F_{23}(u_2), \end{aligned} \tag{20}$$

where the F_{ij} are double-well functions as described in $(\tilde{H})$ in Section 2.2, and additionally we assume that F satisfies the symmetry property

$$F(u_1, 1-u_1, 0) = F(1-u_1, u_1, 0); \quad u \in [0,1]. \tag{21}$$

See Lemma 2.1 for a precise statement.

(BL2) In addition to (BL1), we assume $\gamma_1 = \gamma_2$ and that the operator

$$H_b := -\partial_{xx} + \Big(\frac{\partial f_1}{\partial u_1}(\bar{u}_1, 1-\bar{u}_1) + \frac{\partial f_1}{\partial u_2}(\bar{u}_1, 1-\bar{u}_1)\Big) \tag{22}$$

is non-negative. We verify that for the case

$$F(u_1, u_2, u_3) = \frac{\gamma_1}{2} u_1^2 (u_2 + u_3)^2 + \frac{\gamma_2}{2} u_2^2 (u_1 + u_3)^2 + \frac{\gamma_3}{2} u_3^2 (u_1 + u_2)^2,$$

this condition is implied by the condition $\gamma_1 \le 4\gamma_3$. See Lemma 3.1 for a precise statement.

(BL3) In addition to (BL1)-(BL2), we assume some technical *endstate conditions* that will be more natural to state explicitly once we have developed more notation. See Lemma 6.1 for a precise statement.

As a study case, consider the Boyer-Lapuerta system (14) with $m = 2$, $\gamma_j = 1$, $j = 0, 1, 2, 3$, $\tilde{M} = 1$, and

$$F(u_1, u_2, u_3) = \frac{1}{2} u_1^2 (u_2 + u_3)^2 + \frac{1}{2} u_2^2 (u_1 + u_3)^2 + \frac{1}{2} u_3^2 (u_1 + u_2)^2.$$

We show that the operator L obtained by linearization of this system about the transition front solution

$$\begin{pmatrix} \bar{u}_1(x) \\ \bar{u}_2(x) \end{pmatrix} = \frac{1}{1 + e^{\sqrt{3}x}} \begin{pmatrix} 1 \\ e^{\sqrt{3}x} \end{pmatrix}$$

(or any translation of this wave) satisfies Condition 1.

Outline of the paper. In Section 2 we discuss the existence and structure of transition front solutions in both gradient and Boyer-Lapuerta systems. In Section 3 we discuss the spectrum associated with our linearized operator L for $\lambda \neq 0$, and in Section 4 we develop preliminary ODE results required for analyzing the Evans function at $\lambda = 0$. In Section 5 we develop our stability condition (Condition

1) generally, and in Section 6 we analyze this condition for the two study cases mentioned above.

2. **Existence and Structure of Transition Waves.** We look for stationary solutions $\bar{u}(x)$ for (1) that satisfy $\bar{u}(\pm\infty) = u_\pm \in \mathcal{M}$. (We recall that our notation $\mathcal{M}$ is defined in (H1).) Upon substitution of $\bar{u}(x)$ into (1), and after integrating twice and using $f(u_\pm) = 0$, we find

$$-\Gamma\bar{u}_{xx} + f(\bar{u}) = 0. \tag{23}$$

We set $U = \bar{u}$ and $V = \bar{u}_x$, and write this as a first order system

$$\begin{aligned} U' &= V \\ V' &= \Gamma^{-1} f(U). \end{aligned} \tag{24}$$

Upon linearization about the endstates $(u_\pm, 0)$ we obtain

$$\begin{pmatrix} \tilde{U} \\ \tilde{V} \end{pmatrix}' = \begin{pmatrix} 0 & I \\ \Gamma^{-1} f'(u_\pm) & 0 \end{pmatrix} \begin{pmatrix} \tilde{U} \\ \tilde{V} \end{pmatrix}.$$

The associated eigenvalues are $\{-\sqrt{\nu_j^\pm}\}_{j=1}^m$ and $\{+\sqrt{\nu_j^\pm}\}_{j=1}^m$, where the values $\{\nu_j^\pm\}_{j=1}^m$ are the (necessarily positive) eigenvalues of $\Gamma^{-1} f'(u_\pm)$. Clearly, the points $(u_\pm, 0)$ both have an m-dimensional unstable manifold and an m-dimensional stable manifold. In this way we see that $\bar{u}(x)$ must correspond with a connection either between the m-dimensional unstable manifold of $(u_-, 0)$ and the m-dimensional stable manifold of $(u_+, 0)$ or vice versa.

2.1. **Gradient Systems.** In the standard case that f can be written as the gradient of some function, $f = F'$, system (24) has been the subject of considerable study. In particular, existence of transition front solutions in this case has been established by Alikakos and Fusco, whose result we stated in Theorem 1.1 of our introduction. (See [2] and also the related analysis of Stefanopoulos [31]).

Notes on the proof of Theorem 1.1. Aside from the brief observations we make here, this theorem was established in [2] Theorem 3.6.

While the analysis of [2] is carried out with Γ taken as the identity matrix, we can reduce our equation to their case by setting $\bar{v} := \Gamma^{1/2}\bar{u}$ and

$$F_0(\bar{v}) := F((\Gamma^{-1/2})\bar{v}).$$

That is, we now have

$$-\bar{v}_{xx} + F_0'(\bar{v}) = 0.$$

Clearly, F_0 has precisely $m+1$ local minima at $\{\Gamma^{1/2}\xi_j\}_{j=1}^{m+1}$, and

$$F_0(t\Gamma^{1/2}u_- + (1-t)\Gamma^{1/2}u_+) = F(u_- t + u_+(1-t)) > 0.$$

Under these conditions, Theorem 3.6 of [2], along with Extension Theorem 3.8 from the same paper, assert the existence of a weak $W^{1,2}_{\text{loc}}(\mathbb{R})$ solution to (23). According to Theorem 4.2 in [14] (also Theorem 4.4 on p. 277 of [13]), this solution must have Holder continuous derivatives, and so consequently it must agree with the Picard solution for system (24). Our claimed regularity is immediate. □

Here, we note for later reference,

$$E'(\bar{u})(\varphi) = \int_{\mathbb{R}} (F'(\bar{u}) - \Gamma \bar{u}_{xx}, \varphi) dx, \tag{25}$$

and

$$E''(\bar{u})(\psi, \phi) = \int_{\mathbb{R}} \Big((-\Gamma \partial_x^2 + F''(\bar{u}))\psi, \varphi \Big) dx. \tag{26}$$

In particular, the assertion that $\bar{u}$ is a minimizer of E ensures that the operator

$$H := -\Gamma \partial_x^2 + F''(\bar{u})$$

is non-negative.

We observe that the transition front guaranteed by Theorem 1.1 may not be a unique minimizer of E, and so this does not guarantee that $\bar{u}(x)$ correponds with a transverse connection in system (24). On the other hand, the derivative statement in our Condition 1 can be regarded as a transversality condition. For the case $m = 2$, and for bulk free energy densities of form

$$F(u_1, u_2) = |h(u_1 + iu_2)|^2, \tag{27}$$

where h is analytic on $\mathbb{C}$, Alikakos, Betelu, and Chen have shown that the transition fronts guaranteed by Theorem 1.1 are unique: i.e., given a valid pair of endstates u_- and u_+ there is precisely one transition wave $\bar{u}(x)$ that solves (23) and satisfies $\bar{u}(\pm\infty) = u_\pm$. (See [1].) Generically, these can be reversed, so that there will also be a solution so that $\bar{u}(\pm\infty) = u_\mp$. For example, if we would like to work with the case of three minima located at the standard points $(0, 0)$, $(1, 0)$, and $(0, 1)$, we can take

$$h(z) = z(z - 1)(z - i),$$

and the theorem of Alikakos, Betelu, and Chen guarantees we have a unique solution. (Of course, this corresponds with a sixth order bulk free energy density polynomial rather than the more standard quartic.) This uniqueness guarantees transversality, and also that the waves we investigate numerically by the methods of Section 6 are the waves guaranteed by Theorem 1.1.

For the case (common for numerical simulations)

$$F(u_1, u_2) = u_1^2 u_2^2 + u_1^2(1 - u_1 - u_2)^2 + u_2^2(1 - u_1 - u_2)^2,$$

the three minima occur at $(0, 0)$, $(1, 0)$, and $(0, 1)$. We can numerically approximate a transition wave solution for (1) with $f = F'$ by solving a boundary value problem with values given close to these endpoints. A transition front computed in this way (connecting $(1, 0)$ to $(0, 1)$) is given in Figure 2.1.

2.2. **Boyer–Lapuerta Systems.** Following [5], we consider particularly the case of (14) with $m = 2$, corresponding with a ternary alloy. In this case the transition wave solves

$$\begin{aligned} -\gamma_j \bar{u}_{j_{xx}} + \gamma_0 \sum_{i=1}^{3} \frac{1}{\gamma_i} \Big(F_{u_j}(\bar{u}) - F_{u_i}(\bar{u}) \Big) = 0; \quad j = 1, 2 \\ \bar{u}_3 + \sum_{j=1}^{2} \bar{u}_j - 1 = 0. \end{aligned} \tag{28}$$

We assume that if one of the three components is absent then the resulting bulk free energy density will be an appropriate bulk free energy density for the resulting two-component system. More precisely, we assume (20), where F_{jk} denotes an

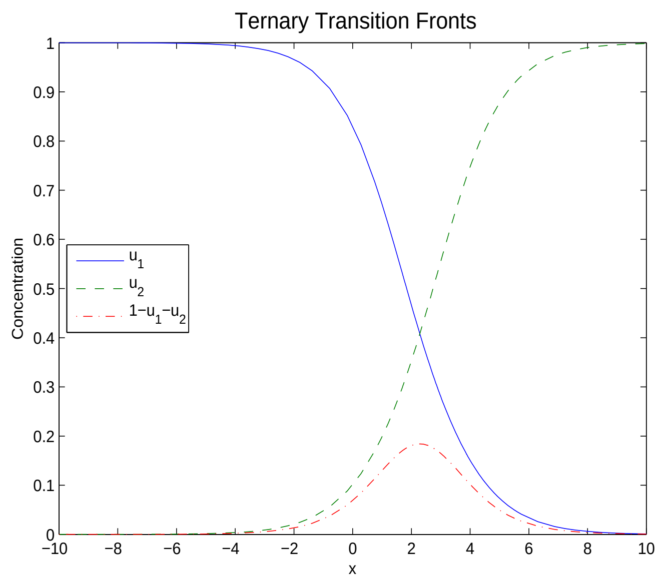

FIGURE 1. Transition front solution for a ternary Cahn-Hilliard system.

appropriate bulk free energy density for a single Cahn–Hilliard equation describing a binary alloy with components i and j. That is, we assume $F_{jk}(u)$ satisfies the assumptions of [22]:

$(\tilde{H})$ $F_{jk} \in C^4(\mathbb{R})$ has a double-well form: there exist real numbers $\alpha_1 = 0 < \alpha_2 < \alpha_3 < \alpha_4 < 1 = \alpha_5$ so that F_{jk} is strictly decreasing on $(-\infty, 0)$ and $(\alpha_3, 1)$ and strictly increasing on $(0, \alpha_3)$ and $(1, +\infty)$, and additionally F_{jk} is concave up on $(-\infty, \alpha_2) \cup (\alpha_4, +\infty)$ and concave down on (α_2, α_4).

In addition we assume that if component j is absent in (28) (i.e., $\bar{u}_j \equiv 0$), then the equation for $\bar{u}_j$ will become simply $\bar{u}_{j_{xx}} = 0$. This clearly imposes the conditions

$$\begin{aligned}
&\sum_{i=1}^{3} \frac{1}{\gamma_i}\Big(F_{u_3}(u_1, 1-u_1, 0) - F_{u_i}(u_1, 1-u_1, 0)\Big) = 0\\
&\sum_{i=1}^{3} \frac{1}{\gamma_i}\Big(F_{u_2}(u_1, 0, 1-u_1) - F_{u_i}(u_1, 0, 1-u_1)\Big) = 0\\
&\sum_{i=1}^{3} \frac{1}{\gamma_i}\Big(F_{u_1}(0, u_2, 1-u_2) - F_{u_i}(0, u_2, 1-u_2)\Big) = 0.
\end{aligned} \tag{29}$$

Since the argument for existence is the same for each pair of coordinates, we focus on the case in which component 3 is absent. First, from (20), we see that

$$F_{12}'(u_1) = F_{u_1}(u_1, 1-u_1, 0) - F_{u_2}(u_1, 1-u_1, 0).$$

Also, according to (29), we have

$$F_{u_3}(u_1, 1-u_1, 0) = \frac{\gamma_2}{\gamma_1+\gamma_2} F_{u_1}(u_1, 1-u_1, 0) + \frac{\gamma_1}{\gamma_1+\gamma_2} F_{u_2}(u_1, 1-u_1, 0).$$

Combining these observations, we find

$$\sum_{i=1}^{3} \frac{1}{\gamma_i} \Big(F_{u_1}(u_1, 1-u_1, 0) - F_{u_i}(u_1, 1-u_1, 0) \Big) = \Big(\frac{1}{\gamma_2} + \frac{\gamma_1/\gamma_3}{\gamma_1+\gamma_2} \Big) F_{12}'(u_1),$$

and likewise

$$\sum_{i=1}^{3} \frac{1}{\gamma_i} \Big(F_{u_2}(u_1, 1-u_1, 0) - F_{u_i}(u_1, 1-u_1, 0) \Big) = -\Big(\frac{1}{\gamma_1} + \frac{\gamma_2/\gamma_3}{\gamma_1+\gamma_2} \Big) F_{12}'(u_1).$$

In this way system (28) becomes a coupling of two equivalent equations

$$-\bar{u}_{j_{xx}} + \gamma_0 \Big(\frac{1}{\gamma_1 \gamma_2} + \frac{1/\gamma_3}{\gamma_1+\gamma_2} \Big) F_{12}'(\bar{u}_j) = 0, \tag{30}$$

for $j = 1, 2$. This equation is the standing wave equation for the Cahn–Hilliard equation associated with the binary alloy with components 1 and 2, and it's well-known (see, for example, [22]) that such equations have precisely two transition front solutions (up to a shift), $\bar{u}_1(x)$ and $\bar{u}_2(x) = \bar{u}_1(-x)$. In the symmetric case (21) we have $F_{12}(\bar{u}_1) = F_{12}(1-\bar{u}_1)$, and so $\bar{u}_2(x) = 1 - \bar{u}_1(x)$. In this way, we find that transition fronts for Boyer-Lapuerta systems with $m = 2$ have the form

$$\bar{u}(x) = \begin{pmatrix} \bar{u}_1(x) \\ 1 - \bar{u}_1(x) \end{pmatrix}, \tag{31}$$

where $\bar{u}_1(x)$ is a transition front for a binary alloy. We note particularly that, unlike the gradient case, the third component associated with these waves is always identically 0.

We summarize these observations in the following lemma.

Lemma 2.1. *For equation (28), suppose F satisfies (20) (with $\tilde{H}$), as well as (21). Then (28) has a transition wave solution with form (31), unique up to a shift and an exchange of coordinates.*

3. **Spectral Analysis for $\lambda \neq 0$.** When the Cahn–Hilliard system (1) is linearized about a standing wave solution $\bar{u}(x)$ the resulting linear equation is

$$v_t = \Big(M(\bar{u})(-\Gamma v_{xx} + f'(\bar{u})v)_x \Big)_x, \tag{32}$$

with associated eigenvalue problem

$$L\phi = \Big(M(\bar{u})(-\Gamma \phi_{xx} + f'(\bar{u})\phi)_x \Big)_x = \lambda \phi. \tag{33}$$

It follows from Lemma 2 in the Appendix of Chapter 5 in [17] that the essential spectrum of L is determined by the asymptotic operators

$$L_\pm \phi = -M_\pm \Gamma \phi'''' + M_\pm f'(u_\pm)\phi''.$$

In particular, the essential spectrum is determined by the existence of solutions of the form $\phi = e^{ikx}v$, for which we have

$$\Big(-M_\pm \Gamma k^4 - M_\pm f'(u_\pm)k^2 \Big) v = \lambda(k) v,$$

where essential spectrum is restricted to the parametrized curve $\lambda(k)$. Upon multiplication of this last relation by the symmetric, positive definite matrix $M_\pm^{-1}$, and after taking inner product with v, we obtain the relation

$$-\langle v, \Gamma v \rangle k^4 - \langle v, f'(u_\pm)v \rangle k^2 = \lambda(k) \langle v, M_\pm^{-1} v \rangle.$$

Clearly, $f'(u_\pm)$ is always symmetric for gradient systems ($f'(\bar{u}) = F''(\bar{u})$, a Hessian matrix), and we will see below that it is also symmetric for Boyer–Lapuerta systems under symmetry condition (21). (Recall that we assume in (H4) that $f'(u_\pm)$ is symmetric.) Since Γ and $M_\pm^{-1}$ are also symmetric, we see that the essential spectrum must be real-valued, and since these three matrices $f'(u_\pm)$, Γ, and $M_\pm^{-1}$ are all positive definite we find that essential spectrum is confined to the negative real axis, including $\lambda = 0$.

