# *Pointwise Semigroup Methods and Stability of Viscous Shock Waves*

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ABSTRACT. Considered as rest points of ODE on $L^p$, stationary viscous shock waves present a critical case for which standard semigroup methods do not suffice to determine stability. More precisely, there is no spectral gap between stationary modes and essential spectrum of the linearized operator about the wave, a fact that precludes the usual analysis by decomposition into invariant subspaces. For this reason, there have been until recently no results on shock stability from the semigroup perspective except in the scalar or totally compressive case ([Sat], [K.2], resp.), each of which can be reduced to the standard semigroup setting by Sattinger's method of weighted norms. We overcome this difficulty in the general case by the introduction of new, pointwise semigroup techniques, generalizing earlier work of Howard [H.1], Kapitula [K.1-2], and Zeng [Ze,LZe]. These techniques allow us to do "hard" analysis in PDE within the dynamical systems/semigroup framework: in particular, to obtain sharp, global pointwise bounds on the Green's function of the linearized operator around the wave, sufficient for the analysis of linear and nonlinear stability. The method is general, and should find applications also in other situations of sensitive stability.

Central to our analysis is a notion of "effective" point spectrum that can be extended to regions of essential spectrum. This turns out to be intimately related to the Evans function, a well-known tool for the spectral analysis of traveling waves. Indeed, crucial to our whole analysis is the "Gap Lemma" of [GZ,KS], a technical result developed originally in the context of Evans function theory. Using these new tools, we can treat general over- and undercompressive, and even strong shock waves for systems within the same framework used for standard weak (i.e. slowly varying) Lax waves. In all cases, we show that stability is determined by the simple and numerically computable condition that the number of zeroes of the Evans function in the right complex half-plane be equal to the dimension of the stationary manifold of nearby traveling wave solutions. Interpreting this criterion in the conserva-

tion law setting, we quickly recover all known analytic stability results, obtaining several new results as well.

## Contents



## Glossary of Symbols

$\{\bar{u}^\delta\}$, $\ell$: Stationary manifold of traveling waves, and its dimension $\ell$.
$\sigma(A)$: Spectrum of a matrix $A$.
$^*$: Operator or matrix adjoint, or scalar complex conjugate.
$^t$: Matrix transpose.
$Res_{\lambda_0} f$: Residue of $f(\lambda)$ at $\lambda_0$.
$\sigma(L : X)$: Spectrum of operator $L$ with respect to space $X$ ($L^2$ default).
$\sigma_{\text{ess}}(L : X)$: essential spectrum of $L$.
$\sigma_p(L : X)$: point spectrum of $L$.
$\Sigma_{\lambda_0}(L : X)$ : Eigenspace of $L$ at $\lambda_0$ with respect to space $X$.
$\Sigma_{\lambda_0,k}(L : X)$ : Eigenspace of ascent $k$ at $\lambda_0$.
$\text{Ker}(L : X)$: Kernel of $L$ with respect to $X$ ($L^2$ default).
$\sigma_p'(L)$: Effective point spectrum of an operator $L$.
$\Sigma_{\lambda_0}'(L)$: Effective eigenspace of $L$ at $\lambda_0$.
$\Sigma_{\lambda_0,k}'(L)$: Effective eigenspace of ascent $k$ at $\lambda_0$.
$\Lambda$: Region of consistent splitting for $L$.
$D_L(\lambda)$: The Evans function associated with $L$.
$G_\lambda(x, y)$: Resolvent kernel, or elliptic Green's function of $(L - \lambda I)$.
$\Omega_\theta$: Sector on which $G_\lambda$ is meromorphic, $D_L$ analytic.
$P_{\lambda_0}(x, y)$: Projection kernel of $L$ at $\lambda_0$.
$Q_{\lambda_0,k}(x, y)$: $(L - \lambda_0 I)^k P_{\lambda_0}(x, y)$.

$\mathcal{P}_{\lambda_0}$: Spectral projection of $L$ at $\lambda_0$.
$\mathcal{Q}_{\lambda_0,k}(x,y)$: $(L-\lambda_0 I)^k \mathcal{P}_{\lambda_0}$.
$G(x,t;y)$: Parabolic Green's function for $(d/dt - L)$.
$S(x,t;y)$: Scattering component of $G$.
$E(x,t;y)$: Excited component of $G$.
$T(x,t;y)$: Transient component of $G$.
$R(x,t;y)$: Residual component of $G$.
$\chi_S$: Logical indicator function, 1 if statement $S$ is true, else zero.
$\mathbf{O}(\cdot)$, $\mathbf{o}(\cdot)$: Uniform bound or decay with respect to the argument.
$C^{j+\tilde\alpha}$: Hölder continuous $j$th derivative, Hölder exponent $\alpha$.
$C^{(j,k)+(\tilde\alpha,\tilde\beta)}$: Functions $f$ with $D_x^j f$, $D_t^k f \in C^{0+\tilde\alpha}(x) \cap C^{0+\tilde\beta}(t)$.
$(\mathcal{D})$: The Evans function condition, p. 17.

## 1. Introduction.

The stability of viscous shock waves has been much studied, as an issue of obvious physical importance (see, for example, [IO, Sat, MN, G.1, KMN, L.1, SX, LZ.1-2, JGK, FreL, K.2, L.3]). Besides its intrinsic interest, this topic is connected to such fundamental issues in conservation laws as convergence of numerical schemes, validity of matched asymptotic expansion, and convergence in the inviscid limit [SX, GX, Y, HL]. However, to date, stability results for systems are rather isolated, depending on special structure of the shock profile. In particular, almost all existing results for systems rely on approximate decoupling of the linearized equations about the wave, which requires among other things the restrictive assumptions of *weak shock strength*, *identity viscosity matrix*, and *approximately linear profile*. The only (partial) exception is in the "totally compressive" case, for which all characteristics enter the shock as in the scalar case [K.2].[1] Thus, present day theory is *weakly nonlinear*, like other small variation theories such as the Glimm difference scheme or weakly nonlinear optics [Gl,L.4,Hu,HuK]. Yet, perhaps the most interesting cases, of nonclassical and strong shocks, are by nature *strongly nonlinear*, and *highly coupled* [AMPZ.2].

The purpose of the present paper is to present a unified, functional analytic approach to shock wave stability, analogous to that carried out by Sattinger [Sat] in the scalar case, making minimal use of the details of the equations or the shock structure. In particular, we make no special assumptions on shock strength, form of the viscosity matrix, or type of the equations. The usual assumptions on shock structure are replaced, following the dynamical systems approach of, e.g. [He,AGJ], by more primary conditions on the *point spectrum of the linearized operator around the wave.* These conditions can then be verified by separate techniques, for example by energy estimate in the weak shock case,

[1] In fact, the theorem stated in [K.2] is limited to the scalar case by the hypothesis that the viscous profile be unique up to translation. However, the totally compressive case can be easily treated by either the methods of [Sat] or [K.2], by working with the integrated equations as in [JGK]. This would recover for example the result of [FreL] on overcompressive shocks in the cubic model, by the argument given in Section 1.2.4 below.

or numerically in the case of strong shocks [Br]. Recall, in the wider context of stability of general traveling waves, that point spectrum encodes dynamics of the inner wave structure, and essential spectrum the far field behavior [He].

The well-known difficulty with this program is that for shock waves there is no separation between the top eigenvalue $\lambda = 0$ and the essential spectrum of the linearized operator about the wave. The lack of a spectral gap makes this a critical case for which standard semigroup results do not yield stability information. Indeed, stability analysis for systems in general appears to require pointwise estimates on the Green's function of the linearized solution operator [SX, LZ.1-2, L.3]. Moreover, at points such as $\lambda = 0$ that are embedded in the essential spectrum, the point spectrum of a non-normal operator is in general not well-defined, leaving inner shock dynamics unclear.

These issues can be avoided in the scalar case by Sattinger's technique of (exponentially) weighted norms [Sat], which shifts the essential spectrum to the left to recover a spectral gap and therefore exponential decay. Though Sattinger did not carry it out, it is easily verified that his technique applies for systems, also, precisely in the *totally compressive* case considered in [K.2], that is, to shocks for which all characteristics *enter the shock* as in the scalar case. However, physical shocks, in particular the most commonly occurring case identified by Lax, in the system case typically have at least one characteristic *leaving the shock* (see, e.g. [La,Sm]).

More recently, Kapitula [K.1-2] has introduced innovative techniques to deal directly with the difficulties of the essential spectrum, using extension of the resolvent and a graded family of algebraically weighted norms to obtain *algebraic* rates of decay in the absence of a spectral gap. However, his techniques still apply only to the same class of (totally compressive) shocks to which Sattinger's method applies (see discussion, sections 1.1.4 and 1.2.2).

We overcome these difficulties in the general case by the introduction of new, pointwise semigroup methods, by which one can obtain "hard" PDE estimates within the dynamical systems framework: in particular, sharp *global* parabolic estimates of the type required to prove nonlinear stability. Our analysis uses many of the techniques pioneered by Kapitula, in particular the idea of extending the resolvent; however, we depart completely from the weighted norm approach of [Sat,K.1-2,JGK]. Indeed, philosophically, our approach bears more resemblance to the Fourier Transform techniques introduced by Zeng [Ze, LZe] in her study of decay to constant states.

Some distinctions at a technical level: (i) Kapitula's resolvent estimates are intimately intertwined with the normed spaces on which he defines the resolvent, hence do not give pointwise information. Our estimates on the *resolvent kernel* reveal new (scattering) structure that is crucial for the analysis. (ii) Though Kapitula establishes an analytic extension of the resolvent onto an open neighborhood of the resolvent set, he *uses* only $C^1$ extension up to the boundary, performing all his estimates on a single, fixed contour lying in the closure of the resolvent set (indeed, for the algebraically weighted spaces he considers, the resolvent is only defined up to this boundary). We, rather, use the full power of the analytic extension, performing estimates on *moving* contours as in the work of Zeng. In particular, we move our contours *into the essential spectrum* to obtain optimal bounds in certain spatio-temporal regimes.

These pointwise methods lead naturally to a notion of extended, or "effective" point spectrum inside regions of essential spectrum, obeying a modified version of the usual Fredholm theory. The effective point spectrum turns out to be the appropriate notion from the standpoint of asymptotic behavior of PDE. Moreover, we show that it is intimately connected with the *Evans function* of, e.g. [E, J, AGJ, PW]. This, on the one hand explains the meaning of the Evans function in regions of essential spectrum, and on the other verifies continuity of the effective point spectrum under perturbation. The latter observation allows the application of powerful continuation arguments to the question of stability.

We expect these techniques to be of general use in situations of sensitive stability. Applied to the problem of shock stability, they yield a simple, and *computable* necessary and sufficient condition for stability in terms of the index of the Evans function on the unstable complex half-plane. Moreover, in the case of stability, they give at the same time sharp rates of decay in all spatio-temporal regimes.

The dynamical systems approach to shock stability, for general systems, was initiated by Gardner and Zumbrun in [GZ], and we make essential use of ideas and results therein. In particular, the Gap Lemma of the title (discovered independently by Kapitula and Sandstede [KS]) is what makes possible the crucial extension of the resolvent (more precisely, the resolvent kernel) for other than the totally compressive case. The pointwise bounds we obtain here generalize an earlier, scalar analysis carried out by Howard [H.1]. These two results, in turn, have their roots in the Evans function methods of [Sat, AGJ, K.1-2, JGK, PW] and the Fourier Transform analyses of [Ka, Ze, LZe, HoZ.1-2]. Finally, the passage from linearized to nonlinear stability rests on the pointwise Green's function method developed in [L.2, LZ.1-2, SZ, LZe, L.3, LX]. Our analysis thus synthesizes (and depends on) three distinct lines of research.

### *1.1. Background.*

1.1.1. VISCOUS SHOCK WAVES. A *viscous shock wave* is a traveling wave solution

$$u(x,t) = \bar{u}(x - st) \tag{1.1}$$

of a system of viscous conservation laws

$$u_t + f(u)_x = (B(u)u_x)_x; \quad u,\ f \in \mathbb{R}^n, \tag{1.2}$$

tending as $x \to \pm\infty$ to asymptotic states

$$u_\pm = \bar u(\pm\infty). \tag{1.3}$$

Such solutions, and equations, occur in a variety of physical contexts. Here, we make only the assumptions:

(**H0**) $f,\, B \in C^{1+\tilde\alpha}$, $\tilde\alpha > 0$.

(**H1**) $\mathrm{Re}\,\sigma(B) > 0$.

(**H2**) $\sigma(f'(u_\pm))$ real, distinct, and nonzero.

(**H3**) $\mathrm{Re}\,\sigma(-ikf'(u_\pm) - k^2 B(u_\pm)) < -\theta k^2$ for all real $k$, some $\theta > 0$.

(**H4**) There exists a solution $\bar u$ of (1.1)–(1.3), nearby which the set of all solutions connecting the same values $u_\pm$ forms a smooth manifold $\{\bar u^\delta\}$, $\delta \in \mathbb{R}^\ell$.

The zeroth hypothesis gives the minimal smoothness needed to carry out our analysis. Of the remaining, the first corresponds to strict parabolicity of (1.2), and the second to strict hyperbolicity of the endstates $u_\pm$ with regard to the corresponding inviscid equation, $u_t + f(u)_x = 0$. The third condition is the stable viscosity matrix criterion of Majda and Pego, [MP], which loosely speaking corresponds to $L^2$ linearized stability of the constant states $u \equiv u_\pm$ as solutions of (1.2); indeed, it is the condition obtained by Kawashima, [Ka], in his study of decay to constant solutions. The fourth, very weak condition is the only assumption concerning the shock profile $\bar u$.

Assumptions (**H0**)–(**H4**) are sufficiently general as to encompass an open dense subset of known physical examples. [2] Note in particular that we make no assumptions of hyperbolicity or Majda–Pego stability at states $\bar u(x)$ *along the profile* $\bar u$, but only at the end states

$$u_\pm = \lim_{x\to\pm\infty} \bar u(x).$$

This is important in applications to nonclassical shocks occurring in multiphase flow or Van der Waals gas dynamics [AMPZ.2, Z.4].

Substituting (1.1) into (1.2) and integrating from $-\infty$ gives the *traveling wave ODE*

$$\bar u' = B(\bar u)^{-1}(f(\bar u) - s\bar u - f(u_-) + su_-), \tag{1.4}$$

a dynamical system parametrized by $(u_-, s)$. Solutions $\bar u$ correspond to orbits between rest states $u_\pm$. The following lemma proved in [MP], a generalization of Sylvester's Law of Inertia, asserts that $u_\pm$ are hyperbolic also in the ODE sense (for an alternative proof, see Remark 2.3 in section 2). This implies exponential

[2] However, several important examples lie on the boundary, see Section 1.2.5.

approach of $\bar{u}$ to its asymptotic states at $x = \pm\infty$, a fact that will be crucial in our subsequent analysis. However, the inner structure of the shock can be essentially arbitrary.

***Lemma 1.1.*** *Given* (**H1**)–(**H3**), *the stable/unstable manifolds of* $f'(u_\pm)$ *and* $B(u_\pm)^{-1}f'(u_\pm)$ *have equal dimensions. In particular,* $B(u_\pm)^{-1}f'(u_\pm)$ *has no center manifold.*

***Corollary 1.2.*** *Given* (**H0**)–(**H4**), *solutions* $\bar{u}$ *of* (1.4) *are in* $C^{2+\tilde{\alpha}}$, *satisfying*

$$D_x^j(\overline{u}(x) - u_\pm) = \mathbf{O}(e^{-\alpha|x|}), \quad \alpha > 0,\ 0 \le j \le 2. \tag{1.5}$$

1.1.2. ORBITAL STABILITY. From now on, we specialize to the case most convenient for analysis, of a *stationary shock*, $s = 0$. We can always reduce to this case by the normalization $x \to x - st$, $f(u) \to f(u) - su$. Given a particular solution $\bar{u}$ of (1.1)–(1.3), we define the *stationary manifold*

$$\{\bar{u}^\delta\}; \quad \delta \in \mathbb{R}^\ell$$

to be the set of all solutions connecting the same $u_\pm$, with $\bar{u}^0 = \bar{u}$. By (**H4**), $\{\bar{u}^\delta\}$ forms a smooth manifold near $\bar{u}$; we take $\delta \to \bar{u}^\delta$ to be a smooth parametrization of this manifold near $\delta = 0$.

Note that the dimension $\ell$ of the stationary manifold is always at least one since, by translation invariance of (1.2), it must contain at least all shifts $\bar{u}(x - d)$ of the reference solution $\bar{u}$. Besides this requirement, the types of the rest points $u_\pm$, and the value $1 \le \ell \le n$ are, again, essentially arbitrary.

***Example.*** We display below the phase diagram for (1.4) corresponding to an overcompressive shock of the cubic model $u \in \mathbb{R}^2$, $f(u) = |u|^2 u$, $B \equiv I$, a model equation related to magnetohydrodynamics (MHD) [Fr,BH]. (Figure reproduced from [Br] by permission of the author). In this case, the stationary manifold is *two-dimensional* and can be parametrized as $\bar{u}^\delta = \bar{u}_{\delta_2}(x - \delta_1)$, where $\bar{u}_{\delta_2}(0) := (0, \delta_2)^t$.

The existence of arbitrarily nearby stationary solutions $\bar{u}^\delta$ precludes asymptotic stability of $\bar{u}$. The appropriate notion of stability is, rather, *orbital stability*, or approach to the stationary manifold $\{\bar{u}^\delta\}$.

***Definition 1.3.*** *Fix a norm,* $\|\cdot\|$, *and a set* $\mathcal{A}$ *of admissible perturbations. Then* $\bar{u}$ *is orbitally stable with respect to* $\mathcal{A}$ *if* $u(\cdot, t) \to \{\bar{u}^\delta\}$ *as* $t \to \infty$ *whenever* $u(\cdot, 0) - \bar{u} \in \mathcal{A}$.

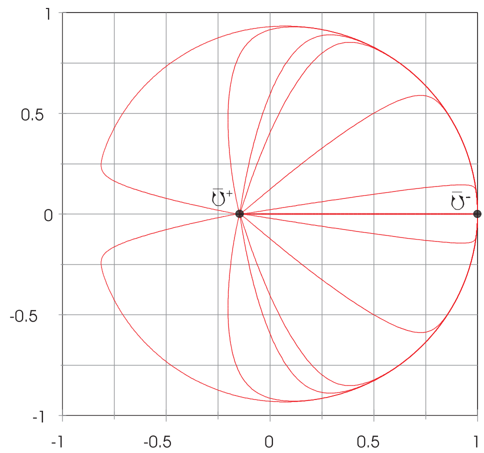


FIGURE 1. SHOCKS FOR THE CUBIC MODEL

***Remark 1.4.*** Note that the solution $u$ is required only to approach the stationary manifold as a whole, and not any particular profile $\bar u^{\delta_0}$. For Lax and overcompressive shocks, this distinction is unimportant, since the invariants given by conservation of mass are sufficient to fix the location of the solution in $\{\bar u^{\delta}\}$. However, in the undercompressive case they are not, and the solution in fact appears to drift indefinitely along the stationary manifold, with position $\sim \log t$ [Z.2].

1.1.3. THE LINEARIZED EQUATIONS. Linearizing (1.2) about a stationary solution $\bar u(x)$ gives the system of *convection–diffusion equations*

$$v_t = Lv := (-Av)_x + (Bv_x)_x, \tag{1.6}$$

where

$$B(x) := B(\bar u(x)); \quad A(x)v := f'(\bar u(x))v - B'(\bar u(x))v\bar u_x.$$

Hypotheses (**H0**)–(**H4**) induce the following consequences at the linearized level:

(**C0**) $A,\, B \in C^{0+\tilde\alpha}$, $\tilde\alpha > 0$, with $(A - A_\pm)$, $(B - B_\pm) = \mathbf{O}(e^{-\alpha|x|})$ as $x \to \pm\infty$, $\alpha > 0$.
(**C1**) $\mathrm{Re}\,\sigma(B) > 0$.
(**C2**) $\sigma(A_\pm)$ real, distinct, nonzero.
(**C3**) $\sigma_{\mathrm{ess}}(L) \subset \{\mathrm{Re}\lambda \le -\theta(\mathrm{Im}\lambda)^2\} \cup \{\mathrm{Re}\lambda \le -\theta\ \mathrm{Im}\lambda\}$, for some $\theta > 0$.

(*Note: Unless otherwise specified,* $\sigma(L)$ *refers to* $L^2$ *spectrum.*)

All of our linear results will be proved at the level of generality of (**C0**)–(**C3**), the assumptions (**H0**)–(**H4**) entering only at the nonlinear level. Conditions (**C0**)–(**C2**) follow in straightforward fashion. Condition (**C3**) follows from the fact that, given (**C0**), $\sigma_{\text{ess}}(L)$ lies on and to the left of the rightmost boundary of $\sigma(L_+) \cup \sigma(L_-)$, where

$$L_{\pm}v := -A_{\pm}v_x + B_{\pm}v_{xx}$$

denote the limiting, constant coefficient operators approached by $L$ as $x \to \pm\infty$, and (by Fourier Transform calculation)

$$\sigma(L_{\pm}) \equiv \sigma_{\text{ess}}(L_{\pm}) = \{\lambda \in \sigma(-ikA_{\pm} - k^2 B_{\pm}) : k \text{ real}\}.$$

This standard result may be found in, e.g., [He,CH], or see the calculations later in this paper. The final step is to observe that $\text{Re}\,\sigma(-ikA_{\pm} - k^2 B_{\pm})$ is less than $-\theta_1 k^2$ by (**H3**), while $\text{Im}\,\sigma(-ikA_{\pm} - k^2 B_{\pm})$ is at most of order $k$ for $k$ bounded, $k^2$ for $k$ large.

Note that the *tangent manifold*, $\text{Span}\{\partial\bar{u}^{\delta}/\partial\delta_j\}$, of $\{\bar{u}^{\delta}\}$ at $\bar{u}$ consists of stationary solutions of (1.6), or equivalently

$$\text{Span}\{\partial\bar{u}^{\delta}/\partial\delta_j\} \subset \text{Ker}(L). \tag{1.7}$$

Just as for the nonlinear equations, this precludes asymptotic stability. Instead, we study linearized orbital stability, defined analogously to Definition 1.3 as approach to the tangent manifold.

***Definition 1.5.*** *Fix a norm, $\|\cdot\|$, and a set $\mathcal{A}$ of admissible perturbations. Then $\bar{u}$ is linearly orbitally stable with respect to $\mathcal{A}$ if $v(\cdot,t) \to \text{Span}\{\partial\bar{u}^{\delta}/\partial\delta_j\}$ as $t \to \infty$ whenever $v(\cdot,0) \in \mathcal{A}$, where $v$ is the solution of* (1.6).

1.1.4. THE POINTWISE GREEN'S FUNCTION METHOD. A fundamental difficulty in the study of stability of viscous shock waves is the accumulation at the imaginary axis, (**C3**), of the essential spectrum of the linearized operator $L$, that is, the lack of a spectral gap between stationary and time-decaying modes of (1.6). From the dynamical systems viewpoint, considering (1.2) as an ODE on an appropriate Banach space, this means that $\{\bar{u}^{\delta}\}$ is a *nonhyperbolic rest manifold* for which standard semigroup methods do not yield stability. Indeed, as is familiar from finite-dimensional ODE, this is a critical case in which one can conclude neither stability nor instability without further investigation. Moreover, stability if it holds is at *algebraic* rather than *exponential* rate. Of course the type of nonhyperbolicity considered here, being entirely connected with essential spectrum, has no counterpart in finite-dimensional ODE, and the resulting, purely

PDE phenomena must be taken into account in the analysis as well. Recall that essential spectrum is associated with far-field behavior.

In the scalar case, and more generally for "totally compressive" systems with the property that all signals are convected inward toward the shock, stability can be successfully treated by the weighted-norm semigroup methods of [Sat, K.1-2, JGK]. This is easily understood at a heuristic level; indeed, the basic idea goes back to the first scalar analysis by Il'in and Oleinik [IO]. Given an increasing weight $W(|x|)$, it is clear that signals convecting inward at rate not less than $a$ will decay in a $W$-weighted norm $\|f(\cdot)\|_W := \|f(\cdot)W(|\cdot|)\|_\infty$ at roughly rate $\sup_x W(|x|)/W(|x|+at)$. Thus, the exponentially weighted norms $W \sim e^{\theta|x|}$ of [Sat] lead to exponential decay

$$\frac{\|v(t)\|_W}{\|v(0)\|_W} \sim e^{-a\theta t}.$$

Similarly, for the algebraically weighted norms $\|f\|_k := \|f(x)(1+|x|)^k\|_\infty$ of [K.1-2, JGK], one expects algebraic decay

$$\frac{\|v(t)\|_k}{\|v(0)\|_{k+2}} \sim \sup_x \frac{(1+|x|)^k}{(1+|x|+at)^{k+2}} \sim (1+t)^{-2},$$

as indeed is the case. (Note: the rate $(1+t)^{-1}$ found in [K.1-2, JGK] is sufficient but not optimal [H.2]). However, the same reasoning shows that these methods are *unsuited* for the treatment of standard systems possessing an outgoing mode. For, weights must be bounded from below for technical reasons, specifically to close a nonlinear analysis. Thus, outgoing modes will appear to *grow* in a weighted norm, or at best remain constant. The semigroup analyses of [Sat, K.1-2, JGK] take into account only the dominant effect of *convection* in the far-field behavior, whereas outgoing modes decay rather from the effects of *diffusion.*

Analysis for general systems has proceeded instead by *direct methods*, bypassing spectral information and obtaining estimates on the solution by other means, for example, the characteristic–weighted-energy methods of [G.1, MN, KMN, L.1, SX]. These take account of convection through the weighting, but also diffusion through energy estimates. The most recent approach, and the one that will be important for us here, is the *pointwise Green's function method* developed in [L.2, LZ.1-2, SZ, LZe, L.3, LX], which in principle takes account of *all* information about far-field behavior. We describe this method below:

Setting $u = \bar{u} + v$, we obtain from (1.2)–(1.6) the perturbation equation

$$v_t - Lv = Q(v, v_x)_x, \tag{1.8}$$

where $Q$ is a quadratic order source term. By Duhamel's principle,

$$
\begin{aligned}
(1.9)\quad v(\cdot,t) &= e^{Lt}v(\cdot,0) + \int_0^t e^{L(t-s)}Q(v,v_x)_x(\cdot,s)\,ds \\
&= \int G(\cdot,t;y)v(y,0)dy + \int_0^t \int G_y(\cdot,t-s;y)Q(v,v_x)(y,s)\,dy\,ds,
\end{aligned}
$$

where $G(x,t;y)$ is the Green's function for (1.6). To simplify the discussion, let

$$B \equiv \text{constant},$$

so that $Q = Q(v)$.