For the point spectrum, we observe that for any $\lambda \neq 0$ and associated eigenfunction $\phi(\cdot;\lambda) \in L^2(\mathbb{R}) \cap C^5(\mathbb{R})$ (the regularity following without loss of generality from the regularity asserted in (H1)-(H3)) we must have $\int_{\mathbb{R}} \phi(x;\lambda)dx = 0$, which justifies our setting

$$\varphi(x;\lambda) := \int_{-\infty}^{x} \phi(y;\lambda)dy. \tag{34}$$

Upon integration our eigenvalue problem becomes

$$M(\bar{u})(-\Gamma\varphi_{xxx} + f'(\bar{u})\varphi_x)_x = \lambda\varphi. \tag{35}$$

We multiply both sides by $M(\bar{u})^{-1}$, then take an L^2 inner product, denoted $\langle\cdot,\cdot\rangle$, with φ to obtain the relation (after one application of integration by parts)

$$\langle -H\varphi_x, \varphi_x\rangle = \lambda\langle\varphi, M(\bar{u})^{-1}\varphi\rangle,$$

where

$$H := -\Gamma\partial_x^2 + f'(\bar{u}(x)). \tag{36}$$

Since Γ is symmetric, the operator H is symmetric so long as the matrix $f'(\bar{u})$ is. As discussed above, $f'(\bar{u})$ is always symmetric for gradient systems and is also symmetric for Boyer–Lapuerta systems under symmetry condition (21). Moreover, since $M(\bar{u}(x))$ is positive definite, $M(\bar{u}(x))^{-1}$ is positive definite, and so if $M(\bar{u}(x))$ is symmetric (slightly more than we assume in general), then the point spectrum for L will be real-valued, and it will be entirely non-positive so long as H is a positive operator. We have already seen in Section 2 that if $\bar{u}$ is a wave guaranteed by Theorem 1.1 then H is non-negative for gradient Cahn–Hilliard systems, and so we have no eigenvalues with positive real part.

We turn next to Boyer-Lapuerta systems for which we consider only $m = 2$ in detail. In this case, we have

$$\Gamma = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}.$$

and

$$\begin{aligned} f_1(u_1,u_2) &= \gamma_0\Big\{\frac{1}{\gamma_2}\Big(F_{u_1}(u_1,u_2,1-u_1-u_2) - F_{u_2}(u_1,u_2,1-u_1-u_2)\Big) \\ &\quad + \frac{1}{\gamma_3}\Big(F_{u_1}(u_1,u_2,1-u_1-u_2) - F_{u_3}(u_1,u_2,1-u_1-u_2)\Big)\Big\} \\ f_2(u_1,u_2) &= \gamma_0\Big\{\frac{1}{\gamma_1}\Big(F_{u_2}(u_1,u_2,1-u_1-u_2) - F_{u_1}(u_1,u_2,1-u_1-u_2)\Big) \\ &\quad + \frac{1}{\gamma_3}\Big(F_{u_2}(u_1,u_2,1-u_1-u_2) - F_{u_3}(u_1,u_2,1-u_1-u_2)\Big)\Big\}. \end{aligned}$$

According to (20) we have

$$F_{12}(u_1) = F(u_1, 1-u_1, 0), \tag{37}$$

and so

$$F_{12}'(u_1) = F_{u_1}(u_1, 1-u_1, 0) - F_{u_2}(u_1, 1-u_1, 0)$$
$$F_{12}''(u_1) = F_{u_1u_1}(u_1, 1-u_1, 0) - 2F_{u_1u_2}(u_1, 1-u_1, 0) + F_{u_2u_2}(u_1, 1-u_1, 0).$$

Under our assumption (21) we know that $\bar{u}(x)$ has the form (31), and in our general notation we have

$$b_{ij}(x) := \frac{\partial f_i}{\partial u_j}(\bar{u}_1, 1-\bar{u}_1),$$

Here, the eigenvalue problem for H can be written as

$$\begin{aligned} -\gamma_1 \phi_{1_{xx}} + b_{11}(x)\phi_1 + b_{12}(x)\phi_2 &= \lambda\phi_1 \\ -\gamma_2 \phi_{2_{xx}} + b_{21}(x)\phi_1 + b_{22}(x)\phi_2 &= \lambda\phi_2. \end{aligned} \tag{38}$$

A direct calculation gives

$$\begin{aligned} b_{12}(x) &= b_{21}(x) \\ b_{11}(x) &= b_{12}(x) + \gamma_0 \frac{\gamma_1\gamma_2 + \gamma_1\gamma_3 + \gamma_2\gamma_3}{(\gamma_1+\gamma_2)\gamma_2\gamma_3} F_{12}''(\bar{u}_1) \\ b_{22}(x) &= b_{21}(x) + \gamma_0 \frac{\gamma_1\gamma_2 + \gamma_1\gamma_3 + \gamma_2\gamma_3}{(\gamma_1+\gamma_2)\gamma_1\gamma_3} F_{12}''(\bar{u}_1). \end{aligned} \tag{39}$$

We will prove the following lemma.

Lemma 3.1. *Suppose $\bar{u}(x)$ denotes a transition front solution of system (14) with $m = 2$, and the following conditions hold:*

(i) $F(u_1, 1-u_1, 0) = F(1-u_1, u_1, 0)$; $\quad u_1 \in [0,1]$;

(ii) $\gamma_1 = \gamma_2$;

(iii) The operator $H_b := -\gamma_1\partial_{xx} + b_{11}(x) + b_{12}(x)$ is non-negative.

Then the spectrum associated with (38) lies entirely on the non-negative real line.

Remark 2. Before proving this lemma, we note that the conditions will be checked below for a standard case. Also, we observe that Condition (iii) clearly holds if $b_{11}(x) + b_{12}(x) \geq 0$ for all $x \in \mathbb{R}$.

Proof. We specialize to the case $\gamma_1 = \gamma_2$, and for notational convenience set

$$\gamma := \gamma_0 \frac{\gamma_1 + 2\gamma_3}{2\gamma_1\gamma_3}.$$

Under this assumption, $b_{11}(x) = b_{22}(x)$. Now, suppose the vector (ϕ_1, ϕ_2) corresponds with an eigenvalue $\lambda \neq 0$ and set $v := \phi_2 - \phi_1$.

First, consider the case $\phi_2 \equiv \phi_1$, so that $v \equiv 0$. Here, (38) simply consists of two copies of the same equation,

$$-\gamma_1\phi_{1_{xx}} + (b_{11}(x) + b_{12}(x))\phi_1 = \lambda\phi_1.$$

If we multiply by ϕ_1 and integrate over $\mathbb{R}$ we obtain

$$\lambda \int_{-\infty}^{+\infty} \phi^2 dx = \int_{-\infty}^{+\infty} \phi_1 H_b \phi_1 dx = \gamma_1 \int_{-\infty}^{+\infty} {\phi_2}_x^2 dx + \int_{-\infty}^{+\infty} (b_{11}(x) + b_{12}(x))\phi_1^2 dx.$$

Clearly, if H_b is a non-negative operator we must have $\lambda \geq 0$.

Next, we consider the case in which ϕ_1 and ϕ_2 are not equivalent, so v is not identically 0. If we subtract the first equation in (38) from the second we obtain

$$-\gamma_1 v_{xx} + (b_{22}(x) - b_{12}(x))v = \lambda v, \tag{40}$$

so v must be an eigenfunction associated with the operator

$$H_1 := -\gamma_1 \partial^2_{xx} + (b_{22}(x) - b_{12}(x)).$$

The potential for this operator is

$$b_{22}(x) - b_{12}(x) = \gamma F''_{12}(\bar{u}_1(x)).$$

Comparing (40) with (30), we see that $v = \bar{u}_1'(x)$ is an eigenfunction of (40) associated with eigenvalue $\lambda = 0$. Since $\bar{u}_1'(x)$ is monotonic we know from standard ODE theory (see, for example, the discussion in [22]) that the operator H_1 has no eigenvalues below $\lambda = 0$. This completes the proof. ☐

As an application, consider the fourth-order Boyer-Lapuerta bulk free energy density (i.e., (19) with $H \equiv 0$, (taken without loss of generality; see the remark following (19) and $G \equiv 0$),

$$F(u_1, u_2, u_3) = \frac{\gamma_1}{2} u_1^2 (u_2 + u_3)^2 + \frac{\gamma_2}{2} u_2^2 (u_1 + u_3)^2 + \frac{\gamma_3}{2} u_3^2 (u_1 + u_2)^2, \tag{41}$$

which (as noted in [5]) is equivalent along the restriction $u_1 + u_2 + u_3 = 1$ to the simpler form

$$F(u_1, u_2, u_3) = \sum_{j=1}^{3} \frac{\gamma_j}{2} u_j^2 (1 - u_j)^2.$$

In this case,

$$F_{12}(u_1) = F(u_1, 1 - u_1, 0) = \gamma_1 u_1^2 (1 - u_1)^2,$$

and

$$\begin{aligned} b_{11}(x) &= \gamma_0 \Big((\frac{1}{\gamma_2} + \frac{1}{\gamma_3}) \frac{1}{2} F''_{12}(\bar{u}_1) + 1 \Big) \\ b_{12}(x) &= \gamma_0 \Big(-\frac{1}{\gamma_1} \frac{1}{2} F''_{12}(\bar{u}_1) + 1 \Big). \end{aligned} \tag{42}$$

In the case $\gamma_1 = \gamma_2$ we have

$$b_{11}(x) + b_{12}(x) = \gamma_0 \Big(\frac{1}{\gamma_3} \frac{1}{2} F''_{12}(\bar{u}_1) + 2 \Big).$$

Here,

$$F''_{12}(\bar{u}_1) = 2\gamma_1 (\bar{u}_1^2 - 4\bar{u}_1(1 - \bar{u}_1) + (1 - \bar{u}_1)^2),$$

so that

$$\min_{\bar{u}_1 \in [0,1]} F''_{12}(\bar{u}_1) = -\gamma_1.$$

In this way, we see that Condition (iii) of Lemma 3.1 holds in this case so long as

$$\gamma_1 \leq 4\gamma_3.$$

4. **ODE Estimates.** The general eigenvalue problem is

$$\Big(M(x)(-\Gamma \phi_{xx} + B(x)\phi)_x \Big)_x = \lambda \phi, \tag{43}$$

where $B(x) := f'(\bar{u}(x))$ and (with a slight abuse of notation) $M(x) := M(\bar{u}(x))$. We set $W_j = \partial_x^{j-1} \phi$, $j = 1, 2, 3, 4$, and regard this equation as the first order system $W' = \mathbb{A}(x; \lambda) W$. As $x \to \pm\infty$, we can write this system as

$$W' = \mathbb{A}_{\pm}(\lambda) W + Q_{\pm}(x; \lambda) W, \tag{44}$$

where

$$\mathbb{A}_{\pm}(\lambda) = \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -\lambda \Gamma^{-1} M_{\pm}^{-1} & 0 & \Gamma^{-1} B_{\pm} & 0 \end{pmatrix}, \tag{45}$$

and there exists $\eta > 0$ so that

$$Q_-(x;\lambda) = \mathbf{O}(e^{-\eta|x|}), x \to -\infty; \quad Q_+(x;\lambda) = \mathbf{O}(e^{-\eta|x|}), x \to +\infty,$$

uniformly for λ sufficiently small. We note that the $4m \times 4m$ matrices $Q_{\pm}$ only have non-zero entries in their last m rows.

While the eigenvalues of $\mathbb{A}_{\pm}(\lambda)$ can be computed directly using, for example, the determinant identity

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A) \cdot \det(D - CA^{-1}B),$$

valid whenever A is a non-singular matrix, it is more straightforward (and is equivalent) to simply look for solutions of the form $\phi = e^{\mu x} r$ for the asymptotic equation,

$$-M_{\pm}\Gamma \phi'''' + M_{\pm} B_{\pm} \phi'' = \lambda \phi.$$

We find

$$\Big(-\mu^4 M_{\pm}\Gamma + \mu^2 M_{\pm} B_{\pm} - \lambda I \Big) r = 0.$$

We will divide our discussion of the growth and decay modes $\mu(\lambda)$ into two cases: (1) *fast rates*, for which $\mu(0) \neq 0$; and (2) *slow rates*, for which $\mu(0) = 0$.

Fast rates. For the fast rates, $\mu(0) = \mu_0 \neq 0$, and for $\lambda = 0$ we have

$$\det(-\mu_0^4 M_{\pm}\Gamma + \mu_0^2 M_{\pm} B_{\pm}) = \mu_0^{2m} \det(M_{\pm}\Gamma) \det(\Gamma^{-1} B_{\pm} - \mu_0^2 I) = 0.$$

Since Γ and $M_{\pm}$ are positive definite, we find that the values for μ_0^2 are the eigenvalues of $\Gamma^{-1}B_{\pm}$. If the eigenvalues of this matrix are distinct, it follows from Theorem XII.1 of [30] that the associated fast $\mu(\lambda)^2$ are analytic. If the eigenvalues of $\Gamma^{-1}B_{\pm}$ are not distinct, we have from (H4) that the fast eigenvalues are nonetheless analytic in λ. In either case, it's clear that since the matrices $B_{\pm}$ are additionally symmetric and positive definite the eigenvalues of $\Gamma^{-1}B_{\pm}$ are positive. I.e.,

$$\Gamma^{-1} B_{\pm} v = \mu_0^2 v \Rightarrow B_{\pm} v = \mu_0^2 \Gamma v \Rightarrow \langle v, B_{\pm} v \rangle = \mu_0^2 \langle v, \Gamma v \rangle \Rightarrow \mu_0^2 = \frac{\langle v, B_{\pm} v \rangle}{\langle v, \Gamma v \rangle} > 0.$$

Our notation will be

$$\sigma(\Gamma^{-1} B_{\pm}) = \{\nu_j^{\pm}\}_{j=1}^m, \tag{46}$$

ordered so that $j < k \Rightarrow \nu_j^{\pm} \leq \nu_k^{\pm}$. We conclude that for $j = 1, \dots, m$ the fast rates $\{\mu_j^{\pm}\}_{j=1}^m$ and $\{\mu_j^{\pm}\}_{j=3m+1}^{4m}$ are given by

$$\begin{aligned} \mu_j^{\pm}(\lambda) &= -\sqrt{\nu_{m+1-j}^{\pm}} + \mathbf{O}(|\lambda|) \\ \mu_{3m+j}^{\pm}(\lambda) &= \sqrt{\nu_j^{\pm}} + \mathbf{O}(|\lambda|), \end{aligned} \tag{47}$$

where for consistency the indices are chosen so that $j < k \Rightarrow \mu_j^{\pm} \leq \mu_k^{\pm}$. The eigenvectors $\{V_j^{\pm}(\lambda)\}_{j=1}^m$ and $\{V_j^{\pm}(\lambda)\}_{j=3m+1}^{4m}$ associated with these eigenvalues have the form

$$V_j^\pm = \begin{pmatrix} r_j^\pm \\ \mu_j^\pm r_j^\pm \\ (\mu_j^\pm)^2 r_j^\pm \\ (\mu_j^\pm)^3 r_j^\pm \end{pmatrix}, \tag{48}$$

where $r_j^\pm(\lambda)$ satisfies

$$\Big(- (\mu_j^\pm)^4 M_\pm \Gamma + (\mu_j^\pm)^2 M_\pm B_\pm - \lambda I \Big) r_j^\pm = 0. \tag{49}$$

Since $\mu_j^\pm = -\mu_{4m+1-j}^\pm$ we clearly have

$$r_j^\pm(\lambda) = r_{4m+1-j}^\pm(\lambda),$$

for $j = 1, \dots, m$. Finally, the leading term $r_j^\pm(0)$ is an eigenvector of $\Gamma^{-1} B_\pm$ associated with the eigenvalue $(\mu_j^\pm(0))^2$.