The basic strategy of the pointwise Green's function method is to convert pointwise bounds on the (linear) Green's function $G$ into pointwise bounds on the nonlinear solution $v$ of (1.8). This is accomplished with the aid of a *template function* (our terminology) $h(x,t)$ with shape roughly proportional to that expected of $|v|$, recording rates of decay in all spatio-temporal regimes. The object is to show that $|v|$ maintains its proportion with $h$ for all time, hence decays at the predicted rates. That is, rather than show decay in a weighted norm, as in the weighted norm and characteristic–weighted-energy methods, we try to show *boundedness* with respect to a weight that *enforces* decay.

We illustrate the method in the simpler, model case that $\mathrm{Ker}(L) = \emptyset$, corresponding to asymptotic stability. The heart of the matter lies in the following straightforward observation.

**Lemma 1.6.** *Let $h(x,t) > 0$ be such that*

$$
\begin{aligned}
(1.10)\qquad &\int |G(x,t;y)|\,h(y,0)dy < Ch(x,t) \quad \text{and} \\
&\int_0^t \int |G_y(x,t-s;y)|\,h(y,s)^2\,dy\,ds \le Ch(x,t),
\end{aligned}
$$

*for all $x$, $t$. If*

$$|v(x,0)| \le \zeta h(x,0)$$

*for $\zeta$ sufficiently small, then $|v| \le C_2\zeta h$ for all $x$, $t$, where $v$ is the solution of* (1.8) *and $C_2$ is independent of $\zeta$.*

*Proof.* Let $M$ be such that $|Q(v)| \le M|v|^2$, and define

$$\zeta(t) := \sup_{y,s\le t} \frac{|v(y,s)|}{h(y,s)}, \quad \zeta(0) = \zeta.$$

Then,

$$|v(x,t)| = \left| \int G(x,t;y)v(y,0)\,dy + \int_0^t \int G_y(x,t-s;y)Q(v)(y,s)\,dy\,ds \right|$$
$$\leq \zeta \int |G(x,t;y)|\,h(y,0)\,dy + M\zeta(t)^2 \int_0^t \int |G_y(x,t-s;y)|\,h(y,s)^2\,dy\,ds,$$

hence, using (1.10),

$$\zeta(t) \leq C_1(\zeta + \zeta(t)^2), \tag{1.11}$$

with $C_1 = \max\{CM, C\}$. Taking $4C_1^2\zeta < 1$, we have $\zeta(t) \leq 2C_1\zeta$ by continuous induction, and the claim follows for $C_2 = 2C_1$. □

Lemma 1.6 reduces the problem of establishing pointwise bounds on $|v|$ to that of finding an appropriate template function $h$ satisfying relations (1.10), that is, a sort of weak "fixed point" of the integral operators on the left-hand side. In principle, a "minimal" such template, given a desired initial restriction $h(\cdot,0)$, could be obtained by iteration; in practice, a similar procedure is carried out by trial and error.

***Example.*** The convected Burgers equation

$$v_t + av_x - v_{xx} = (v^2)_x, \tag{1.12}$$

$a > 0$, $v \in \mathbb{R}^1$, loosely models behavior of a single outgoing mode in (1.8), moving toward the right. Observing that the Green's function

$$G(x,t;y) = (4\pi t)^{-1/2} e^{(x-y-at)^2/4t}$$

of the operator on the left hand side satisfies: $G \geq 0$;

$$|G_y(x,t;y)| \leq Ct^{-1/2} G(x,mt;y)$$

for any $m > 1$; and the semigroup property

$$\int G(x,t-s;z)G(z,s;y)\,dz = G(x,t;y)$$

(hence also $\int G(x,m(t-s);z)G(z,ms;y)\,dz = CG(x,mt;y)$), we easily find that relations (1.10) are satisfied for the template function $h(x,t) := G(x,m(t+1);0)$, for any $m > 1$. For example, the second relation in (1.10) follows from

$$\begin{aligned}
&\int_0^t \int |G_y(x,t-s;y)| G^2(y,m(s+1);0)\,dy\,ds \\
&\quad \leq C \int_0^t (t-s)^{-1/2}(s+1)^{-1/2} \int G(y,m(t-s);y)G(x,m(s+1);0)\,dy\,ds \\
&\quad \leq CG(x,m(t+1);0).
\end{aligned}$$

Thus, from Lemma 1.6, we recover the well-known fact that, for small, exponentially decaying initial data, the solution of (1.12), or "nonlinear diffusion wave," in the terminology of [L.1], decays like a heat kernel. Templates for systems become much more elaborate, see Section 11.

In more realistic applications, the argument outlined in Lemma 1.6 becomes much more complicated. The variable $v$ is augmented with derivatives (or even integrals) in order to reveal cancellation using integration by parts, and the (typically nonempty) kernel of $L$ is projected out by various means, see [LZ.1-2, L.3]. Moreover, when $B \not\equiv$ constant, then $Q = Q(v, v_x)$, and there is the problem of "gaining a derivative." However, the principle of the method remains the same. These details will be discussed in section 11.

To verify (1.10) (or its equivalent, in more complicated situations) requires rather sharp pointwise bounds on the Green's function $G$ and its derivatives. An important aspect of the analysis that we have not touched on so far is the means for generating such bounds. In the constant-coefficient scalar example given above, the Green's function could be found explicitly, but clearly this is not possible in the general, nonconstant-coefficient, system case. The weakly nonlinear approach developed in [SZ, L.3], rather, uses the approximate diagonalization of [G] to express $L$ as $\bar{L} + \mathbf{O}(\varepsilon)\tilde{L}$, where $\bar{L}$ is a *decoupled* operator, and $\varepsilon$ is shock strength (i.e. variation of the shock profile $\bar{u}$). The Green's function $\bar{G}$ for $\bar{L}$ can be constructed by fixed point argument [SZ] or solved approximately to $\mathbf{O}(\varepsilon^{-\alpha|x|})$ [L.3], and the $\tilde{L}$ and Green's function errors treated as additional source terms in (1.9), of *linear* order. This leads to a relation

$$\zeta(t) \leq C_1(\zeta + \zeta(t)^2 + \varepsilon\zeta(t)) \tag{1.13}$$

in place of (1.11), and the argument closes as before, provided *both* $\zeta$ and $\varepsilon$ are sufficiently small, that is for *weak shock strength* as well as weak perturbations. To treat strong shocks, we require, rather, estimates on the *exact Green's function*, $G$.

To summarize:

The pointwise Green's function method gives a very general way to convert Green's function bounds into nonlinear decay estimates, as illustrated in Lemma 1.6. However, the essentially constructive, weakly nonlinear methods that have been used to obtain these bounds are limited to the approximately decoupled case, in particular to weak shock strength.

In order to treat strong, or strongly nonlinear shocks, there is needed a new way of obtaining Green's function bounds that does not rely on weak nonlinearity of the underlying system. *It is this lack in the theory that we address in the present paper.* Surprisingly, we find that the required pointwise bounds after all depend only on the spectrum of the operator $L$. However, it is not the usual spectrum, but an extended, "pointwise" version that plays the key role.

### *1.2. Present work.*

1.2.1. Two classical problems. At this point, our investigation of shock stability reduces to two *linear* problems of classical, Sturm–Liouville type:

For a second-order elliptic operator $L$ satisfying (1.6), (**C0**)–(**C3**):

(i) Determine the point spectrum of $L$, and its relation to the time-asymptotic behavior of $e^{Lt}$.

(ii) Obtain sharp *global* parabolic estimates on the Green's function for (1.6), sufficient for the application of the pointwise Green's function method described in the previous section.

However, to our knowledge, there exist no such classical results in the present context. The asymptotic behavior of $e^{Lt}$ for a non-normal operator $L$ is in general not determined by the top eigenvalue of $L$, nor is the point spectrum well-defined at eigenvalues (such as $\lambda = 0$ in our case) that are embedded in the essential spectrum. Any such result must therefore depend to some extent on the specialized structure of (**C0**)–(**C3**).

Likewise, existing global parabolic bounds are restricted to the scalar, selfadjoint case, in which the Green's function bounds are precisely the small-time bounds given by the heat equation, cf. [St]. Such bounds are clearly not valid in the non-selfadjoint case $A \not\equiv 0$, as the dominant large-time effect in the far field is then *convection* at rates $a_j^{\pm}$ equal to the eigenvalues of $A_{\pm}$, rather than simple diffusion. Indeed, the bounds we derive here are considerably more complicated, and obtained by quite different methods. However, there is an interesting philosophical similarity between the Nash–Moser technique of obtaining pointwise bounds by variably weighted energy estimates (see, e.g., discussion in [St], p. 229, or [Mo, Na]) and our technique of variable contours. From the point of view of [Sat], moving into the essential spectrum is roughly equivalent to choosing a new weighted norm.

The resolution of problems (i)–(ii) is the primary focus of this paper. We remark that all our analysis goes through also in the case of mixed and non–

divergence-form operators of the forms

$$Lv := Cv - (Av)_x + (Bv_x)_x,$$
$$Lv := Cv - Av_x + Bv_{xx}, \quad \text{or}$$
$$Lv := CV - Av_x + (Bv_x)_x,$$

with Hölder coefficients $A$, $B$, $C$, the only requirement being that the coefficient $C(x)$ of the zeroth order term go to zero as $x \to \pm\infty$. The general case $C_\pm \not\equiv 0$ can be treated similarly, but leads to slightly different results. In each case, we obtain *both optimal regularity* (cf. short-time theory, Section 11.3) *and optimal bounds* on the Green's function.

1.2.2. POINTWISE SEMIGROUP METHODS. In standard semigroup theory, and indeed in the larger spectral theory of nonnormal operators, the principal object of study is the *resolvent*, $(L - \lambda I)^{-1}$. In our search for pointwise information, we are led naturally to study instead the *resolvent kernel* $G_\lambda(x, y)$, defined formally by

$$G_\lambda(\cdot, y) := (L - \lambda I)^{-1}\delta_y, \tag{1.14}$$

or equivalently

$$(L - \lambda I)^{-1} f(x) = \int G_\lambda(x, t; y) f(y)\, dy, \tag{1.15}$$

that is, the elliptic Green's function associated with $(L - \lambda I)$. This seemingly slight change in perspective pays surprising dividends in flexibility and power. At the same time, it greatly simplifies the analysis. The reason in each case is the same: the Green's function is a *universal* object, whereas the resolvent is *restricted*, with consequent loss of information, by the imposition of a function space and norm.

The most fundamental property of the resolvent is that it is *analytic on the resolvent set*, as can be seen by straightforward Neumann expansion [Kat]. This brings to bear the full power of complex analysis. At an isolated eigenvalue $\lambda_0$ of $L$, the spectral projection operator can be defined by

$$\mathcal{P}_{\lambda_0} = \mathrm{Res}_{\lambda_0}(L - \lambda I)^{-1}.$$

Assuming that $L$ is sectorial, as can be verified in the present case by standard energy estimates [He], we have the spectral resolution formula,

$$e^{Lt} = \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t}(L-\lambda I)^{-1}\,d\lambda, \tag{1.16}$$

for the solution operator $e^{Lt}$ of

$$v_t = Lv; \quad v(0) = v_0, \tag{1.17}$$

where $\Gamma = \partial\{\lambda : \mathrm{Re}\lambda > \theta_1 - \theta_2|\mathrm{Im}\,\lambda|\}$ is the boundary of an appropriate sector containing the spectrum of $L$, $\theta_2 > 0$. (Indeed, (1.16) defines an analytic semigroup, though this will not be important for us).

Let us recall for a moment the standard semigroup approach in the case that $L$ has an isolated simple eigenvalue at $\lambda = 0$, separated by *positive spectral gap* $\eta > 0$ from the remainder of the spectrum,

$$\sigma(L)\setminus\{0\} \subset \{\lambda : \mathrm{Re}\lambda \le -\eta\}, \quad \eta > 0.$$

Defining

$$\tilde\Gamma = \partial\{\lambda : \mathrm{Re}\,\lambda > \theta_1 - \theta_2\,\mathrm{Im}\,\lambda \text{ and } \mathrm{Re}\,\lambda \le -\eta/2\}, \tag{1.18}$$

(Figure 2), we have by (1.16), together with Cauchy's theorem, that

$$\begin{aligned} e^{Lt} &= \frac{1}{2\pi i}\int_{\tilde\Gamma} e^{\lambda t}(L-\lambda I)^{-1}\,d\lambda + \mathrm{Res}_0\, e^{\lambda t}(L-\lambda I)^{-1} \\ &= \mathbf{O}(e^{-\eta t/2}) + \mathcal{P}_0, \end{aligned}$$

where the time-exponential decay follows from the uniform bound on the resolvent afforded by sectoriality, together with the decay given by $|e^{\lambda t}| \le e^{\mathrm{Re}\lambda t}$. Thus, we see immediately that the solution $e^{Lt}v_0$ of (1.17) converges time-exponentially to the projection $\mathcal{P}_0 v_0$ of $v_0$ onto $\mathrm{Ker}(L)$. That is, the projection $\mathcal{P}_0$ *captures the asymptotic behavior of the solution.*

In the case of our interest, when $L$ has *no spectral gap*, this method fails. For, the resolvent by definition is undefined on the essential spectrum, hence $\Gamma$ cannot be moved past the value $\lambda = 0$. Neither is the projection $\mathcal{P}_0$ defined, nor is it clear that the solution operator is even bounded. To further illustrate the confusion at the spectral level, the dimensions of $\mathrm{Ker}(L)$ and $\mathrm{Ker}(L^*)$ typically *do not match*: in the generic, Lax case, these are 1 and 0 respectively with respect to $L^p$, $p < \infty$, and $n+1$ and $n$ with respect to $L^\infty$ (see Chapter IV of [LZ.2], or Section 10 of this paper).

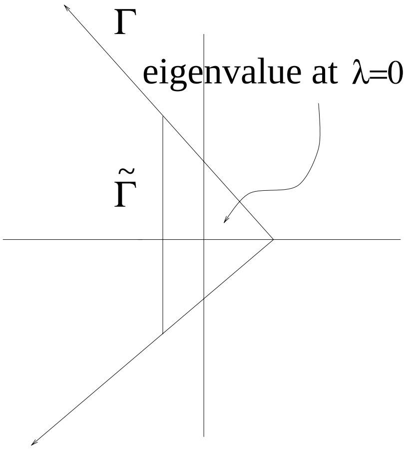


FIGURE 2. THE CONTOURS $\Gamma$ AND $\tilde{\Gamma}$

All of these difficulties are easily resolved from the pointwise, Green's function perspective, at least in principle. Here, we sketch the main ideas informally. Precise statements will be given later on.

***Extension of the resolvent kernel.*** The first, and crucial, observation is that, unlike the resolvent $(L - \lambda I)^{-1}$,

***Result 0.*** (*Prop. 4.6*) *The resolvent kernel $G_\lambda$ can be meromorphically extended into the essential spectrum of $L$, in particular onto some sector*

$$\Omega_\theta := \{\lambda : \mathrm{Re}(\lambda) \geq -\theta_1 - \theta_2 |\mathrm{Im}(\lambda)|\}; \quad \theta_1,\ \theta_2 > 0. \tag{1.19}$$

This remarkable fact, a corollary of the Gap Lemma of [GZ, KS], gives quantitative sense to the statement that $G_\lambda$ encodes more information than does $(L - \lambda I)^{-1}$. Put another way, Result 0 expresses the fact that the essential spectrum is *not an intrinsic barrier.* This is related to Sattinger's observation in the scalar case that essential spectrum can be shifted to the left by the choice of an appropriate weighted norm [Sat]. In fact, Result 0 follows from Sattinger's argument in the decoupled case $B \equiv I$, by the choice of appropriate weights in each scalar field. But, in general, the extension of $G_\lambda$ may not correspond to the resolvent for any choice of norm. A closer analogy is the extension of the resolvent carried out by Kapitula, [K.1-2], as an operator between two *differently* normed spaces. However, this is not sufficiently explicit for our needs.

***Extension of the spectrum.*** The spectral projection operator can in turn be extended by defining the *projection kernel*,

$$P_{\lambda_0}(x,y) := \mathrm{Res}_{\lambda_0} G_\lambda(x,y).$$

The operator $\mathcal{P}_{\lambda_0}$ is then defined by

$$\mathcal{P}_{\lambda_0} f(x) := \int P_{\lambda_0}(x,y) f(y)\, dy,$$

where $f$ is any suitably rapidly decaying function. In particular, for $\lambda = 0$, $\mathcal{P}$ can be shown to be defined on all $f \in L^1$, provided that 0 is semisimple ($G_\lambda$ has a pole of order one at 0). This determines an extended, "effective point spectrum" $\sigma_p'(L)$ consisting of the poles of $G_\lambda$, where the "effective eigenspace" $\Sigma_{\lambda_0}'(L)$ at $\lambda_0$ is defined to be the range of $\mathcal{P}_{\lambda_0}$ on test functions $f \in C_0^\infty$.

***Pointwise bounds.*** In similar fashion, we can replace the spectral resolution formula (1.16) with its pointwise analog,

$$G(x,t;y) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} G_\lambda(x,y)\, d\lambda, \tag{1.20}$$

relating the parabolic Green's function $G$ to the elliptic Green's function $G_\lambda$. As usual, this is obtained formally by applying both sides of (1.16) to $\delta_y(x)$. Rigorous justification requires only bounds on $G_\lambda$ sufficient to exchange the order of integration in (1.16), (1.15). These follow from the *coercivity* of $(L - \lambda I)$, much as do the sectorial bounds on the resolvent (Section 7). Alternatively, the same bounds on $G_\lambda$ can be used to verify (1.20) directly, by showing that (distributional) $x$- and $t$-derivatives can be moved inside the integral (see Corollary 7.4).

Using (1.20), we can determine the asymptotic behavior of the Green's function, and, more generally, the behavior for all $x$, $y$, $t$. For example, consider the trivial case in which $x$ and $y$ are bounded and $t \to \infty$. In this case, we can proceed exactly as in the classical setting. Defining $\tilde\Gamma$ as in (1.18), with $\eta$ so small that the (necessarily isolated) pole $\lambda = 0$ is the only singularity of $G_\lambda$ between $\Gamma$ and $\tilde\Gamma$, and observing by compactness (plus large $|\lambda|$ bounds) that $|G_\lambda|$ is bounded, we obtain similarly as before that

$$G(x,t;y) = \mathbf{O}(e^{-\eta t/2}) + P_0(x,y), \tag{1.21}$$

provided $\lambda = 0$ is a simple pole/semisimple eigenvalue. Thus, the Green's function in this regime decays exponentially to the projection kernel at $\lambda = 0$. That is, the effective projection $\mathcal{P}_0$ *captures the asymptotic dynamics of the inner shock layer.*

Bounds in other regimes are much more complicated. In general, $G_\lambda$ does not remain bounded as $\lambda$ crosses the essential spectrum boundary, but grows exponentially as $|x|$, $|y| \to \infty$; this is precisely why the resolvent becomes unbounded, (1.15). On the other hand, the *rate* of exponential growth gives a quantitative measure of *how* unbounded is the resolvent. This allows the possibility of estimating $G$ by balancing the spatial growth of $G_\lambda$ against the temporal decay of $e^{\lambda t}$ as $\Gamma$ is moved into the negative complex half-plane (and thus into the essential spectrum). Our approach is, roughly speaking, to choose an optimal contour for given $x, y, t$ by the *Riemann saddlepoint method*, [R,CH,DeB], selecting a *minimax* path for the modulus of the integrand in (1.20). However, this optimal path is different for different modes of the solution; thus, we must first effect a spectral decomposition into the various scalar modes. This cannot be done globally in $\lambda$, but it *can* be done in the critical small $|\lambda|$ region governing large time behavior, via careful expansion of $G_\lambda$ near $\lambda = 0$ (Section 7). A more detailed discussion of our method for obtaining pointwise bounds is given in the introductory material of Section 8.2 (highly recommended).

Our argument in the scalar case reduces to essentially that of [H.1], and in the constant-coefficient case to that of [LZe]. Here, however, we find interesting new scattering and excitation effects not present in either of these previous contexts. These make the behavior difficult to guess a priori. The power of our method is that it requires no such apriori information to apply, but rather gives an *algorithm* by which one can *deduce* the behavior of solutions. This should make it applicable in much more general circumstances (see Section 1.2.5).

1.2.3. MAIN RESULTS. We now describe our main results, again in informal fashion. The procedure of the previous section defines an effective point spectrum $\sigma_p'(L)$, and eigenspace $\Sigma'(L)$ on any domain of meromorphicity of $G_\lambda$. By definition, these agree with the standard versions, $\sigma_p(L)$ and $\Sigma(L)$, away from $\sigma_{\text{ess}}(L)$.

***Result 1.*** (*Prop. 5.3*) *The effective point spectrum is well-behaved in the sense that it obeys a modified Fredholm theory. In particular,* $\dim\ \Sigma'_{\lambda_0}(L)$ *is finite and equal to* $\dim\ \Sigma'_{\lambda_0^*}(L^*)$, *and* $P_{\lambda_0}$ *decomposes into a sum of right and left eigenfunction pairs,*

$$P_{\lambda_0} = \sum_j \varphi_j(x)\pi_j(y), \quad \varphi_j \in \Sigma'_{\lambda_0}(L), \quad \pi_j \in \Sigma'_{\lambda_0^*}(L^*).$$

A tool that has proved quite useful in locating point spectrum is the Evans function [E, J, AGJ, PW, GZ, KS] (defined rigorously in Section 4). This is an analytic function $D_L(\lambda)$ that plays a role for unbounded operators analogous to that played by the characteristic polynomial $\det\ (L - \lambda I)$ for a finite-dimensional operator $L$; indeed, $D$ is almost, but not quite, equal to $\det G_\lambda^{-1}$ (compare

(4.9) with (4.30)). Its origins come, rather, from the study of the eigenvalue equation and topological methods in classical Sturm–Liouville and scattering theory [Ti,RS]. Away from the essential spectrum, the zeroes of $D_L$ correspond to eigenvalues of $L$, counting algebraic multiplicity [AGJ]. Up to now, however, there has been some mystery as to the meaning of zeroes of $D_L$ occurring *inside* the essential spectrum [PW].

***Result 2.*** (*Prop. 6.2(ii)*) *Within $\Omega_\theta$, zeroes of the Evans function $D_L(\lambda)$ correspond, counting algebraic multiplicity, with effective eigenvalues of $L$.*

Results 1-2 address the first half of problem (i): the determination of point spectrum. Note that Result 2 has the important consequence that effective point spectrum is *continuous* with respect to continuous perturbations of $L$. The relation to asymptotic behavior is given by the *Evans function criterion*:

***Result 3.*** (*Prop. 9.2*) *$L^p$ linear orbital stability of $\bar{u}$ with respect to $L^1$, $p > 1$, is equivalent to:*

($\mathcal{D}$) *$D_L(\lambda)$ has precisely $\ell$ zeroes in $\{\mathrm{Re}\lambda \geq 0\}$, where $\ell$ is the dimension of the stationary manifold. Alternatively, $L$ has $\ell$ effective eigenvalues in $\{\mathrm{Re}\lambda \geq 0\}$.*

The proof of condition ($\mathcal{D}$) is surprisingly involved, depending on the following detailed estimates on the linearized solution operator about the wave. That is, it depends on the resolution of problem (ii), the determination of sharp global bounds on the Green's function $G(x,t;y)$ of (1.6).

Before stating the result, we describe our possibly confusing shorthand notation. To save space, the choice of sign $\pm$ or $\mp$ associated with an index $j$ is held consistent (i.e. top or bottom) throughout each summand, following the choice made in the limits of the sum, along with the choice of inequality $\gtrless$ or $\lessgtr$. Thus, for example, in the second sum appearing in the equation for $S$ below, the term corresponding to limits $a_k^+ < 0$, $a_j^- < 0$ would be

$$\chi_{\{t \geq |y/a_k^+|\}} \mathbf{O}(t^{-1/2} e^{-(x - a_j^-(t - |y/a_k^+|))^2/Mt})$$
$$\times (r_j^- \chi_{\{x<0\}} + \mathbf{O}(e^{-\eta|x|}))(l_k^+ \chi_{\{y>0\}} + \mathbf{O}(e^{-\eta|y|})).$$

This convention is followed throughout the paper.

***Result 4.*** (*Thm. 8.3*) *Let ($\mathcal{D}$) hold, as well as ($\mathbf{C0}$)–($\mathbf{C3}$). Then, $G = S + E + R$, where*

$$(1.22)\qquad S(x,t;y) = \sum_{k,\pm} \mathbf{O}(t^{-1/2} e^{-(x-y-a_k^{\pm} t)^2/Mt}) \times (r_k^{\pm} \chi_{\{x \gtrless 0\}} + \mathbf{O}(e^{-\eta|x|}))(l_k^{\pm} \chi_{\{y \gtrless 0\}} + \mathbf{O}(e^{-\eta|y|}))$$

$$+ \sum_{a_k^{\pm} \lessgtr 0, a_j^{\pm} \gtrless 0} \chi_{\{t \geq |y/a_k^{\pm}|\}} \mathbf{O}(t^{-1/2} e^{-(x-a_j^{\pm}(t-|y/a_k^{\pm}|))^2/Mt}) \times (r_j^{\pm} \chi_{\{x \gtrless 0\}} + \mathbf{O}(e^{-\eta|x|}))(l_k^{\pm} \chi_{\{y \gtrless 0\}} + \mathbf{O}(e^{-\eta|y|}))$$

*comprises the* scattering *modes,*

$$(1.23)\qquad E(x,t;y) = \sum_{k, y \gtrless 0} \chi_{\{|x-y| \leq |a_k^{\pm} t|\}} \varphi_k^{\pm}(x) \pi_k^{\pm}(y) + \sum_{k, y \gtrless 0} \mathbf{O}(e^{-(x-y-a_k^{\pm} t)^2/Mt} + e^{-(x-y+a_k^{\pm} t)^2/Mt}) e^{-\eta|x|} \pi_k^{\pm}(y)$$

*comprises the excited modes, and $R$ is a faster decaying residual term.*

*Here, $\eta > 0$, $M > 0$ is a suitably large constant, $a_j^{\pm}$ denote the eigenvalues of $A_{\pm}$ and $r_j^{\pm}$ and $l_j^{\pm}$ the corresponding right and left eigenvectors, and $\varphi_k^{\pm} \in \Sigma_0'(L)$, $\pi^{\pm} \in \Sigma_0'(L^*)$. Similar bounds hold for spatial derivatives $G_x$ and $G_y$.*

Figure 3a, below, depicts typical scattering terms, consisting of Gaussian signals originating at $y$ and scattering from the shock layer. Note that signals propagate with asymptotic characteristic speed $a_k^{\pm}$ until they reach the shock location $x = 0$, then travel with outgoing asymptotic characteristic speed $a_j^{\pm} \gtrless 0$ thereafter. That is, they propagate at the speeds predicted by the linearized equations about the corresponding *inviscid* shock. This schematic description validates heuristic and numerical conclusions drawn in [ZPM].