Slow rates. For the slow rates, for which $\mu(0) = 0$, we set $\omega = \mu^2$ so that our characteristic equation becomes

$$\det \Big(- \omega^2 M_\pm \Gamma + \omega M_\pm B_\pm - \lambda I \Big) = 0. \tag{50}$$

In this case, when $\lambda = 0$ we have that $\omega_0 = 0$ is repeated m times, and so Theorem XII.1 of [30] does not apply directly. Instead, we work with the scaled variable ζ, defined so that $\omega = \lambda \zeta$. In this way, (50) becomes

$$\det \Big(- \lambda^2 \zeta^2 M_\pm \Gamma + \lambda \zeta M_\pm B_\pm - \lambda I \Big) = 0.$$

Upon dividing by λ^m, we have

$$\det \Big(- \lambda \zeta^2 M_\pm \Gamma + \zeta M_\pm B_\pm - I \Big) = 0,$$

where now setting $\lambda = 0$ we find the values of $\zeta(0)$ are precisely the eigenvalues of $B_\pm^{-1} M_\pm^{-1}$. If the eigenvalues of $B_\pm^{-1} M_\pm^{-1}$ are distinct, we can conclude, again from Theorem XII.1 of [30], that the $\zeta(\lambda)$ are analytic in λ. We have, then,

$$\omega(\lambda) = \sum_{j=1}^{\infty} a_j \lambda^j,$$

and so the slow modes $\mu^\pm(\lambda)$ have the form $\sqrt{\lambda} h(\lambda)$, where h is an analytic function in λ (for $|\lambda|$ sufficiently small) with $h(0) \neq 0$. Calculating almost precisely as for the case of $\Gamma^{-1} B_\pm$ we find that the eigenvalues of $B_\pm^{-1} M_\pm^{-1}$ are all real and positive. (We recall from (H1) and (H3) that $B_\pm^{-1}$ and $M_\pm^{-1}$ are both symmetric positive definite matrices.) Our notation will be

$$\sigma(M_\pm B_\pm) = \{\beta_j^\pm\}_{j=1}^m,$$

where again our choice of ordering is $j < k \Rightarrow \beta_j \leq \beta_k$. We conclude that for each $j = 1, \dots, m$ we have $\omega_j(\lambda) = \frac{\lambda}{\beta_j^\pm} + \mathbf{O}(|\lambda|^2)$, and so the slow rates are $\{\mu_j^\pm\}_{j=m+1}^{3m}$

$$\begin{aligned} \mu_{m+j}^\pm(\lambda) &= -\sqrt{\frac{\lambda}{\beta_j^\pm}} + \mathbf{O}(|\lambda|^{3/2}), \quad j = 1, \dots, m \\ \mu_{2m+j}^\pm(\lambda) &= \sqrt{\frac{\lambda}{\beta_{m+1-j}^\pm}} + \mathbf{O}(|\lambda|^{3/2}), \quad j = 1, \dots, m. \end{aligned} \tag{51}$$

Similarly as with the case of fast modes, if the eigenvalues of $B_\pm^{-1}M_\pm^{-1}$ are not distinct, we have directly from (H4) that for $|\lambda|$ sufficiently small we can write $\mu(\lambda) = \sqrt{\lambda}h(\lambda)$ for some function $h(\lambda)$ that is analytic in λ.

The eigenvectors $\{V_j^\pm(\lambda)\}_{j=m+1}^{3m}$ associated with these eigenvalues have the form (48) where $r_j^\pm(\lambda)$ satisfies (49). Since $\mu_j^\pm = -\mu_{4m+1-j}^\pm$ for $j = 1, \dots, 2m$ we clearly have

$$r_j^\pm(\lambda) = r_{4m+1-j}^\pm(\lambda),$$

for $j = m+1, \dots, 2m$. Finally, the values $\{\beta_j\}_{j=1}^m$ and $\{r_j^-(0)\}_{j=2m+1}^{3m}$, along with $\{r_j^+(0)\}_{j=m+1}^{2m}$ are sufficiently important to our later calculations that we summarize their roles in the following remark.

Remark 3. We take

$$\sigma(M_\pm B_\pm) = \{\beta_j^\pm\}_{j=1}^m,$$

where our choice of ordering is $j < k \Rightarrow \beta_j \le \beta_k$. The eigenvectors $\{r_k^-(0)\}_{k=2m+1}^{3m}$ and $\{r_k^+(0)\}_{k=m+1}^{2m}$ associated with these eigenvalues have the following index correspondences, for $j = 1, \dots, m$:

$$M_- B_- r_{2m+j}^-(0) = \beta_{m+1-j}^- r_{2m+j}^-(0)$$
$$M_+ B_+ r_{m+j}^+(0) = \beta_j^+ r_{m+j}^+(0).$$

In particular, for the case $m = 2$ (convenient for devising simple examples), we see that $r_5^-(0)$ and $r_6^-(0)$ are respectively associated with β_2^- and β_1^-, while $r_3^+(0)$ and $r_4^+(0)$ are respectively associated with β_1^+ and β_2^+.

We are now prepared to state our basic ODE lemma.

Lemma 4.1. *Under Conditions (C1)–(C2), along with (H4), there exist values $\eta > 0$ and $r_0 > 0$ so that for a choice of linearly independent solutions of the eigenvalue problem (43), we have the following estimates, uniformly in the set $\{\lambda : \lambda \in B(0, r_0), Arg\lambda \neq \pi\}$:*

(I) For $x \le 0$ and $k = 0, 1, 2, 3$ we have:

(i) For $j = 1, \dots, 2m$

$$\partial_x^k \phi_j^-(x;\lambda) = e^{\mu_{2m+j}^-(\lambda)x}\Big(\mu_{2m+j}^-(\lambda)^k r_{2m+j}^-(\lambda) + \mathbf{O}(e^{-\eta|x|})\Big);$$

(ii) For $j = 1, \dots, m$

$$\partial_x^k \psi_j^-(x;\lambda) = e^{\mu_j^-(\lambda)x}\Big(\mu_j^-(\lambda)^k r_j^-(\lambda) + \mathbf{O}(e^{-\eta|x|})\Big);$$

(iii) For $j = m+1, \dots, 2m$

$$\partial_x^k \psi_j^-(x;\lambda) = \frac{1}{\mu_j^-(\lambda)}\Big(\mu_j^-(\lambda)^k e^{\mu_j^-(\lambda)x} - (-\mu_j^-(\lambda))^k e^{-\mu_j^-(\lambda)x}\Big) r_j^-(\lambda) + \mathbf{O}(e^{-\eta|x|});$$

(II) For $x \ge 0$ and $k = 0, 1, 2, 3$ we have:

(i) For $j = 1, \dots, 2m$

$$\partial_x^k \phi_j^+(x;\lambda) = e^{\mu_j^+(\lambda)x}\Big(\mu_j^+(\lambda)^k r_j^+(\lambda) + \mathbf{O}(e^{-\eta|x|})\Big);$$

(ii) For $j = 1, \dots, m$

$$\partial_x^k \psi_j^+(x;\lambda) = \frac{1}{\mu_{2m+j}^+}\Big((\mu_{2m+j}^+)^k e^{\mu_{2m+j}^+ x} - (-\mu_{2m+j}^+)^k e^{-\mu_{2m+j}^+ x}\Big) r_{2m+j}^+ + \mathbf{O}(e^{-\eta|x|});$$

(iii) For $j = m+1, \dots, 2m$

$$\partial_x^k \psi_j^+(x;\lambda) = e^{\mu_{2m+j}^+(\lambda)x}\Big(\mu_{2m+j}^+(\lambda)^k r_{2m+j}^+(\lambda) + \mathbf{O}(e^{-\eta|x|})\Big).$$

Remark 4. Before proving Lemma 4.1, we make several remarks.

1. The dependence of μ_{2m+j}^+ and r_{2m+j}^+ on λ has been suppressed in (IIii) for notational brevity.

2. The fast decay modes are $\{\phi_j^-\}_{j=m+1}^{2m}$ and $\{\phi_j^+\}_{j=1}^{m}$. Likewise, the slow decay modes are $\{\phi_j^-\}_{j=1}^{m}$ and $\{\phi_j^+\}_{j=m+1}^{2m}$.

3. The rates of growth and decay can be characterized for convenient reference as follows: for $j = 1, \dots, m$,

$$\begin{aligned}
\mu_j^\pm(\lambda) &= -\sqrt{\nu_{m+1-j}^\pm} + \mathbf{O}(|\lambda|)\\
\mu_{m+j}^\pm(\lambda) &= -\sqrt{\frac{\lambda}{\beta_j^\pm}} + \mathbf{O}(|\lambda|^{3/2}),\\
\mu_{2m+j}^\pm(\lambda) &= \sqrt{\frac{\lambda}{\beta_{m+1-j}^\pm}} + \mathbf{O}(|\lambda|^{3/2}),\\
\mu_{3m+j}^\pm(\lambda) &= \sqrt{\nu_j^\pm} + \mathbf{O}(|\lambda|).
\end{aligned} \tag{52}$$

4. We recall that for $j = 1, \dots, 2m$

$$\begin{aligned}
\mu_j^\pm(\lambda) &= -\mu_{4m+1-j}^\pm(\lambda)\\
r_j^\pm(\lambda) &= r_{4m+1-j}^\pm(\lambda).
\end{aligned} \tag{53}$$

5. For $j = 1, \dots, m$ and $j = 3m+1, \dots, 4m$ the leading terms $r_j^\pm(0)$ are eigenvectors of $\Gamma^{-1}B_\pm$ associated with the eigenvalue $(\mu_j^\pm(0))^2$. Likewise, for $j = m+1, \dots, 3m$ the leading terms $r_j^\pm(0)$ are eigenvectors of $B_\pm^{-1}M_\pm^{-1}$ with indices as specified in Remark 3.

6. The choice we take for our slow growth modes (the difference modes) serves to keep our slow growth linearly independent from our slow decay modes when $\lambda = 0$. This idea was taken from [4]. See also [20], where the idea is used in the case of single Cahn-Hilliard equations on $\mathbb{R}$, and [21], where the idea is used in the case of single Cahn-Hilliard equations on $\mathbb{R}^n$. For an alternative approach in a similar setting (degenerate viscous shock profiles), see [18, 19, 24].

Proof of Lemma 4.1. First, the cases (Ii), (Iii), (IIi), and (IIiii) can be established by a standard calculation almost identical to the one carried out in the proof of Proposition 3.1 of [32].

The cases (Iiii) and (IIii) are clearly similar, and so we work through the details only for (Iiii). We begin by noting that the m slow modes $\{\phi_j^-(x;0)\}_{j=m+1}^{2m}$ each correspond with a solution that neither grows nor decays as $x \to -\infty$. We let $\Phi_0(x)$ denote the $m \times m$ matrix constructed by taking each of these modes as a column. Looking for solutions of the form

$$\phi(x;\lambda) = \Phi_0(x)w(x;\lambda),$$

and using the decay rates in (C1)-(C2), we find that w solves

$$\begin{aligned} -M(x)\Gamma\Phi_0(x)w'''' + M(x)B(x)\Phi_0(x)w'' - \lambda\Phi_0(x)w \\ = \mathbf{O}(e^{-\eta|x|})w' + \mathbf{O}(e^{-\eta|x|})w'' + \mathbf{O}(e^{-\eta|x|})w'''. \end{aligned} \tag{54}$$

Since $M(x)$ is uniformly positive definite and Γ is positive definite, and since $\Phi_0(x)$ is invertible by construction, we can multiply this equation by $\Phi_0(x)^{-1}\Gamma^{-1}M(x)^{-1}$ and consider asymptotic limits of the coefficient matrices to obtain the form

$$\begin{aligned} -w'''' + {\Phi_0^-}^{-1}\Gamma^{-1}B_-\Phi_0^- w'' - \lambda {\Phi_0^-}^{-1}\Gamma^{-1}M_-^{-1}\Phi_0^- w \\ = \mathbf{O}(e^{-\eta|x|})w' + \mathbf{O}(e^{-\eta|x|})w'' + \mathbf{O}(e^{-\eta|x|})w''', \end{aligned} \tag{55}$$

where $\Phi_0^- := \Phi_0(-\infty)$. We set $v_j = \partial^{j-1}w$, $j = 1, \dots, 4$, and obtain the first order system

$$v' = \mathbb{A}_0^-(\lambda)v + \mathbb{E}_0(x;\lambda)v,$$

where

$$\mathbb{A}_0^-(\lambda) = \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -\lambda{\Phi_0^-}^{-1}\Gamma^{-1}M_\pm^{-1}\Phi_0^- & 0 & {\Phi_0^-}^{-1}\Gamma^{-1}B_\pm\Phi_0^- & 0 \end{pmatrix},$$

and

$$\mathbb{E}_0(x;\lambda) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda\mathbf{O}(e^{-\alpha|x|}) & \mathbf{O}(e^{-\alpha|x|}) & \mathbf{O}(e^{-\alpha|x|}) & \mathbf{O}(e^{-\alpha|x|}) \end{pmatrix}.$$

Here, we recall that $\alpha = \min\{\alpha_B, \alpha_M\}$ (see (C1) and (C2)). The eigenvalues of $\mathbb{A}_0^-(\lambda)$ are precisely the same as for $\mathbb{A}_-(\lambda)$ (i.e., the $\{\mu_j^-(\lambda)\}_{j=1}^{4m}$), while the eigenvectors are

$$v_j^- = \begin{pmatrix} {\Phi_0^-}^{-1} r_j^\pm \\ \mu_j^\pm {\Phi_0^-}^{-1} r_j^\pm \\ (\mu_j^\pm)^2 {\Phi_0^-}^{-1} r_j^\pm \\ (\mu_j^\pm)^3 {\Phi_0^-}^{-1} r_j^\pm \end{pmatrix}.$$

Let $\mu_j^-(\lambda)$ denote any slow decay rate for $x < 0$ (i.e., $j \in \{2m+1, \dots, 3m\}$), and set $v = e^{\mu_j^-(\lambda)x}z$, so that

$$z' = (\mathbb{A}_0^-(\lambda) - \mu_j^-(\lambda)I)z + \mathbb{E}_0(x;\lambda)z. \tag{56}$$

Let $\bar{\eta} > 0$ be any constant so that $\bar{\eta} < \alpha$. Fix, in addition, constants η_1 and η_2 so that $\bar{\eta} < \eta_1 < \eta_2 < \alpha$. Let P_0 denote a projection operator projecting vectors in $\mathbb{R}^{4m}$ onto the eigenspace spanned by the eigenvectors of $\mathbb{A}_0^-$ that are associated with eigenvalues $\tilde{\mu}$ so that

$$\mathrm{Re}(\tilde{\mu}) < \mathrm{Re}(\mu_j^-) + \eta_2,$$

and let Q_0 denote a projection operator projecting vectors in $\mathbb{R}^{4m}$ onto the eigenspace spanned by the eigenvectors of $\mathbb{A}_0^-$ that are associated with eigenvalues $\tilde{\mu}$ so that

$$\mathrm{Re}(\tilde{\mu}) \geq \mathrm{Re}(\mu_j^-) + \eta_2 > \mathrm{Re}(\mu_j^-) + \eta_1.$$

Clearly, we have that for any $v \in \mathbb{R}^m$, $v = P_0 v + Q_0 v$. Integrating, we find

$$\begin{aligned} z(x;\lambda) = \mathcal{T} v := v_j^-(\lambda) + \int_{-\infty}^{x} e^{(\mathbb{A}_0^-(\lambda) - \mu_j^-(\lambda) I)(x-y)} P_0 \mathbb{E}_0(y;\lambda) z(y;\lambda) dy \\ - \int_x^{-M} e^{(\mathbb{A}_0^-(\lambda) - \mu_j^-(\lambda) I)(x-y)} Q_0 \mathbb{E}_0(y;\lambda) z(y;\lambda) dy. \end{aligned} \tag{57}$$