Figure 3b depicts the excitation of a single stationary mode by a signal originating from $y$. Note that the final time-asymptotic state, determined by projection against the left eigenfunction $\pi$, appears gradually, within the cone of influence $\chi_{\{|a_k^{\pm} t| \geq |y|\}}$ of $y$. The conglomerate time-asymptotic state $\mathcal{P}_0 v_0$ is determined as the superposition of modes $\varphi_k$ excited by the different signals originating from $y$, $k = 1, \ldots, n$. This is an extremely detailed picture of behavior, that would be difficult either to see by numerics or guess from heuristic considerations.

More detailed bounds, including derivative estimates, are given in Theorem 8.3. Comparison with exact solution (Example 8.6) verifies that these bounds are sharp.

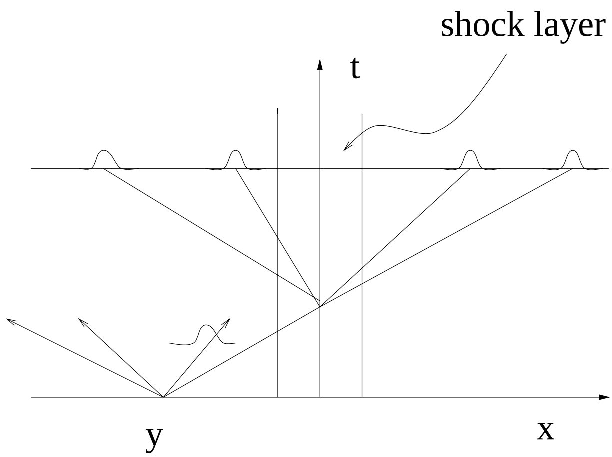


Figure 3a. Scattering from the shock layer

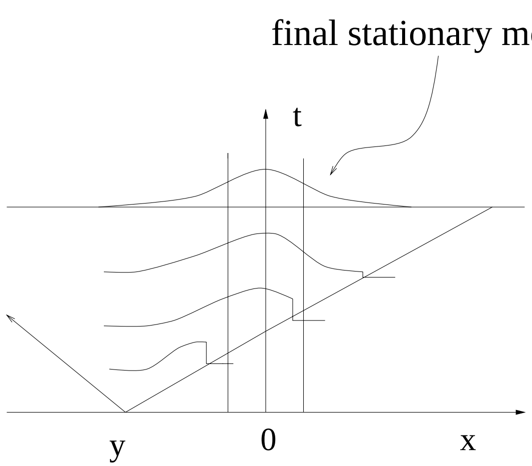


Figure 3b. Excitation of a stationary mode

***Interpretation of*** $(\mathcal{D})$. The abstract Evans function condition can be phrased more directly in conservation law terms, as the standard spectral requirement $\sigma(L)\cap\{\mathrm{Re}\,\lambda \geq 0\} = \{0\}$, augmented with a single *transversality condition* equivalent to $(d/d\lambda)^\ell D_L(0) \neq 0$ (see Section 10). For Lax shocks, this transversality condition takes the form of the *Liu–Majda* condition,

$$\{r_j^\pm : a_j^\pm \lessgtr 0\} \cup (u_+ - u_-) \text{ is a basis for } \mathbb{R}^n, \tag{1.24}$$

an algebraic condition corresponding to *linearized stability of the corresponding inviscid shock*, and also to the property that *the asymptotic state of the perturbed (viscous) wave is determined by perturbation mass*. Here, $a_j^\pm$, $r_j^\pm$ are the eigenvalues and eigenvectors of $A_\pm = f'(u_\pm)$. For Lax and undercompressive shocks, the transversality condition is equivalent to linearized stability of the inviscid shock; for Lax and overcompressive shocks, to mass-determination of the viscous asymptotic state.

Pursuing this connection further in the case of Lax and overcompressive shocks, we discover the useful alternative conditions:

***Result 5.*** (*Props. 10.5-10.6*) *For Lax and overcompressive shocks, linearized orbital stability is equivalent to linearized asymptotic stability with respect to zero mass perturbations plus independence of "outgoing" eigenvectors of $A_\pm$. Alternatively, the Evans function for the "integrated equations" has no zeroes in the nonnegative complex half-plane* $\{\mathrm{Re}\lambda \geq 0\}$.

Zero-mass stability can be treated by more standard, energy methods; indeed, zero-mass results exist for many cases in which the general problem remains open [G.1, MN, KN, Fri]. Likewise, computation of the "integrated" Evans function is numerically better conditioned than computation of the usual Evans function [Br].

1.2.4. APPLICATIONS TO NONLINEAR STABILITY. Using the Green's function bounds of Result 4/Proposition 8.3, one can establish the following theorem on *nonlinear stability*:

***Result 6****.* (*Prop. 11.1, Thm. 11.7*) *Condition* $(\mathcal{D})$ *implies nonlinear $L^p$ orbital stability, $p > 1$, of $\bar{u}$ with respect to*

$$\mathcal{A}_\zeta := L^1 \cap C^{0+\alpha} \cap \{f : |f(x)| \leq \zeta(1+|x|)^{-r}\}, \tag{1.25}$$

*for $r$ sufficiently large and $\zeta$ sufficiently small.*

The converse should also be possible to prove, but involves more complicated analysis of *wave-splitting* and convergence to Riemann patterns (see Section 12). The technical condition $(u_0 - \bar{u}) \in C^{0+\alpha}$ can be dropped when $B \equiv$ *constant*.

The asterisk indicates that we do not prove Result 6 in full generality here. That analysis involves separate issues outside the focus of this paper. We content ourselves with the observation that, for Lax and overcompressive shocks, our

Green's function bounds are sufficient to apply the pointwise Green's function argument of [L.3] virtually without modification. Indeed, the analysis somewhat simplifies, since we need not deal with diagonalization and other errors associated with construction of approximate Green's functions. This topic is discussed in Section 11. From the conclusions of [L.3], we immediately obtain Result 6 with $r = 3/2$, along with detailed pointwise bounds on the nonlinear solution, *for general Lax and overcompressive shocks.* Likewise, the $2 \times 2$ *undercompressive* case can, with slight modifications, be treated by the argument of [LZ.2], though we do not prove this here.

To treat the general ($n \times n$) undercompressive case, or values of $r < 3/2$, requires a refined analysis incorporating the wave-tracing technique described in Section 9. These topics will be treated in detail in future work [Z.2]. The case $r = 1$ is important for the study of stability of *Riemann patterns*, since the tail of a rarefaction wave decays as $1/|x|$ [SZ].

Using the two observations above, and Result 5 of the previous section, we easily recover the existing analytic results on stability, and several new results as well.

***Weak shocks.*** In the case of weak shocks, $(\mathcal{D})$ can be confirmed by energy estimates, either directly from the eigenvalue ODE, or using Result 5 to bootstrap from existing zero mass results for the linearized PDE (1.6). We remark that essentially the same calculations are involved in energy estimates for the ODE as for the PDE.

For example, from the original, zero-mass result of Goodman, [G], we obtain nonlinear stability of weak, genuinely nonlinear Lax shocks, with condition

$$R(u)B(u)R(u)^{-1} > 0 \tag{1.26}$$

on the viscosity matrix, a condition which implies (**H1**) and (**H3**). (Condition (1.26) is equivalent to (**H1**) and (**H3**) in the symmetrizable case, but not always [MP]). Thus, we extend the result of [L.3] to nearly arbitrary, variable viscosity matrices.

Likewise, we can extend to the case of weak, non-genuinely nonlinear Lax shocks using the very recent zero-mass result of Fries [Fri.1] (note: Fries has since extended this result to the *nonzero mass* case, using arguments in the spirit of [SX], [Fri.2]). Such shocks arise, for example in elasticity and coplanar MHD. In fact, the argument also applies to a class of special overcompressive shocks analogous to radial overcompressive shocks in the cubic model [Fre.3]. This gives rise, by continuity of effective point spectrum with respect to perturbations in $L$ (result 2), to a result analogous to that of [FreL] for the cubic model, but for the full equations of MHD: namely, orbital stability of a small *band* of nearby overcompressive shocks.

Finally, we mention that the zero-mass results of [MN,KMN] give nonlinear stability of relatively strong shocks for gamma-law gas dynamics with artificial (but quasilinear), strictly parabolic viscosity. However, the strength of the shock depends on $\gamma$ through appropriate rescaling.

***Shocks of*** $2 \times 2$ ***cubic and quadratic models.*** The results of [LZ.1-2] and [FreL] for under- and overcompressive shocks concern perturbations of linear profiles of the quadratic complex Burgers and cubic models, respectively. We recover these results by the observation that in both cases the linearized equations decouple, giving linearized stability from standard scalar results. The general case then follows, again, by continuity of effective point spectrum.

***Strong shocks and numerical verification.*** For strong (large amplitude) shocks, and strongly nonlinear nonclassical shocks, stability does not hold in general [FreZ, GZ]. Evidently, the structure and behavior of such shocks can be rather arbitrary, making their analytic treatment problematic. Nonetheless, the Evans function condition $(\mathcal{D})$ gives a *numerically computable* criterion by which we can assess their stability. For example, Brin has developed an efficient and accurate code exploiting the analyticity of $D_L$, which assesses $(\mathcal{D})$ by calculating the winding number of $D_L$ around the nonnegative complex half-plane [Br]. His approach appears also to offer the possibility of numerical proof.

***Direct continuation arguments.*** The hybrid analyses described so far do not exploit the full power of the spectral machinery. In the future, it may be possible to use direct arguments to replace standard methods altogether. For example, there is outlined in [GZ] a program for studying stability of weak shocks by continuation from the "zero shock strength" limit, or constant solution. This would have the advantage of treating Lax, over- and undercompressive shocks on the same footing. An equally intriguing possibility is the analytic treatment of *strong shocks* in certain cases. For example, a standard Evans function argument is:

***Claim.*** *Assuming* (**H0**)–(**H4**), *let*

(i) $\bar{u}_s(x-st)$ *be a family of viscous Lax shock solutions of* (1.2) *depending smoothly on the shock speed* $s \geq 0$, *with* $u_-$ *fixed,* $\bar{u}_0 \equiv u_-$, *for which*

(ii) *the Liu–Majda condition* (1.24) *holds for all* $s$. *Further, suppose that*

(iii) $L_s$, *the linearized operator about the wave, has no pure imaginary eigenvalues for any* $s$.

*Then, each one of the entire family of shocks* $\bar{u}_s$ *is nonlinearly stable.*

*Proof.* By the result for weak Lax shocks, $\bar{u}_s$ is stable for $s$ sufficiently small, in particular $\sigma(L_s) \cap \{\mathrm{Re}\lambda \geq 0\} = \{0\}$. The Liu–Majda condition, by Result 5, implies that there is exactly $\ell = 1$ effective eigenvalue at $\lambda = 0$ for all $s$, corresponding to the eigenfunction $\bar{u}_x$, while by assumption there are no effective eigenvalues on the imaginary axis. Thus, no effective eigenvalues can cross the imaginary axis, and so $\sigma(L_s) \cap \{\mathrm{Re}\lambda \geq 0\} = \{0\}$ for all $s$, giving the result. □

For gamma-law gas dynamics, $\gamma \neq 3$, (i) and (ii) are always satisfied, as pointed out in [Gi] and [Se.1], respectively. It might be hoped that structural properties such as symmetrizability or existence of an entropy would imply (iii). The question of existence or non-existence of imaginary eigenvalues, (iii), is a fundamental one for vector-valued non-normal operators.

1.2.5. Extensions/open problems. There are many directions in which one can generalize the basic theory outlined above.

***Symmetrizable hyperbolic–parabolic systems.*** An important direction for extension of the theory is to relax the requirements of strict hyperbolicity and parabolicity in hypotheses (**H2**) and (**H1**) to include the boundary case of symmetrizable hyperbolic–parabolic systems [Ka]. This important class of systems includes such physical examples as gas dynamics and MHD, both of which lie on the boundary of (**H1**)–(**H2**). In this setting, the assumption of strict hyperbolicity in (**H2**) is easily removed (see Remark 2.3). However, the inclusion of "real", semidefinite viscosity necessitates modification of our arguments for pointwise Green's function bounds. An approach that looks promising is to proceed by vanishing viscosity approximation, keeping careful track of constants. A new feature in the argument is that we must perform a spectral decomposition (i.e. expansion of $G_\lambda$) at $|\lambda| = \infty$ as well as $\lambda = 0$, in order to isolate singular (i.e. exponentially decaying delta-function) components in the solution; for related arguments, see [Ka, LZe, HoZ]. The passage from linear to nonlinear stability likewise becomes problematic, due to incomplete parabolic smoothing; however, this issue has already been addressed in the study of decay to constant states, [LZe], by a combination of Green's function and energy methods. It can be hoped that these methods will carry over to the nonconstant coefficient case.

***Sonic shocks.*** Another interesting boundary case is that of "sonic" shocks, in which one or more of the $a_j^\pm$ vanish. This violation of (**H2**) is less generic than the previous one, in the sense that it occurs in any given system only for *certain shocks*, whereas strict hyperbolicity can be violated for *all* shocks of certain *systems* (e.g. MHD). In this case, the viscous profile decays *algebraically* to its endstates $u_\pm$ [Ni], making the estimates more subtle. As above, we must proceed by a limiting argument, letting $a_j^\pm \to 0$, and taking advantage of certain cancellations that occur. This seems quite interesting from the general point of view of variable coefficient convection–diffusion equations with slowly decaying coefficients. Examples such as the Kolmogorov equation, [Bra], show that the effects of a slowly decaying tail can in general be quite subtle. We remark that the zero-mass results required to verify $(\mathcal{D})$ have already been obtained by Nishihara [N.1].

***Unstable endstates.*** Interestingly, Majda–Pego stability of the endstates $u_\pm$, (**H3**), is not necessary for stability of a shock wave under sufficiently localized (i.e. exponentially decaying) perturbations [Z.3]. That is, a connecting shock wave can stabilize unstable constant states. This stabilizing effect is due to compressivity.

We emphasize that our approach is not inherently limited to the present setting of parabolic conservation laws with second order diffusion, or to a single spatial dimension. Indeed, the main strength of the method is its general applicability, *without* the foreknowledge of asymptotic behavior. We mention a few wider applications below.

***Dispersive undercompressive shocks.*** Recently, there has been considerable interest in nonclassical shock waves arising through the combined effects of diffusion and dispersion [W, JMS, HL.1-2]. However, their stability analysis has been carried out so far only in the scalar case, [D], using the method of weighted norms. Using pointwise semigroup methods, we can obtain pointwise bounds on the Green's function of the linearized equation about the wave [HZ]. The result is that signals propagate mainly in oscillatory Gaussian wave packets, with an additional $\mathbf{O}(e^{-\eta(t+|x-z_{jk}^{\pm}(y,t)|)})$ correction, $\eta > 0$, where $z_{jk}$ denotes the center of the packet (Note that this correction is significant only for short time, reflecting the faster propagation of dispersive signals). We hope to use these bounds in the future to carry out a complete nonlinear stability analysis.

***Multi-dimensional planar shocks.*** Consider a planar shock solution $u(x,t) = \bar{u}(x_1)$ of a conservation law in several space dimensions,

$$u_t + \sum_j f^j(u)_{x_j} = \sum_{j,k} (B^{jk}(u)u_{x_j})_{x_k}, \quad u \in \mathbb{R}^n, \ x \in \mathbb{R}^d.$$

Taking the Fourier transform in the transverse variable $\tilde{x} := (x_2, \ldots, x_d)$, the linearized equation about the wave becomes

$$\hat{v}_t = L_\xi \hat{v},$$

where $L_\xi := L_0 - \sum_j iA^j(x_1)\xi_j - \sum_{j,k} B^{jk}(x_1)\xi_j\xi_k$ and $L_0$ is the one-dimensional linearized operator around the wave, i.e.

$$L_0 f = -(A^1(x_1)f)_{x_1} + (B^{11}(x_1)f_{x_1})_{x_1}$$

(cf. (1.6)). Combining the spectral resolution formula (1.20) and inverse Fourier

Transform, we obtain the Green's function representation

$$G(x,t;y) = \frac{1}{(2\pi i)^d} \int_{\xi \in \mathbb{R}^{d-1}} \int_\Gamma e^{i\xi\cdot(\tilde{x}-\tilde{y})} e^{\lambda t} G_{\lambda,\xi}(x_1,y_1)\, d\lambda\, d\xi, \tag{1.27}$$

where $G_{\lambda,\xi}$ is the resolvent kernel, or elliptic Green's function, for the operator $(L_\xi - \lambda I)$. Using this representation, the behavior of the Green's function can be *deduced* by Taylor expansion/spectral decomposition of $G_{\lambda,\xi}$ in the critical neighborhood about $(\lambda,\xi) = (0,0)$, and a combination of pointwise semigroup methods and "Paley–Wiener" methods as in [HoZ.1-2], as prescribed by the Riemann saddlepoint method in the combined variable $(\lambda,\xi)$. However, the behavior is much more complicated than in the constant coefficient case studied in [HoZ.1-2], due to scattering from the shock layer. This will be the topic of future investigation. Note that we have described a way to *derive* the scattering coefficients of impinging signals, which in contrast to the one-dimensional case are difficult to deduce from heuristic considerations (indeed, they can be quite complicated, as evidenced in [S]).

The role of the effective spectrum is somewhat easier to describe. The stability condition $(\mathcal{D})$ in the multi-dimensional case becomes

$(\mathcal{D}_\xi)$ *$L_\xi$ has no effective eigenvalues in $\{Re\,\lambda \geq 0\}$ for real $\xi \neq 0$, and $\ell$ effective eigenvalues for $\xi = 0$,* [3]

where $\ell$ as usual denotes the dimension of the stationary manifold $\{\bar{u}^\delta\}$ for the one-dimensional problem. Note, for small $|\xi|$, that $L_\xi$ has precisely $\ell$ effective eigenvalues in a neighborhood of $\lambda = 0$, by continuity of effective spectrum. Condition $(\mathcal{D}_\xi)$ implies that these have real part bounded by $-\eta|\xi|^{2m}$ (even to lowest order, because otherwise the real part would cross the imaginary axis for small real $\xi$), for some $\eta > 0$, and integer $m \geq 1$. Consider the simplest case that the order of contact is $2m = 2$ and all eigenvalues are distinct for $0 < |\xi| \leq \rho$ (generic for $\ell = 1$ or spatial dimension $d \leq 2$). Then, the eigenvalues are given by analytic functions $\lambda_h$, $h = 1, \dots \ell$,

$$\begin{aligned} \mathrm{Re}\,\lambda_h(\xi) &= -\sum_{j,k} d_h^{jk} \xi_j \xi_k + \mathbf{O}(|\xi|^3), \\ \mathrm{Im}\,\lambda_h &= -\sum_j c_h^j \xi_j, \end{aligned}$$

[3] This is necessary not for convergence but for convergence at some uniform rate, see the description of the front evolution, below.

$c_h^j$, $d_h^{jk}$ real, $d_h^{jk} > 0$, and the corresponding right and left eigenfunctions by

$$\varphi_{h,\xi}(x_1) = \frac{\partial \bar{u}^\delta}{\partial \delta_h}(x_1) + \mathbf{O}(|\xi|)$$

and

$$\pi_{h,\xi}(y_1) = \pi_h(y_1) + \mathbf{O}(|\xi|).$$

By the same argument used to derive (1.21), the near-field contribution of excited modes is the inverse Fourier transform of

$$\sum_{1\le h\le \ell} e^{\lambda_h(\xi)t}\mathcal{P}_{\lambda_0,\xi}\hat{v}(\cdot,\cdot,0) = \sum_{1\le h\le \ell} e^{\lambda_h(\xi)t}\int_{-\infty}^{+\infty} \varphi_{h,\xi}(x_1)\pi_{h,\xi}(y_1)\hat{v}(y_1,\xi,0)\,dy_1$$

plus a time-exponentially decaying error, where $v_0(x) = u(x,0) - \bar{u}(x_1)$ denotes the initial perturbation of the wave.

Combining with the above expansions of $\varphi_{h,\xi}$, $\pi_{h,\xi}$, and $\lambda_h$, we find that, to lowest order, the near-field contribution of excited modes is given by

$$\sum_{1\le h\le \ell} \alpha_h(\tilde{x},t)\frac{\partial \bar{u}^\delta}{\partial \delta_h}(x_1), \tag{1.28}$$

where

$$\hat{\alpha}_h(\xi,t) := e^{t(-i\sum_j c_h^j \xi_j - \sum_{j,k} d_h^{jk}\xi_j\xi_k)}\hat{\alpha}_h(\xi,0)$$

and

$$\hat{\alpha}_h(\xi,0) := \int_{-\infty}^{+\infty} \pi_h(y_1)\hat{v}_0(y_1,\xi)\,dy_1.$$

Reinterpreting, the local deformation in position and shape of the shock front at each point $\tilde{x}$ is given, (1.28), by a linear combination of the possible deformations $\partial\bar{u}^\delta/\partial\delta_h$ in the one-dimensional profile, where the coefficients $\alpha_h(\tilde{x},t)$ satisfy the linear, constant coefficient system of *decoupled convection–diffusion equations*

$$(\alpha_h)_t + \sum_j c_h^j(\alpha_h)_{x_j} = \sum_{j,k} d_h^{jk}(\alpha_h)_{\tilde{x}_j,\tilde{x}_k}, \tag{1.29}$$

with initial conditions

$$\alpha_h(\tilde{x},0) = \int_{-\infty}^{+\infty} \pi_h(y_1)v_0(y_1,\tilde{x})\,dy_1 \tag{1.30}$$

given by the *time-asymptotic* deformation for the one-dimensional problem obtained by fixing $\tilde{x}$. (The description of behavior in the far field involves also the cone of influence of $y$, similarly as in the one-dimensional case (1.23)). In the general case that eigenvalues cross, the coefficients $\alpha_h$ instead satisfy a *coupled, nonlocal* convection–diffusion system, determined by appropriate Fourier multipliers similarly as in [HoZ], definition on p. 648. The description of front evolution given in equations (1.28)–(1.30) generalizes observations made in [G.2, K.3, GM].

Note, as in standard perturbation theory, [Kat], that

$$
\begin{aligned}
c_h^j &= \langle \pi_h, A^j \varphi_h \rangle \\
&= \int_{-\infty}^{+\infty} \pi_h(x_1)^t A^j(x_1) \frac{\partial \bar{u}^\delta}{\partial \delta_h}(x_1)\, dx_1.
\end{aligned}
\tag{1.31}
$$

In the special case of a Lax shock with constant viscosity coefficients, it holds (see Section 10) that $\ell = 1$, $\partial \bar{u}^\delta / \partial \delta_1 = \bar{u}_{x_1}$, $\pi_1 \equiv$ constant, and $A^j = Df^j(\bar{u}(x_1))$, and we obtain simply

$$
c_h^j = \pi_1^t (f^j(u_+) - f^j(u_-)),
$$

where the vector $\pi_1^t$ is uniquely determined by the properties that $\pi^t r_j^\pm = 0$ for $a_j^\pm \gtrless 0$ and $\pi^t(u_+ - u_-) = 1$ (see condition (1.24)). This reduces in the two-dimensional, scalar case to the result obtained in [G.2, GM].

***Other waves.*** Further potential applications are to combustion waves, especially (undercompressive) weak detonation and deflagration waves, and to shocks occurring in relaxation systems. In both cases, note that shock profiles have the property of exponential decay to their endstates $u_\pm$, making these natural applications of the theory. However, there are also interesting new issues to resolve, connected with damping effects.

Also, Serre has recently pointed out that many *discrete* shock profiles also fall naturally within the Evans function framework, in the case that shock speed is rational [Se.2]. For such shocks, one can use the pointwise semigroup method to estimate the discrete Green's function of the linearized difference equation about the wave. (Note: for *irrational speeds*, our techniques do not directly apply, and indeed the behavior can be much more complicated [LY.1-2]).

***Other directions.*** Above, we have described various directions in which to push the analytic, PDE framework for stability of traveling waves. However, in our view, the most exciting areas for future study are those that are now opened up by the PDE theory at the phenomenological and ODE level. In particular, we mention:

(i) Investigation of *mechanisms* for stability and instability of viscous shock waves by careful study of the eigenvalue equations and their structure.

(ii) Determination of existence or nonexistence of imaginary eigenvalues in physically interesting settings.
(iii) Development of efficient and general computational tools along the lines of [Br], suitable for pushbutton stability analysis of a variety of waves.
(iv) Numerical proof on a practical time scale.
(v) Stability of strong shocks.

**Plan of the paper.** With the exception of the material on nonlinear stability in Section 11, this paper is intended to be self-contained. What semigroup/spectral theory we need is developed within, in the very concrete pointwise setting. Likewise, no prior knowledge of conservation laws is assumed or needed, except in certain remarks. We have divided the paper into four parts, which are expected to be of interest to different subsets of readers. These can to some extent be read independently of each other. In particular, the spectral and Evans function theory of Part II is carried out separately from the rest of the paper, and in a quite general context. We hope that this material will be of use to workers in areas other than conservation laws.

We begin in Section 2 with the study of the asymptotic systems at $\pm\infty$ of the eigenvalue equation $Lw = \lambda w$. In particular, we develop an asymptotic expansion near $\lambda = 0$ for the normal modes, analogous to that obtained by Kawashima [Ka] in the Fourier setting. In Section 3, we give an exposition of the Gap Lemma of [GZ, KS], taking care to determine minimal regularity hypotheses for operators in divergence form. This sets the stage for the construction in Section 4 of the elliptic Green's function $G_\lambda$ in terms of the asymptotic modes at $x = \pm\infty$, and the meromorphic extension of $G_\lambda$ into the essential spectrum.

In Section 5, we define the effective point spectrum, and show that it obeys a modified Fredholm theory. In Section 6, we establish the key relation between the effective eigenvalues and the Evans function. The results in these sections, being of wider interest, are carried out for rather general operators $L$.

In Section 7, we derive pointwise bounds on $G_\lambda$ in the three regimes of small, medium and large $|\lambda|$. The critical small $|\lambda|$/large time estimate is obtained by direct expansion of the description of $G_\lambda$ developed in Section 4, giving the decomposition into scattering modes. The intermediate $|\lambda|$ estimate $|G_\lambda| \le C$, reflecting but not implied by the fact that the resolvent is (by definition) bounded on the resolvent set, is obtained by a straightforward compactness argument. The large $|\lambda|$/short time estimate, corresponding to classical bounds for the Laplacian, is closely related to coercivity; it is obtained by a rescaling argument similar to those in [GZ, AGJ]. In Section 8, we carry out the main calculation of the paper, obtaining sharp bounds on $G$ by integrating the above bounds on $G_\lambda$ along appropriate saddlepoint contours.