The operator $\mathcal{T}$ is easily shown to be a contraction on $L^\infty(-\infty, -M)$ (see p. 780 of [32]), and so a standard iteration confirms that $z \in L^\infty(-\infty, -M)$. The important point here is that

$$\mathbb{E}_0(y;\lambda) v_j^-(\lambda) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathbf{O}(|\lambda^{1/2}|) \mathbf{O}(e^{-\alpha|y|}) \end{pmatrix}.$$

In particular, the error is uniformly $\mathbf{O}(|\lambda^{1/2}|)$. In this way, we find

$$z(x;\lambda) = v_j^-(\lambda) + \sqrt{\lambda}\mathbf{O}(e^{-\bar\eta|x|}),$$

and consequently

$$v_j(x;\lambda) = e^{\mu_j^-(\lambda)x}\Big(v_j^-(\lambda) + \sqrt{\lambda}\mathbf{O}(e^{-\bar\eta|x|}\Big).$$

Finally,

$$\phi_j^-(x;\lambda) = \Phi_0^-(x) v_j(x;\lambda) = \Phi_0(x) e^{\mu_j^-(\lambda)x}\Big({\Phi_0^-}^{-1} r_j^- + \sqrt{\lambda}\mathbf{O}(e^{-\bar\eta|x|})\Big).$$

Proceeding in exactly the same way for the growth rate $\mu_{4m+1-j}^- = -\mu_j^-$, we can construct a solution of the eigenvalue problem (43) that, for $\lambda \neq 0$, grows as $x \to -\infty$,

$$\bar\psi_{4m+1-j}^-(x;\lambda) = \Phi_0(x) e^{-\mu_j^-(\lambda)x}\Big({\Phi_0^-}^{-1} r_j^- + \sqrt{\lambda}\mathbf{O}(e^{-\bar\eta|x|})\Big).$$

For $j = 2m+1, \dots, 3m$ (and so for $4m+1-j \in \{m+1, \dots, 2m\}$), set

$$\begin{aligned} \psi_{4m+1-j}^-(x;\lambda) &:= \frac{1}{-\mu_j^-(\lambda)}\Big(\bar\psi_{4m+1-j}^-(x;\lambda) - \phi_j^-(x;\lambda)\Big) \\ &= \frac{\Phi_0(x)}{-\mu_j^-(\lambda)}\Big((e^{-\mu_j^-(\lambda)x} - e^{\mu_j^-(\lambda)x}){\Phi_0^-}^{-1} r_j^- + \sqrt{\lambda}\mathbf{O}e^{-\bar\eta|x|}\Big). \end{aligned} \tag{58}$$

We now obtain the claimed estimate on $\psi_{4m+1-j}^-(x,\lambda)$ by expanding $\Phi_0(x)$ as $\Phi_0(x) = \Phi_0^- + \mathbf{O}(e^{-\bar\eta|x|})$ and observing that

$$\mathbf{O}(e^{-\bar\eta|x|})\Big(e^{-\mu_j^-(\lambda)x} - e^{\mu_j^-(\lambda)x}\Big) = \sqrt{\lambda}\mathbf{O}(e^{-\eta|x|}),$$

for any $\eta > 0$ so that $\eta < \bar\eta$.

The derivative estimates are obtained in a similar manner from the definition of $\psi_{4m+j-1}^-(x;\lambda)$ and the derivative estimates on ϕ_j^- and $\bar\psi_{4m+j-1}^-$. □

Remark 5. Since $\bar u'(x)$ is a solution of the eigenvalue problem (43) with $\lambda = 0$ there exist constants $\{e_j^-\}_{j=m+1}^{2m}$ and $\{e_j^+\}_{j=1}^{m}$ so that

$$\sum_{j=m+1}^{2m} e_j^- \phi_j^-(x;0) = \bar u'(x) = \sum_{j=1}^{m} e_j^+ \phi_j^+(x;0).$$

By subtracting away slow-decaying modes, we can choose the $\phi_j^\pm$, without changing our stated estimates, so that there exist integers $J^- \in \{m+1, m+2, \dots, 2m\}$ and $J^+ \in \{1, 2, \dots, m\}$ so that

$$\phi^-_{J^-}(x;0) = \bar{u}'(x) = \phi^+_{J^+}(x;0).$$

In some cases it's convenient to take the convention $J^- = 2m$ and $J^+ = 1$, but we must keep in mind that this changes our labeling in Lemma 4.1.

It's clear from our expressions for $\{\mu_j^\pm(\lambda)\}$ and $\{r_j^\pm(\lambda)\}$ that while the fast eigenvalue-eigenvector pairs are analytic in λ the slow eigenvalue-eigenvector pairs are not. We see from the discussion leading up to (51) that these slow rates are analytic as functions of $\rho := \sqrt{\lambda}$. This analyticity and its consequences will be important in our analysis of the Evans function, and we summarize some useful observations in the following two lemmas.

Lemma 4.2. *Let Conditions (C1) and (C2) hold, along with (H4), and let the functions $\{\phi_j^\pm(x;\lambda)\}_{j=1}^{2m}$ and $\{\psi_j^\pm(x;\lambda)\}_{j=1}^{2m}$ be as in Lemma 4.1. Let $\{\bar{\phi}_j^\pm(x;\rho)\}_{j=1}^{2m}$ and $\{\bar{\psi}_j^\pm(x;\rho)\}_{j=1}^{2m}$ denote functions obtained by formally replacing $\sqrt{\lambda}$ with ρ in the expressions for $\{\phi_j^\pm(x;\lambda)\}_{j=1}^{2m}$ and $\{\psi_j^\pm(x;\lambda)\}_{j=1}^{2m}$. Then the functions $\{\bar{\phi}_j^\pm(x;\rho)\}_{j=1}^{2m}$ and $\{\bar{\psi}_j^\pm(x;\rho)\}_{j=1}^{2m}$ are analytic in ρ in a neighborhood of $\rho = 0$, and for the operators*

$$\begin{aligned} \mathcal{T}_1\phi &:= M(x)(-\Gamma\phi'' + B(x)\phi)' \\ \mathcal{T}_2\phi &:= -\Gamma\phi'' + B(x)\phi \end{aligned} \tag{59}$$

we have the following relations:

(I) For all modes $\{\phi_j^\pm\}_{j=1}^{2m}$ we have

$$\mathcal{T}_1\bar{\phi}_j^\pm(x;0) = 0.$$

(II) For $x \le 0$

(i) For $j = 1, \dots, m$ (slow modes),

$$\mathcal{T}_2\bar{\phi}_j^-(x;0) = B_- r^-_{2m+j}(0);$$

(ii) For $j = m+1, \dots, 2m$ (fast modes),

$$\mathcal{T}_2\bar{\phi}_j^-(x;0) = 0;$$

(III) For $x \ge 0$

(i) For $j = 1, \dots, m$ (fast modes),

$$\mathcal{T}_2\bar{\phi}_j^+(x;0) = 0;$$

(ii) For $j = m+1, \dots, 2m$ (slow modes),

$$\mathcal{T}_2\bar{\phi}_j^+(x;0) = B_+ r_j^+(0).$$

Proof of Lemma 4.2. If we write the eigenvalue problem (43) in terms of ρ and the $\bar{\phi}_j^\pm$ we have

$$\Big(M(x)(-\Gamma(\bar{\phi}_j^\pm)'' + B(x)\bar{\phi}_j^\pm)'\Big)' = \rho^2\bar{\phi}_j^\pm. \tag{60}$$

We set $\rho = 0$ and integrate once to obtain

$$\mathcal{T}_1\bar{\phi}_j^\pm(x;0) = c_j^\pm,$$

for constants $c_j^\pm$. We observe from the estimates of Lemma 4.1 that $\partial_x \bar{\phi}_j^\pm(x;0)$ goes to 0 as $x \to \pm\infty$, so we find in these limits that $c_j^\pm = 0$ for all $j = 1, 2, \dots, 2m$. This establishes (I).

We now have $\mathcal{T}_1 \bar{\phi}_j^\pm(x;0) = 0$ for all modes, and since $M(x)$ is invertible for all values of x this implies

$$\mathcal{T}_2 \bar{\phi}_j^\pm(x;0) = d_j^\pm,$$

for new constants $d_j^\pm$. If we now take limits as $x \to \pm\infty$ we find that for fast modes $d_j^\pm = 0$, while for slow modes

$$d_j^\pm = \lim_{x\to\pm\infty} B(x)\bar{\phi}_j^\pm(x;0).$$

Claims (II) and (III) are immediate now from the estimates of Lemma 4.1. □

In the next lemma we gather useful relations regarding derivatives of the modes with respect to ρ.

Lemma 4.3. *Under the assumptions of Lemma 4.2 and using the notation of that lemma, we have the following:*

(I) For $x \leq 0$

(i) For $j = 1, \dots, m$ (slow modes),

$$\mathcal{T}_1(\partial_\rho \bar{\phi}_j^-)(x;0) = M_- B_- \frac{r_{2m+j}^-(0)}{\sqrt{\beta_{m+1-j}^-}} = \sqrt{\beta_{m+1-j}^-}\, r_{2m+j}^-(0);$$

(ii) For $j = m+1, \dots, 2m$ (fast modes),

$$\partial_\rho \bar{\phi}_j^- \Big|_{\rho=0} = 0$$

and for $j = J^-$ we also have

$$\mathcal{T}_1 \partial_{\rho\rho} \bar{\phi}_{J^-}(x;0) = 2(\bar{u}(x) - u_-),$$

where J^- is as in Remark 5.
(II) For $x \geq 0$

(i) For $j = 1, \dots, m$ (fast modes),

$$\partial_\rho \bar{\phi}_j^+ \Big|_{\rho=0} = 0$$

and for $j = J^+$ we also have

$$\mathcal{T}_1 \partial_{\rho\rho} \bar{\phi}_{J^+}(x;0) = -2(u_+ - \bar{u}(x)),$$

where J^+ is as in Remark 5.
(ii) For $j = m+1, \dots, 2m$ (slow modes),

$$\mathcal{T}_1 \partial_\rho \bar{\phi}_j^+(x;0) = -M_+ B_+ \frac{r_j^+(0)}{\sqrt{\beta_{j-m}^+}} = -\sqrt{\beta_{j-m}^+}\, r_j^+(0).$$

Proof of Lemma 4.3. We begin with the fast modes, which are analytic in λ and so are analytic as functions of ρ^2. The first parts of claims (Iii) and (IIi) follow immediately from this analyticity.

We take two derivatives of the eigenvalue equation (60) with respect to ρ, and set $\rho = 0$. For the particular modes $\bar{\phi}^{\pm}_{J\pm}$ this gives

$$\Big(M(x)(-\Gamma((\partial_{\rho\rho}\bar{\phi}^{\pm}_{J\pm})''(x;0) + B(x)\partial_{\rho\rho}\bar{\phi}^{\pm}_{J\pm}(x;0))'\Big)' = 2\bar{u}'(x).$$

The second part of Claim (Iii) follows from integration of the equation for ϕ^{-}_{J-} on $(-\infty, x]$, while the second part of Claim (IIi) follows from integration of the equation for ϕ^{+}_{J+} on $[x,\infty)$.

For the slow modes, we take a single ρ-derivative of the eigenvalue equation (60) and set $\rho = 0$. This gives

$$\mathcal{T}_1 \partial_\rho \bar{\phi}^{\pm}_j(x;0) = c^{\pm}_j, \tag{61}$$

for constants $c^{\pm}_j$. We will evaluate $c^{\pm}_j$ in each case by taking the limit as $x \to \pm\infty$. In order to compute

$$\lim_{x\to\pm\infty} (\partial_\rho \bar{\phi}^{\pm}_j)'(x;0),$$

(here, prime denotes differentiation with respect to x) we first observe from Lemma 4.1 that

$$\begin{aligned} \partial_x \bar{\phi}^{-}_j(x;\rho) = e^{(\frac{\rho}{\sqrt{\beta^{-}_{m+1-j}}} + \mathbf{O}(\rho^3))x} \\ \times \Big((\frac{\rho}{\sqrt{\beta^{-}_{m+1-j}}} + \mathbf{O}(\rho^3))(r^{-}_{2m+j}(0) + \mathbf{O}(\rho^2)) + \mathbf{O}(e^{-\eta|x|})\Big), \end{aligned} \tag{62}$$

where we have used analyticity to expand (analytic extensions of) μ^{-}_{2m+j} and r^{-}_{2m+j} in powers of ρ. We now take a derivative of (62) with respect to ρ and set $\rho = 0$. This gives

$$\partial_{x\rho} \bar{\phi}^{-}_j(x;\rho)\Big|_{\rho=0} = \frac{r^{-}_{2m+j}(0)}{\sqrt{\beta^{-}_{m+1-j}}} + \mathbf{O}(e^{-\eta|x|}),$$

from which we immediately see

$$\lim_{x\to-\infty} (\partial_\rho \bar{\phi}^{\pm}_j)'(x;0) = \frac{r^{-}_{2m+j}(0)}{\sqrt{\beta^{-}_{m+1-j}}}.$$

Likewise, if we start with $\partial^2_x \bar{\phi}^{-}_j(x;\rho)$, we find

$$\lim_{x\to-\infty} (\partial_\rho \bar{\phi}^{\pm}_j)''(x;0) = 0,$$

and (Ii) now follows by taking $x \to -\infty$ in (61). Claim (IIii) can be established similarly. □

5. **Analysis of the Evans Function.** We note at the outset that, for notational convenience, we will make two notational changes for this and the following section. First, we will work with ODE modes depending on ρ, but we will drop the bar notation from Section 4. Second, we will take the index convention

$$\phi^{-}_{2m}(x;0) = \bar{u}_x(x) = \phi^{+}_1(x;0). \tag{63}$$

To be clear, this generally will not be in agreement with the estimates of Lemma 4.1, and in principle requires an entirely new labeling scheme. We will see, however, that the present analysis only needs to distinguish between fast and slow modes, and the range of indices for this dichotomy will not be changed in the new convention.

We will need to specify a variety of vectors and matrices in terms of the $\phi_j^{\pm}$, and we summarize our notation for these here. We will set:

$$w_j^{\pm} = \begin{pmatrix} \phi_j^{\pm} \\ (\phi_j^{\pm})' \end{pmatrix} \quad \Phi_j^{\pm} = \begin{pmatrix} \phi_j^{\pm} \\ (\phi_j^{\pm})' \\ (\phi_j^{\pm})'' \\ (\phi_j^{\pm})''' \end{pmatrix} \quad \Phi^{\pm} = (\Phi_1^{\pm}, \Phi_2^{\pm}, \dots, \Phi_{2m}^{\pm}). \tag{64}$$

We note that $\Phi^{\pm}$ thus denotes the $4m \times 2m$ matrix in which $\Phi_j^{\pm}$ comprises the j^{th} column.

With this notation in place, we can write the Evans function associated with (1) and the wave $\bar{u}(x)$ as

$$D_a(\rho) = \det(\Phi^+(0;\rho), \Phi^-(0;\rho)). \tag{65}$$

(When the Evans function is regarded as a function of λ, we will designate it $D(\lambda)$ as in most standard literature.) In this context, the Evans function is simply a Wronskian computed with all asymptotically decaying solutions of the eigenvalue problem (60). Our designation of D as an Evans function, as opposed to Wronskian, is taken because the tools we will use to analyze $D_a(\rho)$ are taken in large part from Evans function literature. For a brief overview of this literature, and references, see, for example, p. 771 of [20].

Since an eigenfunction of (60) must, by definition, decay at both $\pm\infty$, we are assured that any eigenfunction is a linear combination both of the $2m$ modes that decay as $x \to -\infty$ and the $2m$ modes that decay as $x \to +\infty$. In this way, the Evans function clearly vanishes at all eigenvalues, and so serves as a characteristic function for the operator L. In addition, it can often be shown that, away from essential spectrum, the degree to which the Evans function vanishes corresponds with the geometric multiplicity of the eigenvalue (see particularly [3, 16]). In the current setting $\lambda = 0$ belongs to the essential spectrum, and so we don't necessarily expect this property to hold, and in fact since

$$\frac{d^j}{d\rho^j} D_a(\rho)\Big|_{\rho=0} = 0, \quad j = 0, 1, \dots, m,$$

(as we will verify below) we know that it does not.

In the case of Cahn–Hilliard systems (1), we have seen in Section 3 that it's often possible to establish that the entire spectrum must lie on the negative real axis, including a leading eigenvalue at $\lambda = 0$. In these cases stability is determined by the nature of the eigenvalue at $\lambda = 0$. In the remainder of this paper we will focus on the condition

$$\frac{d^{m+1}}{d\rho^{m+1}} D_a(\rho)\Big|_{\rho=0} \neq 0.$$

(see Condition 1). We note that it has been shown in [20] that for $m = 1$ this condition is sufficient to establish nonlinear stability. The generalization of this result to systems will be published in a companion paper [23].

Our goal in evaluating the Evans function will be to work mainly with solutions of the twice-integrated equation

$$-\Gamma\phi'' + B(x)\phi = 0. \tag{66}$$

It's clear from consideration of the associated asymptotic equation that all solutions of this equation either grow or decay at exponential rate at $-\infty$ and likewise at $+\infty$.

Also, it's clear that we have m modes that decay at each end and m that grow. In general, we have the following useful lemma.

Lemma 5.1. *Let (H0)-(H4) hold. Then the fast decay modes $\{\phi_j^-(x;0)\}_{j=m+1}^{2m-1}$, $\{\phi_j^+(x;0)\}_{j=2}^{m}$, along with $\phi_{2m}^-(x;0) = \bar{u}_x(x) = \phi_1^+(x;0)$, and a single mode $\psi(x)$ that grows at both $\pm\infty$ form a complete basis of solutions for (66).*

Proof. Since (by our transversality assumption (H2)) $\bar{u}_x$ is the only solution to (66) that decays at both $\pm\infty$, and since all solutions either grow or decay at exponential rate at $-\infty$ and likewise at $+\infty$, we know that the modes $\{\phi_j^-(x;0)\}_{j=m+1}^{2m-1}$ must all grow at $+\infty$, while the modes $\{\phi_j^+(x;0)\}_{j=2}^{m}$ must all grow at $-\infty$. This provides us with m modes that decay at $-\infty$, m modes that decay at $+\infty$, $m-1$ modes that grow at $-\infty$, and $m-1$ modes that grow at $+\infty$. We complete the basis by taking any solution to (66) that grows at both $\pm\infty$ at the two rates that are not accounted for in the decay modes. □

Our next lemma describes the behavior of the slow modes $\{\phi_j^-(x;0)\}_{j=1}^{m}$ and $\{\phi_j^+(x;0)\}_{j=m+1}^{2m}$ in terms of the fast modes. For notational convenience, we augment the notation (64) with

$$w(x) = \begin{pmatrix} \psi(x) \\ \psi'(x) \end{pmatrix}.$$

If we regard (66) as a first order system

$$w_j^{\pm\prime}(x;0) = \begin{pmatrix} 0 & I \\ \Gamma^{-1}B(x) & 0 \end{pmatrix} w_j^{\pm}(x;0),$$

we can choose a $2m \times 2m$ fundamental matrix

$$\bar{\Phi}(x) := (w_1^+(x;0), w_2^+(x;0), \dots, w_m^+(x;0), w_{m+1}^-(x;0), w_{m+2}^-(x;0), \dots, w(x)).$$

Clearly (by Abel's formula) $\det \bar{\Phi}(x)$ is constant, and it is non-zero by assumption/construction (i.e., under the assumption that $\bar{u}'(x)$ is the only solution of (66) that decays at both $\pm\infty$). We'll write

$$\det \bar{\Phi}(x) = \Delta_0. \tag{67}$$

Lemma 5.2. *Let (H0)-(H4) hold. Then:*

(I) For each slow mode $\phi_j^-(x;0)$, $j \in \{1, 2, \dots, m\}$, after possibly (and without loss of generality) subtracting off modes that decay at $-\infty$ we have

$$w_j^-(0;0) = c_{(2m)j}^- w(0) + \sum_{k=2}^{m} c_{kj}^- w_k^+(0;0),$$

where for $k = 2, \dots, m$ and $k = 2m$

$$c_{kj}^- = \frac{1}{\Delta_0} \int_{-\infty}^{0} \det \bar{\Phi}_{kj}^-(x;0) dx,$$

and $\bar{\Phi}_{kj}^-(x;0)$ denotes the matrix obtained by replacing the k-th column of $\bar{\Phi}(x)$ with

$$(0, -\Gamma^{-1}B_- r_{2m+j}^-(0))^{tr}.$$

(II) For each slow mode $\phi_j^+(x;0)$, $j \in \{m+1, m+2, \dots, 2m\}$, after possibly (and without loss of generality) subtracting off modes that decay at ∞ we have

$$w_j^+(0;0) = c_{(2m)j}^+ w(0) + \sum_{k=m+1}^{2m-1} c_{kj}^+ w_k^-(0;0),$$

where for $k = m+1, \dots, 2m-1$

$$c_{kj}^+ = -\frac{1}{\Delta_0} \int_0^\infty \det \bar{\Phi}_{kj}^+(x;0) dx,$$

and $\bar{\Phi}_{kj}^+(x;0)$ denotes the matrix obtained by replacing the k-th column of $\bar{\Phi}(x)$ with

$$(0, -\Gamma^{-1} B_+ r_j^+(0))^{tr}.$$

Remark 6. The only coefficients that will be critical for our later analysis will be $c_{(2m)j}^\pm$, and for brevity in the expressions we will write these as simply $c_j^\pm$.

Proof. Let $\phi_j^-(x;0)$ be any slow mode at $-\infty$, and recall from Lemma 4.2 that

$$-\Gamma \phi_j^{-\prime\prime}(x;0) + B(x)\phi_j^-(x;0) = B_- r_{2m+j}^-(0).$$

If we write this as a system for $w_j^-(x;0)$ we have the standard representation in terms of the homogeneous basis,

$$\begin{aligned} w_j^-(x;0) = c_j^- w(x) + \sum_{k=1}^m c_{jk}^- w_k^+(x;0) + \sum_{k=m+1}^{2m-1} c_{jk}^- w_k^-(x;0) \\ + \bar{\Phi}(x) \int_0^x \bar{\Phi}(y)^{-1} \begin{pmatrix} 0 \\ -\Gamma^{-1} B_- r_{2m+j}^-(0) \end{pmatrix} dy. \end{aligned} \tag{68}$$

The $\{w_k^-(x;0)\}_{k=m+1}^{2m-1}$ and $w_1^+(x;0) = \bar{u}_x(x)$ decay at exponential rate as $x \to -\infty$, and so we can eliminate these without loss of generality, and without changing the estimates of Lemma 4.1. (We recall that the slow modes have not been redefined.) Setting, for $j = 1, 2, \dots, 2m$,

$$J_j(x) = \int_0^x \bar{\Phi}(y)^{-1} \begin{pmatrix} 0 \\ -\Gamma^{-1} B_- r_{2m+j}^-(0) \end{pmatrix} dy, \quad J_j = \begin{pmatrix} J_{1j} \\ J_{2j} \\ \vdots \\ J_{(2m)j} \end{pmatrix},$$

we have

$$\begin{aligned} &\bar{\Phi}(x) \int_0^x \bar{\Phi}(y)^{-1} \begin{pmatrix} 0 \\ -\Gamma^{-1} B_- r_{2m+j}^-(0) \end{pmatrix} dy \\ &= \sum_{k=1}^m J_{kj} w_k^+(x;0) + \sum_{k=m+1}^{2m-1} J_{kj} w_k^-(x;0) + J_{(2m)j} w(x). \end{aligned} \tag{69}$$

According to Cramer's Rule, $\Delta_0 J_{kj}$ is an integral from 0 to x of the determinant of the matrix obtained by replacing the k-th column in $\bar{\Phi}(x)$ with

$$\begin{pmatrix} 0 \\ -\Gamma^{-1} B_- r_{2m+j}^-(0) \end{pmatrix}.$$

In particular, for $k = 2, \dots, m$, and $k = 2m$,

$$\begin{aligned} J_{kj}(x) &= \int_0^x \{\bar{\Phi}(y)^{-1} \begin{pmatrix} 0 \\ -\Gamma^{-1} B_- r^-_{2m+j}(0) \end{pmatrix} \}_k dy \\ &= \int_0^{-\infty} \{\bar{\Phi}(y)^{-1} \begin{pmatrix} 0 \\ -\Gamma^{-1} B_- r^-_{2m+j}(0) \end{pmatrix} \}_k dy \\ &\quad - \int_x^{-\infty} \{\bar{\Phi}(y)^{-1} \begin{pmatrix} 0 \\ -\Gamma^{-1} B_- r^-_{2m+j}(0) \end{pmatrix} \}_k dy \\ &= \int_0^{-\infty} \{\bar{\Phi}(y)^{-1} \begin{pmatrix} 0 \\ -\Gamma^{-1} B_- r^-_{2m+j}(0) \end{pmatrix} \}_k dy + \mathbf{O}(e^{-\eta|x|}), \end{aligned}$$

where we have observed that if $w(x)$ or any of the $w_k^+(x;0)$ is replaced in $\bar{\Phi}(x)$ with a constant vector then the determinant of the resulting matrix will decay at exponential rate as $x \to -\infty$, since all modes that decay at $-\infty$ will be present, while one mode that grows at $-\infty$ will be omitted (and the $2m$ modes sum to zero). It is important to observe here that since the vector $(-\Gamma^{-1} B_- r^-_{2m+j}(0))^{tr}$ is replacing the k-th column of $\bar{\Phi}$, and since $\bar{\Phi}$ comprises a full basis of solutions to (66), so that, in particular, the sum of rates associated with the columns in $\bar{\Phi}(x)$ is 0 for all x, we must have that the $\mathbf{O}(e^{-\eta|x|})$ term in our expression for J_{kj} decays at precisely the rate at which the replaced mode $w_k^+(x;0)$ grows at $-\infty$. In this way, the remainder terms

$$\mathbf{O}(e^{-\eta|x|}) w_k^+(x;0)$$

in (69) do not grow as $x \to -\infty$, and we have

$$\begin{aligned} &\bar{\Phi}(x) \int_0^x \bar{\Phi}(y)^{-1} \begin{pmatrix} 0 \\ -\Gamma^{-1} B_- r^-_{2m+j}(0) \end{pmatrix} dy \\ &= \sum_{k=1}^m J^-_{kj} w_k^+(x;0) + J^-_{(2m)j} w(x) + R_j(x), \end{aligned} \tag{70}$$

where

$$J^-_{kj} := \lim_{x \to -\infty} J_{kj}(x).$$

and $R_j(x)$ does not grow as $x \to -\infty$. Comparing now (68) and (70) we see that for each $k = 2, 3, \dots, m$, and for $k = 2m$,

$$c_{jk} = -J^-_{kj}.$$

Claim (I) now follows by setting $x = 0$. Claim (II) is proved similarly. □

5.1. **The Case m = 2.** In order to clarify our approach, we first analyze the specific case $m = 2$. We will adopt the notation

$$D_a(\rho) = W(\phi_1^+, \phi_2^+, \phi_3^+, \phi_4^+, \phi_1^-, \phi_2^-, \phi_3^-, \phi_4^-), \tag{71}$$

where according to (63)

$$\phi_4^-(x;0) = \bar{u}_x(x) = \phi_1^+(x;0),$$

and we recall that the fast modes are ϕ_1^+, ϕ_2^+, ϕ_3^- and ϕ_4^-. Clearly, $D_a(0) = 0$. In computing $D_a'(0)$, we sum eight Wronskian determinants, each of the form (71) with a ρ derivative on a single term. If this derivative falls on a slow mode then $\phi_4^-(x;0)$ and $\phi_1^+(x;0)$ are both still present and the determinant is zero, while if the ρ derivative falls on a fast mode that entire column becomes zero by analyticity (see Lemma 4.3). In this way it's clear that $D_a'(0) = 0$.

We now consider $D_a''(0)$. In this case, we can either have two ρ derivatives on any individual term, or we can have one ρ derivative on each of two terms. In the former case, if the ρ derivatives are not on either ϕ_4^- or ϕ_1^+ then the term is zero. In the latter case, if the ρ derivatives don't fall on either ϕ_4^- or ϕ_1^+ then the determinant is likewise zero, while if it falls on one of these the determinant is zero by analyticity. This leaves only the case where two ρ derivatives fall on either ϕ_4^- or ϕ_1^+, and in either of these cases $\bar{u}_x$ will replace whichever of these modes is undifferentiated. If we combine the two terms obtained this way, we find

$$D_a''(0) = W(\partial_{\rho\rho}(\phi_1^+ - \phi_4^-), \phi_2^+, \phi_3^+, \phi_4^+, \phi_1^-, \phi_2^-, \phi_3^-, \bar{u}_x). \tag{72}$$

The right-hand side of (72) is the determinant of an 8×8 matrix for which the final two rows consist of third order x derivatives of the specified function. In the case of undifferentiated terms, these third order derivatives can be replaced, using Lemma 4.2, with a linear combination of lower order derivatives. More precisely, we have, in all cases,

$$\phi_j^{\pm\prime\prime\prime} = \Gamma^{-1} B'(x) \phi_j^{\pm} + \Gamma^{-1} B(x) \phi_j^{\pm\prime}. \tag{73}$$

In addition, if we set $z = \partial_{\rho\rho}(\phi_1^+ - \phi_4^-)$ and use Parts (Iii) and (IIi) from Lemma 4.3, we have

$$z''' = \Gamma^{-1} B'(x) z + \Gamma^{-1} B(x) z' + 2\Gamma^{-1} M(x)^{-1}[u], \tag{74}$$

where we recall $[u] = u_+ - u_-$. Using row operations, we can reduce the matrix in (72) to one for which the last two rows in the first column are $2\Gamma^{-1} M(x)^{-1}[u]$, while the last two rows in each of the remaining columns are both zero. The determinant of such a matrix is clearly zero, and so we have $D_a''(0) = 0$.