In Section 9, we introduce a wave-tracing scheme to obtain sharp results on linearized orbital stability. In particular, we verify the necessity and sufficiency of condition $(\mathcal{D})$. In Section 10, we investigate the meaning of the transversality condition $(d/d\lambda)^\ell D_L(0) \ne 0$ in the context of conservation laws, deriving useful consequences for Lax and overcompressive shocks. These are used in Section 11 to verify the conditions needed for the nonlinear stability argument of Liu [L.3]. This argument was carried out in [L.3] for artificial viscosity $B \equiv constant$; we extend to the full, variable-viscosity case using the "pointwise" smoothing property of the parabolic solution operator. We conclude in Section 12 with an examination of *neutral instability*, and the phenomenon of *wave–splitting*, carried out within the context of the effective point spectrum.

***Note.*** The results in this paper were announced in their entirety July 1997 in the Fifth Workshop on Partial Differential Equations at IMPA, Rio de Janeiro, Brazil, and presented in detail November 1997 in the PDE Conference in honor of Olga Oleinik at Ames, Iowa, USA, and December 4 and 18, 1997 in the hyperbolic waves meetings at Mittag–Leffler Institute and KTH, Stockholm, Sweden. (An exception is the discussion of open problems, Section 1.2.5, which in the talks were only listed).

On the latter occasion, we learned of results in a similar direction obtained independently by Kreiss and Kreiss [KK], and they of ours. We refer the reader to their (subsequently appearing) paper for an interesting alternative approach to stability of viscous shock waves, not involving pointwise bounds. Their approach, however, is so far limited to the case of perturbations with zero mass, Lax, or "Lax-like"[4] shocks, and artificial viscosity $B \equiv I$. They identify a sufficient condition for zero-mass stability, but do not discuss necessity or verification of this condition.

## Part I. Preliminaries.

### 2. The Asymptotic Eigenvalue Equations.

The eigenvalue equation $Lw = \lambda w$ associated with (1.6) is

$$(Bw')' - (Aw)' = \lambda w. \tag{2.1}$$

Written as a first-order system in the variable $W = (w, w')^t$, this becomes

$$W' = \mathbb{A}(\lambda, x)W, \tag{2.2}$$

where

$$\mathbb{A} := \begin{pmatrix} 0 & I \\ \lambda B^{-1} + B^{-1}A' & -B^{-1}B' + B^{-1}A \end{pmatrix}. \tag{2.3}$$

[4] Undercompressive shocks of degree one, with the special property that asymptotic shock location is determined by mass of the initial perturbation (Assumption 2, [KK]). See Section 10, or [LZ.1-2], for a discussion of undercompressive shock waves.

We begin by studying the limiting, constant coefficient systems $L_\pm w = \lambda w$ of (2.1) at $\pm\infty$,

$$B_\pm w'' - A_\pm w' = \lambda w, \tag{2.4}$$

or, written as a first-order system,

$$W' = \mathbb{A}_\pm(\lambda) W, \tag{2.5}$$

where

$$\mathbb{A}_\pm(\lambda) := \begin{pmatrix} 0 & I \\ \lambda B_\pm^{-1} & B_\pm^{-1} A_\pm \end{pmatrix}, \tag{2.6}$$

$$B_\pm := B(\pm\infty), \quad A_\pm := A(\pm\infty).$$

The normal modes of (2.5) are $V_j^\pm e^{\mu_j^\pm x}$, $j = 1, \dots, 2n$, where $\mu_j^\pm$, $V_j^\pm$ are the eigenvalues and eigenvectors of $\mathbb{A}_\pm$; these are easily seen to satisfy

$$V_j = \begin{pmatrix} v_j \\ \mu_j v_j \end{pmatrix}, \quad v_j \in \mathbb{C}^n \tag{2.7}$$

and

$$(\lambda_j B_\pm^{-1} + \mu_j B_\pm^{-1} A_\pm - \mu_j^2 I) v_j = 0. \tag{2.8}$$

Note that as roots of the algebraic equation det $(\lambda B^{-1} + \mu B^{-1} A - \mu^2 I) = 0$, the $\mu_j$ are *holomorphic functions* of $\lambda$. The $V_j$ are holomorphic as well, except at points where there are generalized eigenvectors $V_j^k$. These exceptional points are *isolated*, as zeroes of the holomorphic function $(\partial/\partial\mu)$det $(\lambda B^{-1} + \mu(\lambda) B^{-1} A - \mu(\lambda)^2 I)$. The location of these points is not particularly important in our analysis. All we require is that $\lambda = 0$ is not among them: more precisely, that the $\mu_j$ for which $\mu_j(0) = 0$ are analytic near $\lambda = 0$, with corresponding analytic $V_j(\lambda)$. To simplify our discussion, we make an additional assumption ensuring that *all* $\mu_j$ are analytic at the origin:

(**A1**) $\sigma(B_\pm^{-1} A_\pm)$ is distinct.

This can be achieved by an arbitrarily small perturbation in the matrix function $B(u)$. As the estimates in this paper are insensitive to such perturbations, we obtain the general case in the limit.

The essential spectrum $\sigma_{\text{ess}}(L)$ is confined (see [He], or the results of Section 4) to the *complement* of the set

$$\Lambda = \bigcap \Lambda_h^{\pm}; \quad h = 1, \ldots, n, \tag{2.9}$$

where $\Lambda_h^{\pm}$ denote the open sets bounded on the left by the algebraic curves $\lambda_h^{\pm}(k)$ determined by the eigenvalues of $(-kiA_{\pm} - k^2 B_{\pm})$, $k \in \mathbb{R}^1$. Equivalently, $\lambda_h^{\pm}(k)$ represent the curves along which $\mu_j^{\pm}(\lambda) = ki$ for some $j$, that is, $\lambda_h^{\pm}(k)$ are the curves across which $\text{Re}(\mu_j^{\pm})$ change sign. It is not difficult to see by the methods of Section 4 that $\partial\Lambda$ is in fact contained in $\partial\sigma_{\text{ess}}(L)$. In the Evans function literature (see, e.g., [AGJ, JGK, GZ]), $\Lambda$ is called the *region of consistent splitting* (see (ii), Proposition 2.1, below).

Note that $(-kiA_{\pm} - k^2 B_{\pm})$ is the symbol of $L_{\pm}$ under Fourier Transform. Thus, $\sigma_{\text{ess}}(L) \subset \{\text{Re}\lambda \le 0\}$ is equivalent to

$$\text{Re}\,\sigma(-kiA_{\pm} - k^2 B_{\pm}) \le 0, \tag{2.10}$$

which is equivalent to $L^2$ linearized stability of the constant state solutions $u \equiv u_{\pm}$, i.e. $L^2$ stability of $v_t = L_{\pm} v$.

Linearized stability of constant states was studied by Kawashima, [Ka], who showed (**H3**) to be equivalent, for $B_{\pm} \ge 0$ and $A_{\pm}$, $B_{\pm}$ *simultaneously symmetrizable*, to the condition

$$\text{diag}\,[R_{\pm} B_{\pm} R_{\pm}^{-1}] = \begin{pmatrix} \beta_1^{\pm} & & 0 \\ & \ddots & \\ 0 & & \beta_n^{\pm} \end{pmatrix} > 0, \tag{\textbf{K3}}$$

where $R_{\pm}$ is the matrix of right eigenvectors of $A_{\pm}$, i.e.

$$R_{\pm} A_{\pm} R_{\pm}^{-1} = \begin{pmatrix} a_1^{\pm} & & 0 \\ & \ddots & \\ 0 & & a_n^{\pm} \end{pmatrix}.$$

Indeed, it is easily checked, for *any* (not necessarily symmetrizable) $A_{\pm}$, $B_{\pm}$, that $\lambda_h^{\pm}$ has the spectral expansions

$$\lambda_h^{\pm}(k) = -ika_h^{\pm} - \beta_h^{\pm} k^2 + \mathbf{O}(k^3), \quad |k| \le 1, \tag{2.11}$$

$$\lambda_h^{\pm}(k) = -b_h k^2 + \mathbf{O}(k), \quad |k| \ge 1 \tag{2.12}$$

for $k$ near 0 and $\infty$, respectively, where $b_h$ are the eigenvalues of $B$. Thus, (**H3**) in any case implies (**K3**), which was shown in [MP] to play a role also in existence and structure of classical traveling wave solutions. For the role of (**K3**) in non-classical shock structure, see [AMPZ.2].

For our analysis, we will require the following analog of Kawashima's calculation, essentially inverting expansion (2.11).

***Proposition 2.1.*** *Let* (**C0**)–(**C3**) *hold, and let* $\beta_j^\pm$ *be as in* (**K3**). *Then,*

(i) $\beta_j^\pm > 0$; *and*

(ii) *At all except possibly countably many isolated points* $\{\lambda_j\}$ *of* $\Lambda$, *there locally exist analytic choices* $\mu_1^\pm, \cdots, \mu_n^\pm \le 0 \le \mu_{n+1}^\pm, \cdots, \mu_{2n}^\pm$ *(here ordering is by real parts) and* $V_1^\pm, \cdots, V_{2n}^\pm$ *for the eigenvalues and eigenvectors of* $\mathbb{A}_\pm(\lambda)$, $\mathbb{A}_\pm$ *defined as in* (2.6). *Moreover, there exist analytic choices near* $\lambda = 0$ *satisfying the following asymptotic descriptions:*

$$\mu_j^\pm(\lambda), V_j^\pm(\lambda) = \begin{cases} -\lambda/a_j^\pm + \lambda^2\beta_j^\pm/{a_j^\pm}^3 + \mathbf{O}(\lambda^3), \quad \begin{pmatrix} r_j^\pm + \mathbf{O}(\lambda) \\ -\lambda r_j^\pm/a_j^\pm + \mathbf{O}(\lambda^2) \end{pmatrix} & \text{if } a_j^\pm > 0 \\[2ex] \gamma_j^\pm + \mathbf{O}(\lambda), \quad \begin{pmatrix} s_j^\pm \\ \gamma_j^\pm s_j^\pm \end{pmatrix} + \mathbf{O}(\lambda) & \text{if } a_j^\pm < 0, \end{cases} \tag{2.13}$$

$$\mu_{n+j}^\pm(\lambda), V_{n+j}^\pm(\lambda) = \begin{cases} -\lambda/a_j^\pm + \lambda^2\beta_j^\pm/{a_j^\pm}^3 + \mathbf{O}(\lambda^3), \quad \begin{pmatrix} r_j^\pm + \mathbf{O}(\lambda) \\ -\lambda r_j^\pm/a_j^\pm + \mathbf{O}(\lambda^2) \end{pmatrix} & \text{if } a_j^\pm < 0 \\[2ex] \gamma_j^\pm + \mathbf{O}(\lambda), \quad \begin{pmatrix} s_j^\pm \\ \gamma_j^\pm s_j^\pm \end{pmatrix} + \mathbf{O}(\lambda) & \text{if } a_j^\pm > 0, \end{cases} \tag{2.14}$$

*for* $j = 1, \cdots, n$, *where* $a_1^\pm \le a_2^\pm \le \cdots \le a_n^\pm$ *and* $r_j^\pm$ *are the eigenvalues and eigenvectors of* $A_\pm$, *and* $\gamma_1^\pm \le \gamma_2^\pm \le \cdots \le \gamma_n^\pm$ *and* $s_j^\pm$ *are those of* $B_\pm^{-1}A_\pm$.

*Proof.* Because the $\mu_j^\pm$ are holomorphic, they are analytic and distinct at all but countably many isolated points. Where they are distinct, the eigenvectors $V_j^\pm$ are clearly (locally) analytic as well, by standard matrix perturbation theory. Since Re $(\mu_j^\pm)$ does not change sign on $\Lambda$ by definition, it is sufficient to verify the splitting into $n$ positive real part and $n$ negative real part roots for real $\lambda \to \infty$. Taking $\lambda \to +\infty$, we find from (2.7) that the $\mu_j^\pm$ approximately satisfy

$$\det\,(B_\pm^{-1} - (\mu^2/\lambda)I) = 0,$$

which has roots $\mu = \pm\sqrt{\lambda/b_j^\pm}$, where $b_j^\pm$ are the eigenvalues of $B_\pm$. This confirms the $n$–$n$ splitting of roots on $\Lambda$.

The expansion about $\lambda = 0$ follows by standard bifurcation theory [GH]. Substituting $\lambda = 0$ into (2.8) to obtain

$$\mu_j(B_\pm^{-1}A_\pm - \mu_j I)v_j = 0,$$

we find that there is a root $\mu = 0$ of multiplicity $n$, and $n$ distinct roots $\mu = \gamma_j^\pm$, $v = s_j^\pm$. Corresponding to each distinct root, there is an analytic $\mu_j^\pm(\lambda)$ which trivially satisfies $\mu_j^\pm = \gamma_j^\pm + \mathbf{O}(\lambda)$ and $V_j^\pm = \begin{pmatrix} s_j^\pm \\ \gamma_j^\pm s_j^\pm \end{pmatrix} + \mathbf{O}(\lambda)$.