We turn now to the critical term, for $m = 2$, $D_a'''(0)$. In this case we have a sum of determinants, where the summands can be categorized into three general cases: (1) three derivatives on a single term; (2) two derivatives on one term and one derivative on another; and (3) one derivative on each of three different terms. In the first case, if the three derivatives do not fall on either ϕ_4^- or ϕ_1^+ then the determinant is clearly zero, while if the three derivatives do fall on ϕ_4^- or ϕ_1^+ then the determinant is zero by analyticity. Likewise, in the third case if no single derivative falls on either ϕ_4^- or ϕ_1^+ then the determinant is clearly zero, while if a single derivative falls on either ϕ_4^- or ϕ_1^+ then the determinant is zero by analyticity. For the second case, if no derivative falls on either ϕ_4^- or ϕ_1^+ then the determinant is clearly zero, and if the single derivative falls on either ϕ_4^- or ϕ_1^+ then the determinant is zero by analyticity. This leaves only cases in which two derivatives fall on either ϕ_4^- or ϕ_1^+. In addition, if the single derivative falls on any fast mode the determinant is zero, leaving only cases in which we have two ρ derivatives on either ϕ_4^- or ϕ_1^+ and a single ρ derivative on one of the four slow modes. Finally, there are three ways to obtain each of these last terms, and so we have

$$\begin{aligned} \frac{1}{3} D_a'''(0) &= W(\partial_{\rho\rho}(\phi_1^+ - \phi_4^-), \phi_2^+, \partial_\rho \phi_3^+, \phi_4^+, \phi_1^-, \phi_2^-, \phi_3^-, \bar{u}_x)\Big|_{(x,\rho)=(0,0)} \\ &+ W(\partial_{\rho\rho}(\phi_1^+ - \phi_4^-), \phi_2^+, \phi_3^+, \partial_\rho \phi_4^+, \phi_1^-, \phi_2^-, \phi_3^-, \bar{u}_x)\Big|_{(x,\rho)=(0,0)} \\ &+ W(\partial_{\rho\rho}(\phi_1^+ - \phi_4^-), \phi_2^+, \phi_3^+, \phi_4^+, \partial_\rho \phi_1^-, \phi_2^-, \phi_3^-, \bar{u}_x)\Big|_{(x,\rho)=(0,0)} \\ &+ W(\partial_{\rho\rho}(\phi_1^+ - \phi_4^-), \phi_2^+, \phi_3^+, \phi_4^+, \phi_1^-, \partial_\rho \phi_2^-, \phi_3^-, \bar{u}_x)\Big|_{(x,\rho)=(0,0)} \end{aligned} \tag{75}$$

Since the analysis of each summand in (75) is almost identical to that of the others, we will work through details only for the first. For this calculation, we will use the notation $z = \partial_{\rho\rho}(\phi_1^+ - \phi_4^-)$, and in addition to (73) and (74), we recall from Lemma 4.3 the relation

$$(\partial_\rho \phi_3^+)''' = \Gamma^{-1}B'(x)\partial_\rho \phi_3^+ + \Gamma^{-1}B(x)\partial_\rho(\phi_3^+)' + \Gamma^{-1}M(x)^{-1}\sqrt{\beta_1^+}r_3^+(0). \tag{76}$$

More precisely, the assertion of (IIii) of Lemma 4.3 for ϕ_3^+ is

$$\mathcal{T}_1(\partial_\rho \phi_3^+)(x;0) = -\sqrt{\beta_1^+}r_3^+(0),$$

which is equivalent to (76). We use these relations along with appropriate row reduction to simplify the final two rows in our determinant matrix. In addition, for the fifth and sixth rows, which involve two derivatives, we use the observations that for all fast modes, including $\bar{u}_x$, we have

$$\phi_j^{\pm\prime\prime}(x;0) = \Gamma^{-1}B(x)\phi_j^{\pm}(x;0),$$

while for slow modes

$$\begin{aligned}
\phi_j^{-\prime\prime}(x;0) &= \Gamma^{-1}B(x)\phi_j^{-}(x;0) - \Gamma^{-1}B_- r_{4+j}^-(0), \quad j = 1,2 \\
\phi_j^{+\prime\prime}(x;0) &= \Gamma^{-1}B(x)\phi_j^{+}(x;0) - \Gamma^{-1}B_+ r_j^+(0), \quad j = 3,4.
\end{aligned}$$

After appropriate row operations, we find that the first summand on the right-hand side of (75) is

$$\mathcal{W}_3 = \det\begin{pmatrix} * & \phi_2^+ & * & \phi_4^+ & \phi_1^- & \phi_2^- & \phi_3^- & \bar{u}' \\ * & \phi_2^{+\prime} & * & \phi_4^{+\prime} & \phi_1^{-\prime} & \phi_2^{-\prime} & \phi_3^{-\prime} & \bar{u}'' \\ * & 0 & * & w_{34} & w_{35} & w_{36} & 0 & 0 \\ w_{41} & 0 & w_{43} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{77}$$

where

$$\begin{aligned}
w_{41} &= 2\Gamma^{-1}M(0)^{-1}[u] \\
w_{43} &= \Gamma^{-1}M(0)^{-1}\sqrt{\beta_1^+}r_3^+(0) \\
w_{34} &= -\Gamma^{-1}B_+ r_4^+(0) \\
w_{35} &= -\Gamma^{-1}B_- r_5^-(0) \\
w_{36} &= -\Gamma^{-1}B_- r_6^-(0),
\end{aligned} \tag{78}$$

and the terms indicated with $*$ do not contribute to the value of the determinant. (Our notation $\mathcal{W}_3$ corresponds with the summand in (75) that has a derivative on ϕ_3^+; see our discussion below of the general case for a full account of our labeling convention.) We now exchange the second and third columns and compute a block determinant to find

$$\begin{aligned}
\mathcal{W}_3 = &-\det\begin{pmatrix} \phi_2^+ & \phi_4^+ & \phi_1^- & \phi_2^- & \phi_3^- & \bar{u}' \\ \phi_2^{+\prime} & \phi_4^{+\prime} & \phi_1^{-\prime} & \phi_2^{-\prime} & \phi_3^{-\prime} & \bar{u}'' \\ 0 & w_{34} & w_{35} & w_{36} & 0 & 0 \end{pmatrix} \\
&\times \det\Big(2\Gamma^{-1}M(0)^{-1}[u], \Gamma^{-1}M(0)^{-1}\sqrt{\beta_1^+}r_3^+(0)\Big).
\end{aligned} \tag{79}$$

Lemma 5.2 asserts that

$$\begin{aligned} w_4^+(0;0) &= c_4^+ w(0) + c_{43}^+ w_3^-(0;0) \\ w_1^-(0;0) &= c_1^- w(0) + c_{12}^- w_2^+(0;0) \\ w_2^-(0;0) &= c_2^- w(0) + c_{22}^- w_2^+(0;0). \end{aligned}$$

We substitute these expressions into (79) and note that the expressions $c_{43}^+ w_3^-(0;0)$, $c_{12}^- w_2^+(0;0)$, and $c_{22}^- w_2^+(0;0)$ can all be removed with column operations. In this way, (79) becomes

$$\begin{aligned} \mathcal{W}_3 = &- \det \begin{pmatrix} \phi_2^+ & c_4^+\psi(0) & c_1^-\psi(0) & c_2^-\psi(0) & \phi_3^- & \bar{u}' \\ \phi_2^{+\prime} & c_4^+\psi'(0) & c_1^-\psi'(0) & c_2^-\psi'(0) & \phi_3^{-\prime} & \bar{u}'' \\ 0 & w_{34} & w_{35} & w_{36} & 0 & 0 \end{pmatrix} \\ &\times \det \Big(2\Gamma^{-1} M(0)^{-1}[u], \Gamma^{-1} M(0)^{-1} \sqrt{\beta_1^+} r_3^+(0) \Big). \end{aligned} \tag{80}$$

Assuming now that $c_4^+ \neq 0$, we use appropriate column operations to eliminate $c_1^- w(0)$ and $c_2^- w(0)$. In this way, after an even number of column exchanges (after 6), we find

$$\begin{aligned} \mathcal{W}_3 = &- \det \begin{pmatrix} \phi_2^+ & \phi_3^- & \bar{u}' & c_4^+\psi & 0 & 0 \\ \phi_2^{+\prime} & \phi_3^{-\prime} & \bar{u}'' & c_4^+\psi' & 0 & 0 \\ 0 & 0 & 0 & -\Gamma^{-1} B_+ r_4^+(0) & \bar{w}_{35} & \bar{w}_{36} \end{pmatrix} \\ &\times 2 \det(\Gamma^{-1} M(0)^{-1}) \det \Big([u], \sqrt{\beta_1^+} r_3^+(0) \Big), \end{aligned}$$

where

$$\begin{aligned} \bar{w}_{35} &= \frac{c_1^-}{c_4^+} \Gamma^{-1} B_+ r_4^+(0) - \Gamma^{-1} B_- r_5^-(0) \\ \bar{w}_{36} &= \frac{c_2^-}{c_4^+} \Gamma^{-1} B_+ r_4^+(0) - \Gamma^{-1} B_- r_6^-(0). \end{aligned}$$

Finally, we have, upon computing another block determinant,

$$\begin{aligned} \mathcal{W}_3 = &-2c_4^+ \Delta_0 \det(\Gamma^{-1})^2 \det(M(0)^{-1}) \det([u], \sqrt{\beta_1^+} r_3^+(0)) \\ &\times \det(\frac{c_1^-}{c_4^+} B_+ r_4^+(0) - B_- r_5^-(0), \frac{c_2^-}{c_4^+} B_+ r_4^+(0) - B_- r_6^-(0)). \end{aligned} \tag{81}$$

Proceeding similarly, we find

$$\begin{aligned} \mathcal{W}_4 = &\, 2c_3^+ \Delta_0 \det(\Gamma^{-1})^2 \det(M(0)^{-1}) \det([u], \sqrt{\beta_2^+} r_4^+(0)) \\ &\times \det(\frac{c_1^-}{c_3^+} B_+ r_3^+(0) - B_- r_5^-(0), \frac{c_2^-}{c_3^+} B_+ r_3^+(0) - B_- r_6^-(0)) \\ \mathcal{W}_1 = &\, 2c_3^+ \Delta_0 \det(\Gamma^{-1})^2 \det(M(0)^{-1}) \det([u], \sqrt{\beta_2^-} r_5^-(0)) \\ &\times \det(\frac{c_4^+}{c_3^+} B_+ r_3^+(0) - B_+ r_4^+(0), \frac{c_2^-}{c_3^+} B_+ r_3^+(0) - B_- r_6^-(0)) \\ \mathcal{W}_2 = &-2c_3^+ \Delta_0 \det(\Gamma^{-1})^2 \det(M(0)^{-1}) \det([u], \sqrt{\beta_1^-} r_6^-(0)) \\ &\times \det(\frac{c_4^+}{c_3^+} B_+ r_3^+(0) - B_+ r_4^+(0), \frac{c_1^-}{c_3^+} B_+ r_3^+(0) - B_- r_5^-(0)). \end{aligned}$$

Combining these, we have

$$\begin{aligned} D_a'''(0) &= 6\Delta_0(\det\Gamma^{-1})^2 \det M(0)^{-1} \\ &\times \Big\{ -c_4^+ \det(\frac{c_1^-}{c_4^+}B_+r_4^+ - B_-r_5^-, \frac{c_2^-}{c_4^+}B_+r_4^+ - B_-r_6^-) \det([u], \sqrt{\beta_1^+}r_3^+) \\ &+ c_3^+ \det(\frac{c_1^-}{c_3^+}B_+r_3^+ - B_-r_5^-, \frac{c_2^-}{c_3^+}B_+r_3^+ - B_-r_6^-) \det([u], \sqrt{\beta_2^+}r_4^+) \\ &+ c_3^+ \det(\frac{c_4^+}{c_3^+}B_+r_3^+ - B_+r_4^+, \frac{c_2^-}{c_3^+}B_+r_3^+ - B_-r_6^-) \det([u], \sqrt{\beta_2^-}r_5^-) \\ &- c_3^+ \det(\frac{c_4^+}{c_3^+}B_+r_3^+ - B_+r_4^+, \frac{c_1^-}{c_3^+}B_+r_3^+ - B_-r_5^-) \det([u], \sqrt{\beta_1^-}r_6^-)\Big\}, \end{aligned} \tag{82}$$

where for notational brevity we have suppressed the evaluation of the $r_j^\pm$ at $\lambda = 0$.

We know by construction that $\Delta_0 \neq 0$ (again, under our assumption that $\bar{u}'(x)$ is the only solution of (66) that decays at both $\pm\infty$), and so its value doesn't need to be computed explicitly. In particular, if we set

$$I_j^- := \int_{-\infty}^0 \det\begin{pmatrix} \bar{u}'(x) & \phi_2^+(x;0) & \phi_3^-(x;0) & 0 \\ \bar{u}''(x) & {\phi_2^+}'(x;0) & {\phi_3^-}'(x;0) & -\Gamma^{-1}B_-r_{4+j}^-(0) \end{pmatrix} dx,$$

for $j = 1, 2$, and

$$I_j^+ := -\int_0^\infty \det\begin{pmatrix} \bar{u}'(x) & \phi_2^+(x;0) & \phi_3^-(x;0) & 0 \\ \bar{u}''(x) & {\phi_2^+}'(x;0) & {\phi_3^-}'(x;0) & -\Gamma^{-1}B_+r_j^+(0) \end{pmatrix} dx,$$

for $j = 3, 4$, so that $\Delta_0 c_j^- = I_j^-$, $j = 1, 2$ and $\Delta_0 c_j^+ = I_j^+$, $j = 3, 4$, we have

$$\begin{aligned} D_a'''(0) &= 6(\det\Gamma^{-1})^2 \det M(0)^{-1} \\ &\times \Big\{ -I_4^+ \det(\frac{I_1^-}{I_4^+}B_+r_4^+ - B_-r_5^-, \frac{I_2^-}{I_4^+}B_+r_4^+ - B_-r_6^-) \det([u], \sqrt{\beta_1^+}r_3^+) \\ &+ I_3^+ \det(\frac{I_1^-}{I_3^+}B_+r_3^+ - B_-r_5^-, \frac{I_2^-}{I_3^+}B_+r_3^+ - B_-r_6^-) \det([u], \sqrt{\beta_2^+}r_4^+) \\ &+ I_3^+ \det(\frac{I_4^+}{I_3^+}B_+r_3^+ - B_+r_4^+, \frac{I_2^-}{I_3^+}B_+r_3^+ - B_-r_6^-) \det([u], \sqrt{\beta_2^-}r_5^-) \\ &- I_3^+ \det(\frac{I_4^+}{I_3^+}B_+r_3^+ - B_+r_4^+, \frac{I_1^-}{I_3^+}B_+r_3^+ - B_-r_5^-) \det([u], \sqrt{\beta_1^-}r_6^-)\Big\}, \end{aligned} \tag{83}$$

and we only need to check that the expression in brackets is non-zero.

5.2. **The General Case.** For $m = 3, 4, \dots$, the full expression for $\partial^{m+1}D_a(0)$ has $\binom{2m}{m-1}$ terms, and so we won't give a complete form such as (82).

In this case, we have

$$\phi_{2m}^-(x;0) = \bar{u}'(x) = \phi_1^+(x;0), \tag{84}$$

and we recall from Lemma 5.1 that we can construct a basis of solutions for

$$-\Gamma\phi'' + B(x)\phi = 0$$

by taking the set of solutions $\bar{u}'(x)$, $\{\phi_j^-(x;0)\}_{j=m+1}^{2m-1}$, and $\{\phi_j^+(x;0)\}_{j=2}^m$, $(2m-1$ total) and augmenting it with a (choice of) mode $\psi(x)$ that grows at both $\pm\infty$.

Proceeding as in our discussion of the case $m = 2$ we find that $D_a(0) = D_a'(0) = D_a''(0) = 0$. For $D_a'''(0)$, we can only have a non-zero determinant if two derivatives fall on either ϕ_{2m}^- or ϕ_1^+, leaving only one derivative for the remaining terms. If this remaining derivative falls on a fast mode the determinant is 0 by analyticity, while if it falls on a slow mode we have the determinant of a $4m \times 4m$ matrix with zeros in the last m rows of all except possibly two columns (the differentiated columns). For $m = 3, 4, \dots$, the determinant of such a matrix is clearly 0. In fact, it's clear that we can only possibly obtain a matrix with non-zero determinant if we have at least one derivative on each of m different columns. Since we must have at least two derivatives on either ϕ_{2m}^- or ϕ_1^+ this means that $\partial^k D_a(0)$ will certainly be 0 for all $k = 0, 1, \dots, m$. For $k = m+1$ we require two derivatives on either ϕ_{2m}^- or ϕ_1^+ and one derivative on each of $m-1$ different slow modes. Clearly, there are $\binom{2m}{m-1}$ different sets of $m-1$ slow modes (i.e., this will be the number of summands in $D_a^{(m+1)}(0)$), and (using multinomial coefficients) $\frac{(m+1)!}{2}$ ways to obtain each term.

We will use the notation

$$\mathcal{W}_{j_1, j_2, \dots, j_{m-1}} \tag{85}$$

to denote the term in $D_a^{(m+1)}(0)$ for which single derivatives fall on the slow modes ϕ_{j_1}, $\phi_{j_2}, \dots, \phi_{j_{m-1}}$. We note particularly, that since the slow modes are $\{\phi_j^-\}_{j=1}^m$ and $\{\phi_j^+\}_{j=m+1}^{2m}$ these indices uniquely determine the summand under consideration without a specification of sign. In this way, we have the summation formula

$$\frac{2}{(m+1)!} D_a^{(m+1)}(0) = \sum_{j_1, j_2, \dots, j_{m-1}=1}^{(2m)} \mathcal{W}_{j_1, j_2, \dots, j_{m-1}}, \tag{86}$$

where our notation $\sum_{j_1, j_2, \dots, j_{m-1}=1}^{(2m)}$ indicates the sum in which j_1 goes from 1 to $2m$, j_2 goes from $j_1 + 1$ to $2m$, j_3 goes from $j_2 + 1$ to $2m$, etc., and no two indices ever agree.

As an example, we compute $\mathcal{W}_{45}$ for the case $m = 3$. With our notation, this is

$$\mathcal{W}_{45} = W(\partial_{\rho\rho}(\phi_1^+ - \phi_6^-), \phi_2^+, \phi_3^+, \partial_\rho \phi_4^+, \partial_\rho \phi_5^+, \phi_6^+, \phi_1^-, \phi_2^-, \phi_3^-, \phi_4^-, \phi_5^-, \bar{u}'), \tag{87}$$

evaluated at $(x; \rho) = (0; 0)$, which we rearrange, after an odd number of column exchanges, as

$$\mathcal{W}_{45} = -W(\bar{u}', \phi_2^+, \phi_3^+, \phi_4^-, \phi_5^-, \phi_6^+, \phi_1^-, \phi_2^-, \phi_3^-, \partial_\rho \phi_4^+, \partial_\rho \phi_5^+, \partial_{\rho\rho}(\phi_1^+ - \phi_6^-)), \tag{88}$$

again evaluated at $(x; \rho) = (0; 0)$.

Proceeding as in the case $m = 2$, we find

$$\begin{aligned} \mathcal{W}_{45} &= -2c_6^+ \det \bar{\Phi}(0) \det(\Gamma^{-1})^2 \det(M(0)^{-1}) \\ &\times \det\Big(\frac{c_1^-}{c_6^+} B_+ r_6^+ - B_- r_7^-, \frac{c_2^-}{c_6^+} B_+ r_6^+ - B_- r_8^-, \frac{c_3^-}{c_6^+} B_+ r_6^+ - B_- r_9^-\Big) \\ &\times \det(\sqrt{\beta_1^+} r_4^+, \sqrt{\beta_2^+} r_5^+, [u]). \end{aligned} \tag{89}$$

In principle we must likewise compute 14 more terms, but since our examples will be taken only from the case $m = 2$ we omit a full expression for $D_a^{(4)}(0)$ in this case.

6. **Examples.** In this section we verify Condition 1 for two example cases.

6.1. **Boyer-Lapuerta Systems.** We consider the Boyer-Lapuerta model (14) under the conditions of Lemma 3.1, for which we have already established the absence of any positive eigenvalues. We recall, in particular, that this is the case for $m = 2$, with additionally $\gamma_1 = \gamma_2$, and with $F_{12}(u_1)$ symmetric about $u_1 = \frac{1}{2}$ (i.e., $F_{12}(u_1) = F_{12}(1 - u_1)$ for $u_1 \in [0, 1]$), and note that by a choice of shift (setting $\bar{u}_1(0) = \frac{1}{2}$) this ensures $\bar{u}'(x)$ is even in x.

In order to verify that $\bar{u}'(x)$ is the only solution of (66) that decays at both $\pm\infty$, and also to compute the integrals I_1^-, I_2^-, I_3^+, and I_4^+ we must understand, in addition to $\bar{u}'(x)$, the fast modes $\phi_3^-(x; 0)$ and $\phi_2^+(x; 0)$, and also the growth-growth mode $\psi(x)$. Each of these four vector functions solves the ODE system

$$\begin{aligned} -\gamma_1 \phi_1{}'' + b_{11}(x)\phi_1 + b_{12}(x)\phi_2 = 0 \\ -\gamma_1 \phi_2{}'' + b_{21}(x)\phi_1 + b_{22}(x)\phi_2 = 0, \end{aligned} \tag{90}$$

where, according to (39), $b_{11}(x) = b_{22}(x)$ and $b_{12}(x) = b_{21}(x)$. We already know, by construction, that $\bar{u}'(x)$ is a solution of (90) with form $\phi_2 = -\phi_1$. By substituting this relation into (90) we have

$$-\gamma_1 \phi_1'' + (b_{11}(x) - b_{12}(x))\phi_1 = 0.$$

By reduction of order we have a second solution of the form

$$\psi_1(x) = \bar{u}_1'(x) \int_0^x \frac{1}{\bar{u}_1'(y)^2} dy,$$

which grows at $\pm\infty$.

In order to find two more linearly independent solutions of (90) we look for solutions of the form $\phi_1 = \phi_2$, for which we have

$$-\gamma_1 \phi_1'' + (b_{11}(x) + b_{12}(x))\phi_1 = 0. \tag{91}$$

We have already seen in Lemma 3.1 that that there can be no solutions of (91) that decay at both $\pm\infty$, so we must have one solution that decays at $-\infty$ and grows at $+\infty$ and one solution that decays at $+\infty$ and grows at $-\infty$. We now take the first of these to be $\phi_{13}^-(x; 0)$ and the second to be $\phi_{12}^+(x; 0)$. In this way, we have

$$w_3^-(x; 0) = \begin{pmatrix} \phi_{13}^-(x; 0) \\ \phi_{13}^-(x; 0) \\ \phi_{13}^-{}'(x; 0) \\ \phi_{13}^-{}'(x; 0) \end{pmatrix}; \quad w_2^+(x; 0) = \begin{pmatrix} \phi_{12}^+(x; 0) \\ \phi_{12}^+(x; 0) \\ \phi_{12}^+{}'(x; 0) \\ \phi_{12}^+{}'(x; 0) \end{pmatrix}. \tag{92}$$

(To be clear, it is correct that the first two components of each of these vectors is the same, and similarly for the final two; in what follows, we take advantage of this structure.) In this way we have established a full basis of solutions for (90), and so $\Delta_0 \neq 0$. (Of course, it also follows that $\bar{u}'(x)$ is the only solution of (90) that decays at both $\pm\infty$.)

We next observe that $\bar{u}_1(x) = 1 - \bar{u}_1(-x)$ (by our symmetry assumptions), and using our relations

$$F_{12}(\bar{u}_1) = F(\bar{u}_1, 1 - \bar{u}_1, 0) = F(1 - \bar{u}_1, \bar{u}_1, 0),$$

we find that the potential

$$V_+(x) := b_{11}(x) + b_{12}(x) = \frac{1}{2} F_{12}''(\bar{u}_1(x)) + 2, \tag{93}$$

is even as a function of x. If $\phi_{13}^{-}(x;0)$ solves (91), then we can take as a second solution

$$\phi_{12}^{+}(x;0) = \phi_{13}^{-}(-x;0). \tag{94}$$

Now consider the integral

$$I_1^- = \int_{-\infty}^{0} \det \begin{pmatrix} \bar{u}'(x) & \phi_2^+(x;0) & \phi_3^-(x;0) & 0 \\ \bar{u}''(x) & {\phi_2^+}'(x;0) & {\phi_3^-}'(x;0) & -\Gamma^{-1}B_- r_5^-(0) \end{pmatrix} dx. \tag{95}$$

Since $\det(w_1^+(x;0), w_2^+(x,0), w_3^-(x;0), w(x)) \neq 0$ by construction, we know that the four vectors in the integrand for I_1^- can only be linearly dependent at a value x_0 if there exist constants α_1, α_2, and α_3, depending on x_0, so that

$$\begin{pmatrix} 0 \\ -\Gamma^{-1}B_- r_5^-(0) \end{pmatrix} = \alpha_1 w_1^+(x_0;0) + \alpha_2 w_2^+(x_0;0) + \alpha_3 w_3^-(x_0;0). \tag{96}$$

The first two equations in this relation are

$$\begin{aligned} 0 &= \alpha_1 \bar{u}_1'(x_0) + \alpha_2 \phi_{12}^+(x_0;0) + \alpha_3 \phi_{13}^-(x_0;0) \\ 0 &= -\alpha_1 \bar{u}_1'(x_0) + \alpha_2 \phi_{12}^+(x_0;0) + \alpha_3 \phi_{13}^-(x_0;0), \end{aligned} \tag{97}$$

and since $\bar{u}_1'(x_0) \neq 0$ by monotonicity, we must have $\alpha_1 = 0$. The third and fourth equations are now

$$\begin{aligned} \{-\Gamma^{-1}B_- r_5^-(0)\}_1 &= \alpha_2 {\phi_{12}^+}'(x_0;0) + \alpha_3 {\phi_{13}^+}'(x_0;0) \\ \{-\Gamma^{-1}B_- r_5^-(0)\}_2 &= \alpha_2 {\phi_{12}^+}'(x_0;0) + \alpha_3 {\phi_{13}^+}'(x_0;0), \end{aligned} \tag{98}$$

which can only hold if

$$\{-\Gamma^{-1}B_- r_5^-(0)\}_1 = \{-\Gamma^{-1}B_- r_5^-(0)\}_2. \tag{99}$$

This last condition is independent of x_0, so we can conclude that if (99) does not hold then the integrand defining I_1^- is never zero, and so $I_1^- \neq 0$. On the other hand, if (99) holds then equations (97) and (98) give the system

$$\begin{aligned} 0 &= \alpha_2 \phi_{12}^+(x_0;0) + \alpha_3 \phi_{13}^-(x_0;0) \\ \{-\Gamma^{-1}B_- r_5^-(0)\}_1 &= \alpha_2 {\phi_{12}^+}'(x_0;0) + \alpha_3 {\phi_{13}^-}'(x_0;0), \end{aligned}$$

and if $\{-\Gamma^{-1}B_- r_5^-(0)\}_1 \neq 0$ this system is uniquely solvable for α_2 and α_3, not both zero, since $\phi_{12}^+(x;0)$ and $\phi_{13}^-(x;0)$ are linearly independent solutions of (91). In this way, we see that (96) holds for α_1, α_2, and α_3, not all zero, and so $I_1^- = 0$.

Proceeding similarly for I_2^-, I_3^+, and I_4^+, we have the conditions

$$\begin{aligned} \{\Gamma^{-1}B_- r_{2m+j}^-(0)\}_1 = \{\Gamma^{-1}B_- r_{2m+j}^-(0)\}_2 &\Leftrightarrow I_j^- = 0, j = 1, 2 \\ \{\Gamma^{-1}B_+ r_j^+(0)\}_1 = \{\Gamma^{-1}B_- r_j^+(0)\}_2 &\Leftrightarrow I_j^+ = 0, j = 3, 4. \end{aligned} \tag{100}$$

We next observe, by linearity, that

$$I_1^- + I_2^- = \int_{-\infty}^{0} \det \begin{pmatrix} \bar{u}'(x) & \phi_2^+(x;0) & \phi_3^-(x;0) & 0 \\ \bar{u}''(x) & {\phi_2^+}'(x;0) & {\phi_3^-}'(x;0) & -\Gamma^{-1}B_-(r_5^-(0) + r_6^-(0)) \end{pmatrix} dx.$$

We can conclude immediately from the discussion leading up to (100) that if

$$\{\Gamma^{-1}B_-(r_5^-(0) + r_6^-(0))\}_1 = \{\Gamma^{-1}B_-(r_5^-(0) + r_6^-(0))\}_2, \tag{101}$$

then $I_1^- + I_2^- = 0$.

Likewise, if

$$\{\Gamma^{-1}B_+(r_3^+(0) + r_4^+(0))\}_1 = \{\Gamma^{-1}B_+(r_3^+(0) + r_4^+(0))\}_2, \tag{102}$$

the $I_3^+ + I_4^+ = 0$.

Finally, in order to relate I_1^- to I_3^+ and I_2^- to I_4^+ we use (94), and our assumption of endstate symmetry,

$$\begin{aligned} B_+ r_3^+(0) &= B_- r_5^-(0) \\ B_+ r_4^+(0) &= B_- r_6^-(0). \end{aligned} \tag{103}$$

We compute

$$\begin{aligned} I_3^+ &= -\int_o^\infty \det \begin{pmatrix} \bar{u}'(x) & \phi_2^+(x;0) & \phi_3^-(x;0) & 0 \\ \bar{u}''(x) & {\phi_2^+}'(x;0) & {\phi_3^-}'(x;0) & -\Gamma^{-1}B_+ r_3^+(0) \end{pmatrix} dx \\ &= -\int_0^\infty \det \begin{pmatrix} \bar{u}'(-x) & \phi_3^-(-x;0) & \phi_2^+(-x;0) & 0 \\ -\bar{u}''(-x) & -{\phi_3^-}'(-x;0) & -{\phi_2^+}'(-x;0) & -\Gamma^{-1}B_+ r_3^+(0) \end{pmatrix} dx. \end{aligned} \tag{104}$$

We now perform a single column exchange, and make the substitution $y = -x$ to find

$$I_3^+ = -\int_{-\infty}^0 \det \begin{pmatrix} \bar{u}'(x) & \phi_2^+(x;0) & \phi_3^-(x;0) & 0 \\ \bar{u}''(x) & {\phi_2^+}'(x;0) & {\phi_3^-}'(x;0) & -\Gamma^{-1}B_+ r_3^+(0) \end{pmatrix} dx.$$

In the event of endstate symmetry (103), we have, by comparison with (95),

$$I_3^+ = -I_1^- \quad \text{and} \quad I_4^+ = -I_2^-.$$

We summarize our observations regarding this special case in Lemma 6.1.

Lemma 6.1. *For the Boyer-Lapuerta system (14) with* $m = 2$*, suppose:*

1. Equation assumptions

$$\begin{aligned} F(u_1, 1-u_1, 0) &= F(1-u_1, u_1, 0) \\ \gamma_1 &= \gamma_2 \end{aligned}$$

2. Endstate assumptions

$$\begin{aligned} \{\Gamma^{-1}B_+ r_j^+(0)\}_1 &\neq \{\Gamma^{-1}B_+ r_j^+(0)\}_2, j = 3, 4 \\ \{\Gamma^{-1}B_+(r_3^+(0) + r_4^+(0))\}_1 &= \{\Gamma^{-1}B_+(r_3^+(0) + r_4^+(0)\}_2 \\ B_+ r_3^+(0) = B_- r_5^-(0) \quad &\textit{and} \quad B_+ r_4^+(0) = B_- r_6^-(0). \end{aligned}$$

Then

$$\begin{aligned} D_a'''(0) &= -12(\det \Gamma^{-1})^2 \det(M(0)^{-1}) I_4^+ \det(B_+) \det(r_3^+(0), r_4^+(0)) \\ &\times \det([u], \sqrt{\beta_1^+} r_3^+(0) + \sqrt{\beta_2^+} r_4^+(0) + \sqrt{\beta_2^-} r_5^-(0) + \sqrt{\beta_1^-} r_6^-(0)). \end{aligned}$$

Moreover, the top line in this expression for $D_a'''(0)$ *is non-zero, so spectral stability is entirely determined by the determinant on the second line.*

Proof. The proof of Lemma 6.1 has mostly been carried out in the discussion leading up to it. The last step consists simply in substituting the relations $I_1^- = -I_2^-$, $I_3^+ = -I_3^+$, $I_3^+ = -I_1^-$, and $I_4^+ = -I_2^-$ into (83). □

6.2. **Example case.** As an example case, let's take, for simplicity, the Boyer-Lapuerta model (14) with $m = 2$, $\gamma_0 = \gamma_1 = \gamma_2 = \gamma_3 = 1$, and $\tilde{M}(u) \equiv 1$. That is, our equations are

$$\begin{aligned} u_{j_t} &= \Big(- u_{j_{xx}} + \sum_{i=1}^{3} (F_{u_j}(u) - F_{u_i}(u)) \Big)_{xx}, \quad j = 1, 2, \\ u_3 &= 1 - u_1 - u_2. \end{aligned} \tag{105}$$

We take the bulk free energy function

$$F(u_1, u_2, u_3) = \frac{1}{2} u_1^2 (u_2 + u_3)^2 + \frac{1}{2} u_2^2 (u_1 + u_3)^2 + \frac{1}{2} u_3^2 (u_1 + u_2)^2. \tag{106}$$

In this case, we know from (30) that each component of $\bar{u}(x)$ solves

$$-\bar{u}_{j_{xx}} + \frac{3}{2} F_{12}'(\bar{u}_j) = 0,$$

where

$$F_{12}(\bar{u}_1) = F(\bar{u}_1, 1 - \bar{u}_2, 0) = \bar{u}_1^2 (1 - \bar{u}_1)^2.$$

Clearly, $\bar{u}_1(x)$ solves

$$-\bar{u}_{1_{xx}} + 3\bar{u}_1(1 - \bar{u}_1)(1 - 2\bar{u}_1) = 0,$$

and we find by direct calculation that if we fix a shift by setting $\bar{u}_1(0) = \frac{1}{2}$ then

$$\bar{u}_1(x) = \frac{1}{1 + e^{\sqrt{3}x}}. \tag{107}$$

By construction, we take $\bar{u}_2(x) = 1 - \bar{u}_1(x)$, and

$$[u] = u_+ - u_- = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{108}$$

If we linearize about $\bar{u}(x)$ and write the resulting eigenvalue equation in our standard form

$$\Big(M(\bar{u})(-\Gamma \phi_{xx} + B(x)\phi)_x \Big)_x = \lambda \phi,$$

we have $\Gamma = I$, $M(u) \equiv I$, and according to (42)

$$B(x) = \begin{pmatrix} F_{12}''(\bar{u}_1) + 1 & -\frac{1}{2} F_{12}''(\bar{u}_1) + 1 \\ -\frac{1}{2} F_{12}''(\bar{u}_1) + 1 & F_{12}''(\bar{u}_1) + 1 \end{pmatrix}. \tag{109}$$

Here, $F_{12}''(\bar{u}_1) = 2 - 12\bar{u}_1 + 12\bar{u}_1^2$, so

$$B_\pm = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}. \tag{110}$$

Clearly, this is a case in which the eigenvalues of $\Gamma^{-1} B_\pm$ and $M_\pm B_\pm$ are not distinct, and so we must verify the structure of our eigenvalues $\mu(\lambda)$ directly. A very brief calculation shows that

$$(\mu^4 - 3\mu^2 - \lambda)^2 = 0,$$

and so the fast modes are

$$\mu(\lambda) = \pm \sqrt{\frac{3}{2}(1 + \sqrt{1 - 4\lambda/9})},$$

while the slow modes are

$$\mu(\lambda) = \pm \sqrt{\frac{3}{2}(1 - \sqrt{1 - 4\lambda/9})}.$$

We can conclude from these expressions that the fast modes are analytic in λ and that the slow modes can be written in the form $\sqrt{\lambda}h(\lambda)$, where h is analytic in λ.

We have $\beta_1^{\pm} = \beta_2^{\pm} = 3$, and by convenient choice

$$r_3^+(0) = r_5^-(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad r_4^+(0) = r_6^-(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(See Remark 3 for clarification on the indices.)

We now have all the information we require in order to verify stability. We need only compute

$$\det([u], \sqrt{\beta_1^+} r_3^+(0) + \sqrt{\beta_2^+} r_4^+(0) + \sqrt{\beta_2^-} r_5^-(0) + \sqrt{\beta_1^-} r_6^-(0))$$
$$= \det \begin{pmatrix} -1 & 2\sqrt{3} \\ 1 & 2\sqrt{3} \end{pmatrix} = -4\sqrt{3} \neq 0.$$

Combining this with Lemma 3.1, we conclude that the wave $\bar{u}(x)$ is spectrally stable.

6.3. **Gradient Systems.** In this section we combine our analysis with numerical calculations to provide evidence for the stability of the stationary solution given in Figure 2.1. In the general case, we will generate transition front solutions numerically, so we begin by briefly outlining this calculation.

Beginning with

$$-\Gamma \bar{u}'' + f(\bar{u}) = 0, \tag{111}$$

we observe that the linearization of f about the endstates is simply

$$f(\bar{u}) \approx f'(u_\pm)(\bar{u} - u_\pm).$$

We set

$$\bar{w}^\pm(x) := \begin{pmatrix} \bar{u} - u_\pm \\ \bar{u}'(x) \end{pmatrix} \Rightarrow \frac{d\bar{w}^\pm}{dx} \approx \bar{A}_\pm \bar{w}^\pm, \tag{112}$$

where

$$\bar{A}_\pm = \begin{pmatrix} 0 & I \\ \Gamma^{-1} B_\pm & 0 \end{pmatrix}; \quad x \to \pm\infty, \tag{113}$$

and as usual $B_\pm := f'(u_\pm)$. Recalling our notation for the eigenvalues of $\Gamma^{-1}B_\pm$, $\sigma(\Gamma^{-1}B_\pm) = \{\nu_j^\pm\}_{j=1}^m$, we can express the eigenvalues $\{\bar{\mu}_j^\pm\}_{j=1}^{2m}$ of $\bar{A}_\pm$ as

$$\begin{aligned} \bar{\mu}_j^\pm &= -\sqrt{\nu_{m+1-j}^\pm}; \quad j = 1, \dots, m, \\ \bar{\mu}_{m+j}^\pm &= \sqrt{\nu_j^\pm}; \quad j = 1, \dots, m, \end{aligned} \tag{114}$$

the indices specified so that $j < k \Rightarrow \bar{\mu}_j^\pm \leq \bar{\mu}_k^\pm$.

We let $\bar{r}_j^\pm$ denote the eigenvector of $\Gamma^{-1}B_\pm$ associated with $(\bar{\mu}_j^\pm)^2$, so that for $j = 1, 2, \dots, m$

$$\begin{aligned} \Gamma^{-1} B_\pm \bar{r}_j^\pm &= \nu_{m+1-j}^\pm \bar{r}_j^\pm \\ \Gamma^{-1} B_\pm \bar{r}_{m+j}^\pm &= \nu_j^\pm \bar{r}_{m+j}^\pm. \end{aligned} \tag{115}$$

Clearly, $\bar{r}_j^\pm = \bar{r}_{2m+1-j}^\pm$, $j = 1, \dots, 2m$. The eigenvectors of $\bar{A}_j^\pm$ are

$$\bar{v}_j^\pm = \begin{pmatrix} \bar{r}_j^\pm \\ \bar{\mu}_j^\pm \bar{r}_j^\pm \end{pmatrix}, \quad j = 1, \dots, 2m. \tag{116}$$

Asymptotically, we can express $\bar{w}^-$ as a linear combination of the decay modes $\{e^{\sqrt{\nu_j^-}x}\bar{v}_{m+j}^-\}_{j=1}^m$, while we can express $\bar{w}^+$ as a linear combination of the decay modes $\{e^{-\sqrt{\nu_{m+1-j}^+}x}\bar{v}_j^+\}_{j=1}^m$. In this way we expect

$$\begin{aligned}\begin{pmatrix}\bar{u}-u_-\\ \bar{u}'(x)\end{pmatrix} &= \sum_{j=1}^m s_j^- e^{\sqrt{\nu_j^-}x}\begin{pmatrix}\bar{r}_{m+j}^-\\ \sqrt{\nu_j^-}\,\bar{r}_{m+j}^-\end{pmatrix}\\ \begin{pmatrix}\bar{u}-u_+\\ \bar{u}'(x)\end{pmatrix} &= \sum_{j=1}^m s_j^+ e^{-\sqrt{\nu_{m+1-j}^+}x}\begin{pmatrix}\bar{r}_j^+\\ -\sqrt{\nu_{m+1-j}^-}\,\bar{r}_j^+\end{pmatrix}.\end{aligned} \tag{117}$$

In order to construct $\bar{u}(x)$ approximately, we solve the boundary value problem

$$\begin{pmatrix}\bar{u}\\ \bar{u}'\end{pmatrix}' = \begin{pmatrix}\bar{u}'\\ \Gamma^{-1}f(\bar{u})\end{pmatrix},$$

subject to the boundary conditions

$$\begin{aligned}\begin{pmatrix}\bar{u}(-R)\\ \bar{u}'(-R)\end{pmatrix} &= \begin{pmatrix}u_-\\ 0\end{pmatrix} + \sum_{j=1}^m s_j^- e^{\sqrt{\nu_j^-}(-R)}\begin{pmatrix}\bar{r}_{m+j}^-\\ \sqrt{\nu_j^-}\,\bar{r}_{m+j}^-\end{pmatrix}\\ \begin{pmatrix}\bar{u}(R)\\ \bar{u}'(R)\end{pmatrix} &= \begin{pmatrix}u_+\\ 0\end{pmatrix} + \sum_{j=1}^m s_j^+ e^{-\sqrt{\nu_{m+1-j}^+}x}\begin{pmatrix}\bar{r}_j^+\\ -\sqrt{\nu_{m+1-j}^-}\,\bar{r}_j^+\end{pmatrix},\end{aligned} \tag{118}$$

for some suitably large constant R. In particular, we have a system of $2m$ first order equations with $4m$ boundary conditions, but with additionally $2m$ parameters $\{s_j^\pm\}_{j=1}^m$. Of course the $\{s_j^\pm\}_{j=1}^m$ are not uniquely determined, and correspond with a particular choice of shift. For the sake of expediency, we solve this system with MATLAB's built-in solver *bvp4c*.

As an example, we consider (1) with $M \equiv I$, $\Gamma = I$, and $f(u) = F'(u)$, where

$$F(u_1, u_2) = u_1^2u_2^2 + u_1^2(1-u_1-u_2)^2 + u_2^2(1-u_1-u_2)^2.$$

In this case

$$\begin{aligned}F_{u_1}(u_1,u_2) &= 2u_1u_2^2 + 2u_1(1-u_1-u_2)^2 - 2(u_1^2+u_2^2)(1-u_1-u_2)\\ F_{u_2}(u_1,u_2) &= 2u_1^2u_2 + 2u_2(1-u_1-u_2)^2 - 2(u_1^2+u_2^2)(1-u_1-u_2),\end{aligned}$$

and so the system for $\bar{u}(x)$ is

$$\begin{aligned}\bar{u}_1''(x) &= 2\bar{u}_1\bar{u}_2^2 + 2\bar{u}_1(1-\bar{u}_1-\bar{u}_2)^2 - 2(\bar{u}_1^2+\bar{u}_2^2)(1-\bar{u}_1-\bar{u}_2)\\ \bar{u}_2''(x) &= 2\bar{u}_1^2\bar{u}_2 + 2\bar{u}_2(1-\bar{u}_1-\bar{u}_2)^2 - 2(\bar{u}_1^2+\bar{u}_2^2)(1-\bar{u}_1-\bar{u}_2).\end{aligned}$$

Writing this as a first order system, we find

$$B_- = \begin{pmatrix}2 & 2\\ 2 & 4\end{pmatrix}; \quad B_+ = \begin{pmatrix}4 & 2\\ 2 & 2\end{pmatrix}, \tag{119}$$

with eigenvalues

$$\nu_1^\pm = 3-\sqrt{5}; \quad \nu_2^\pm = 3+\sqrt{5}.$$

The relevant eigenvectors are

$$\begin{aligned}\bar{r}_3^- &= \begin{pmatrix}1\\ \frac{1-\sqrt{5}}{2}\end{pmatrix}; \quad \bar{r}_4^- = \begin{pmatrix}1\\ \frac{1+\sqrt{5}}{2}\end{pmatrix}\\ \bar{r}_1^+ &= \begin{pmatrix}1\\ \frac{-1+\sqrt{5}}{2}\end{pmatrix}; \quad \bar{r}_2^+ = \begin{pmatrix}1\\ -\frac{1+\sqrt{5}}{2}\end{pmatrix}.\end{aligned}$$

With these specifications, we carry out the program outlined above and compute the wave $\bar{u}(x)$ depicted in Figure 2.1.

We find the scaling coefficients to be $c_1^- = -.2014$, $c_2^- = -1.0567$, $c_1^+ = 6.7719$, $c_2^+ = -4.9098 \times 10^3$, though in this case the values c_2^- and c_2^+ are irrelevant (due to integration tolerances) because of the exponentially small multipliers. The important point is the suggestion that the wave is principally a connection between the slowest decaying fast modes.

In order to verify our spectral condition $D_a'''(0) \neq 0$, we must compute I_1^-, I_2^-, I_3^+, and I_4^+. In this case, we will make the computations by numerically evaluating the two fast modes $\phi_3^-(x;0)$ and $\phi_2^+(x;0)$. These modes both solve the ODE system

$$-\phi'' + B(x)\phi = 0, \tag{120}$$

and its first-order form

$$w' = A(x)w; \quad A(x) = \begin{pmatrix} 0 & I \\ B(x) & 0 \end{pmatrix}, \tag{121}$$

where $B(x) = F''(\bar{u}(x))$, and

$$\begin{aligned} \frac{\partial^2 F}{\partial u_1^2}(\bar{u}) &= 2\bar{u}_2^2 + 2(1-\bar{u}_1-\bar{u}_2)^2 - 8\bar{u}_1(1-\bar{u}_1-\bar{u}_2) + 2(\bar{u}_1^2+\bar{u}_2^2) \\ \frac{\partial^2 F}{\partial u_1 u_2}(\bar{u}) &= 4\bar{u}_1\bar{u}_2 - 4(\bar{u}_1+\bar{u}_2)(1-\bar{u}_1-\bar{u}_2) + 2(\bar{u}_1^2+\bar{u}_2^2) \\ \frac{\partial^2 F}{\partial u_2^2}(\bar{u}) &= 2\bar{u}_2^1 + 2(1-\bar{u}_1-\bar{u}_2)^2 - 8\bar{u}_2(1-\bar{u}_1-\bar{u}_2) + 2(\bar{u}_1^2+\bar{u}_2^2). \end{aligned}$$

Since the derivative $\bar{u}'(x)$ decays at the slower rate at both $\pm\infty$, we construct $w_3^-(x)$ and $w_2^+(x)$ as the solutions of (120) that decay at the faster rate respectively at $-\infty$ and $+\infty$. Precisely, we approximate $w_3^-(x)$ and $w_2^+(x)$ by solving (121) with initial conditions

$$\begin{aligned} w_3^-(-R) &= e^{\sqrt{3+\sqrt{5}}(-R)}\Big(1, \frac{1}{2}+\frac{\sqrt{5}}{2}, \sqrt{3+\sqrt{5}}, \sqrt{3+\sqrt{5}}(\frac{1}{2}+\frac{\sqrt{5}}{2})\Big)^{\mathrm{tr}}; \\ w_2^+(R) &= e^{-\sqrt{3+\sqrt{5}}R}\Big(1, -\frac{1}{2}+\frac{\sqrt{5}}{2}, -\sqrt{3+\sqrt{5}}, -\sqrt{3+\sqrt{5}}(-\frac{1}{2}+\frac{\sqrt{5}}{2})\Big)^{\mathrm{tr}}, \end{aligned}$$

where R is a suitably large constant (we took $R = 10$). With $\bar{u}'(x)$, $w_3^-(x)$, and $w_2^+(x)$ approximated in this way, we integrate appropriate determinants to compute $I_1^- = -.0024$, $I_2^- = -.0128$, $I_3^+ = .1264$, and $I_4^+ = .1044$.

In order to evaluate the remaining expressions in $D_a'''(0)$ we recall $\sigma(M_\pm B_\pm) = \{\beta_j^\pm\}_{j=1}^m$, and that for $j = 1, 2, \dots, m$ the slow mode correspondences are

$$\begin{aligned} (M_-B_- - \beta_{m+1-j}^- I)r_{2m+j}^-(0) &= 0 \\ (M_+B_+ - \beta_j^+ I)r_{m+j}^+(0) &= 0. \end{aligned} \tag{122}$$

(See Remark 3) In this case, $M_\pm = I$, and the $B_\pm$ are given in (119), giving $\beta_1^\pm = 3-\sqrt{5}$, $\beta_2^\pm = 3+\sqrt{5}$, and

$$r_5^-(0) = \begin{pmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix}; \; r_6^-(0) = \begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix}; \; r_3^+(0) = \begin{pmatrix} 1 \\ -\frac{1+\sqrt{5}}{2} \end{pmatrix}; \; r_4^+(0) = \begin{pmatrix} 1 \\ \frac{-1+\sqrt{5}}{2} \end{pmatrix}. \tag{123}$$

Finally, we use (83) to directly compute

$$D_a'''(0) = 6(.7004 + 4.7941 + 4.7948 + .6986) = 65.9273,$$

where the numerical terms are given respectively with the expressions in (83), and we note both the symmetry and the rough indication of consistency.

Acknowledgments. The authors are indebted to Didier de Fontaine for generously providing a copy of his thesis results, which initiated the rigorous study of multicomponent systems (and, unfortunately, are not available electronically). The authors are also grateful to the anonymous reviewers for numerous enlightening observations and beneficial recommendations.

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Received xxxx 20xx; revised xxxx 20xx.

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