The remaining $n$ roots bifurcate from $(\lambda, \mu) = (0,0)$. Writing (2.8) as

$$(\lambda_j^\pm + \mu_j A_\pm - \mu_j^2 B_\pm)v_j = 0, \tag{2.15}$$

and linearizing about $(\lambda, \mu) = (0,0)$, we obtain $(\lambda^\pm + \mu A_\pm)v = 0$. Since by assumption $A_\pm$ has a full set of eigenvalue–eigenvector pairs $(a_j^\pm, v_j^\pm)$, this is a bifurcation from a simple root, and we obtain $n$ analytic solutions of form

$$\mu_j^\pm = \lambda(-1/a_j^\pm + c_j^\pm\lambda + \mathbf{O}(\lambda^2)), \quad v_j^\pm = r_j^\pm + \mathbf{O}(\lambda).$$

Substituting in (2.15) and matching coefficients, we find that $c_j^\pm = \beta_j^\pm / {a_j^\pm}^3$, where $\beta_j^\pm$ are defined as in (**K3**). Substituting in (2.7), we obtain the expansion for $V_j^\pm$. Finally, setting $\mu = ki$ in the expansion about $\lambda = 0$ we obtain (2.11). This shows that $\beta_j^\pm > 0$, or else (**H3**) and thus (**C3**) would be violated. □

***Remark 2.2.*** The expansions in the first lines of (2.13)–(2.14) can be derived in a more illuminating way using the principle of effective artificial viscosity. As described in [Ka, HoZ.1], the behavior of the constant coefficient version of (1.6) agrees to first order with that of the system obtained by substituting for $B$ the unique "effective artificial viscosity matrix" $\tilde{B}$ commuting with $A$, given by

$$\tilde{B} = R^{-1}\ \mathrm{diag}\ [RBR^{-1}]R = R^{-1}\begin{pmatrix} \beta_1 & & 0 \\ & \ddots & \\ 0 & & \beta_n \end{pmatrix} R.$$

Making this substitution, we find that equation (2.15) decouples into $n$ scalar equations

$$\det\left(\lambda I + \mu\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} - \mu_2\begin{pmatrix} \beta_1 & & 0 \\ & \ddots & \\ 0 & & \beta_n \end{pmatrix}\right) = 0,$$

or

$$\lambda + \mu a_j - \mu^2 \beta_j = 0. \tag{2.16}$$

Solving (2.16) directly, we obtain the formula of

$$\mu = \frac{a_j \pm \sqrt{a_j^2 + 4\beta_j \lambda}}{2\beta_j},$$

where the root is to be taken as close to zero as possible. Note that this indeed agrees with the first lines of (2.13)–(2.14) to second order in $\lambda$. We thus see very directly the crucial fact that $\mu_j$, $V_j$ are *effectively scalar* near $\lambda = 0$. In particular, the effective viscosity coefficient $\beta_j$ is always *real*, even though the corresponding eigenvalue of $B$ may have imaginary component. This will play a crucial role in the estimates of section 8.

***Remark 2.3.*** Proposition 2.1 holds also in the case that $A$ is nonstrictly hyperbolic, provided that the symmetrizability conditions of Kawashima are satisfied. This follows by an argument similar to that of Lemmas 6.7–6.8 of [LZe]. Some such assumption is needed to ensure analyticity, since the roots $\mu_j$ bifurcating from zero no longer split at first order of the bifurcation (cf. [Kat], pp. 69-70). This is the only place in our analysis where we use the assumption of strict hyperbolicity of $u_\pm$ in (**H2**). Likewise, assumption (**H4**) may be simplified in the case that Kawashima's symmetrizability conditions hold to the condition that $\beta_j > 0$ in (**K3**), since this implies (**H4**) [Ka,MP].

***Remark 2.4.*** Note that the $n$–$n$ splitting into nonpositive and nonnegative roots $u_j$, together with formulae (2.13)–(2.14), imply that

$$\text{sgn Re } (\gamma_j^\pm) = \text{sgn } (a_j^\pm) \tag{2.17}$$

for all $j$. Thus, $A_\pm$ and $B_\pm^{-1} A_\pm$ have the same (complex) signature if $(A_\pm, B_\pm)$ is a stable pair in the sense that $v_t + A_\pm v_x - B_\pm v_{xx}$ has bounded solution operator in $L^2$, in particular if they are mutually symmetrizable and satisfy the

Kawashima stability condition (**K3**). This gives an alternative proof of Lemma 1.1.

**3. Asymptotic Behavior of ODE.** In what follows, we shall have to relate the behavior near $x = \pm\infty$ of solutions of (2.1) to that of solutions of the asymptotic systems (2.4), in a manner that is *analytic in* $\lambda$. Consider a general equation

$$W' = \mathbb{A}(\lambda, x)W. \tag{3.1}$$

It is well known (see [Co], Thm. 4, p. 94) that, provided that

$$\int_0^{\pm\infty} |\mathbb{A} - \mathbb{A}_\pm|\, dx < +\infty, \tag{3.2}$$

there is a one-to-one correspondence between the normal modes $V_j^\pm e^{\mu_j^\pm x}$ of the asymptotic systems

$$W' = \mathbb{A}_\pm(\lambda)W, \tag{3.3}$$

where $V_j^\pm$, $\mu_j^\pm$ are eigenvector and eigenvalue of $\mathbb{A}_-$ (alternatively, $V_j^\pm x^\ell e^{\mu_j^\pm x}$, if $V_j^\pm$ is a generalized eigenvector of order $\ell$) and certain solutions $W_j^\pm$ of (3.1) having the same asymptotic behavior, i.e.

$$W_j^\pm(\lambda, x) = V_j^\pm e^{\mu_j^\pm x}(1 + o(1)) \quad \text{as } x \to \pm\infty \tag{3.4}$$

(alternatively, $W_j^\pm(\lambda, x) = V_j^\pm x^\ell e^{\mu_j^\pm x}(1 + o(1))$). That is, the flows near $\pm\infty$ of (3.1) and (3.3) are homeomorphic.

Such a correspondence is of course highly nonunique, since (3.4) determines $W_j^\pm$ only up to faster decaying modes. However, provided that $\mathrm{Re}(\mu_j^\pm)$ is *strictly separated* from all other $\mathrm{Re}(\mu_k^\pm)$, i.e. that there is a *spectral gap*, the choice defined in [Co], Theorem 4 by fixed point iteration is in fact analytic in $\lambda$, as the uniform limit of an analytic sequence of iterates. The argument breaks down at points $\lambda_0$ where $\mathrm{Re}\ (\mu_j) = \mathrm{Re}\ (\mu_k)$ for some $k \neq j$, since in this case $(\mathrm{Re}\ (\mu_j) - \mathrm{Re}\ (\mu_k))$ does not have a definite sign, and the definition of the fixed point iteration is determined by the signs of all $(\mathrm{Re}\ (\mu_j) - \mathrm{Re}\ (\mu_k))$.

The purpose of the present section is to point out that analyticity can be recovered in the absence of a spectral gap, by virtually the same argument as in [Co] if we substitute for (3.2) the stronger hypothesis:

$$|\mathbb{A} - \mathbb{A}_\pm| = \mathbf{O}(e^{-\alpha|x|}) \quad \text{as } x \to \pm\infty. \tag{3.5}$$

This observation is a special case of the "Gap Lemma" of [GZ], also proved independently in [KS]. The original version was phrased in terms of the projectivized flow associated with (3.1). Here, we give an alternative statement and derivation directly in terms of (3.1), a form more convenient for our needs. At the same time, we determine optimal regularity hypotheses, which will later translate into sharp information on the regularity of the parabolic Green's function $G(x,t;y)$.

***Proposition 3.1.*** *In* (3.1), *let* $\mathbb{A}$ *be* $C^{0+\tilde{\alpha}}$ *in* $x$ *and analytic in* $\lambda$, *with* $|\mathbb{A} - \mathbb{A}_-(\lambda)| = \mathbf{O}(e^{-\alpha|x|})$ *as* $x \to -\infty$ *for* $\alpha > 0$, *and* $\bar{\alpha} < \alpha$. *If* $V^-(\lambda)$ *is an eigenvector of* $\mathbb{A}_-$ *with eigenvalue* $\mu(\lambda)$, *both analytic in* $\lambda$, *then there exists a solution* $W(\lambda, x)$ *of* (3.1) *of form*

$$W(\lambda, x) = V(x,\lambda)e^{\mu x},$$

*where* $V$ *(hence* $W$*) is* $C^{1+\tilde{\alpha}}$ *in* $x$ *and locally analytic in* $\lambda$, *and for each* $j = 0, 1, \dots$ *satisfies*

$$\left(\frac{\partial}{\partial \lambda}\right)^j V(x,\lambda) = \left(\frac{\partial}{\partial \lambda}\right)^j V^-(\lambda) + \mathbf{O}\left(e^{-\bar{\alpha}|x|}\left|\left(\frac{\partial}{\partial \lambda}\right)^j V^-(\lambda)\right|\right), \quad x < 0, \tag{3.6}$$

*Moreover, if* Re $\mu(\lambda) >$ Re$\,\tilde{\mu}(\lambda) - \alpha$ *for all eigenvalues* $\tilde{\mu}$ *of* $\mathbb{A}_-$, *then* $W$ *is uniquely determined by* (3.6), *and* (3.6) *holds for* $\bar{\alpha} = \alpha$.

*Proof.* Setting $W(x) = e^{\mu x}V(x)$, we can rewrite $W' = \mathbb{A}W$ as

$$\begin{aligned} V' &= (\mathbb{A}_- - \mu I)V + \theta V, \\ \theta &:= (\mathbb{A} - \mathbb{A}_-) = \mathbf{O}(e^{-\alpha|x|}), \end{aligned} \tag{3.7}$$

and seek a solution $V(x,\lambda) \to V^-(x)$ as $x \to -\infty$.

Set $\bar{\alpha} < \alpha_1 < \alpha_2 < \alpha$. Fixing a base point $\lambda_0$, we can define on some neighborhood of $\lambda_0$ the complementary $\mathbb{A}_-$-invariant projections $P(\lambda)$ and $Q(\lambda)$, where $P$ projects onto the direct sum of all eigenspaces of $\mathbb{A}_-$ with eigenvalues $\tilde{\mu}$ satisfying

$$\text{Re } (\tilde{\mu}) < \text{Re } (\mu) + \alpha_2, \tag{3.8}$$

and $Q$ projects onto the direct sum of the remaining eigenspaces, with eigenvalues satisfying

$$\text{Re } (\tilde{\mu}) \geq \text{Re } (\mu) + \alpha_2 > \text{Re } (\mu) + \alpha_1. \tag{3.9}$$

By basic matrix perturbation theory (eg. [Kat]) it follows that $P$ and $Q$ are analytic in a neighborhood of $\lambda_0$, with

$$\left|e^{(\mathbb{A}_- - \mu I)x} P\right| = \mathbf{O}(e^{\alpha_2 x}), \quad x > 0, \qquad (3.10)$$

$$\left|e^{(\mathbb{A}_- - \mu I)x} Q\right| = \mathbf{O}(e^{\alpha_1 x}), \quad x < 0.$$

For $M > 0$ sufficiently large, therefore, the map $\mathcal{T}$ defined by

$$\begin{aligned} \mathcal{T}V(x) = V^- &+ \int_{-\infty}^{x} e^{(\mathbb{A}_- - \mu I)(x-y)} P\theta(y)V(y)\,dy \\ &- \int_{x}^{-M} e^{(\mathbb{A}_- - \mu I)(x-y)} Q\theta(y)V(y)\,dy, \end{aligned} \qquad (3.11)$$

is a contraction on $L^\infty(-\infty, -M]$, by (3.10). For, defining $x < x_0 < -M$ by the relation $\alpha_1(x - x_0) = \bar{\alpha}(x + M)$, we have

$$\begin{aligned} |\mathcal{T}V_1 - \mathcal{T}V_2|_{(x)} &\le \mathbf{O}(1)|V_1 - V_2|_\infty \Bigg[\int_{-\infty}^{x} e^{\alpha_2(x-y)} e^{\alpha y}\,dy \\ &\quad + \int_{x}^{x_0} e^{\alpha_1(x-y)} e^{\alpha y}\,dy + \int_{x_0}^{-M} e^{\alpha_1(x-y)} e^{\alpha y}\,dy\Bigg] \\ &= \mathbf{O}(1)|V_1 - V_2|_\infty (e^{\alpha x} + e^{\alpha_1 x} + e^{\alpha_1(x - x_0)}) \\ &= \mathbf{O}(1)|V_1 - V_2|_\infty\, e^{\bar{\alpha} x} \\ &= \mathbf{O}(1)|V_1 - V_2|_\infty\, e^{-\bar{\alpha} M} < \frac{1}{2}. \end{aligned} \qquad (3.12)$$

By iteration, we thus obtain a solution $V \in L^\infty(-\infty, -M]$ of $V = \mathcal{T}V$ with $V = \mathbf{O}(|V^-|)$; since $\mathcal{T}$ clearly preserves analyticity, $V(\lambda, x)$ is analytic in $\lambda$ as the uniform limit of analytic iterates (starting with $V_0 = 0$). Differentiation shows that $V$ is a bounded solution of $V = \mathcal{T}V$ iff it is a bounded solution of (3.7). Further, taking $V_1 = V$, $V_2 = 0$ in (3.12), we obtain from the second to last equality that

$$|V - V^-| = |\mathcal{T}(V) - \mathcal{T}(0)| = \mathbf{O}(1)e^{\bar{\alpha} x}|V| = \mathbf{O}(e^{\bar{\alpha} x})|V^-|, \qquad (3.13)$$

giving (3.6) for $j = 0$. Derivative bounds, $j > 0$, follow by standard interior estimates, or, alternatively, by differentiating (3.11) with respect to $\lambda$ and repeating the same argument. Analyticity, and the bounds (3.6), extend to $x < 0$ by standard analytic dependence for the initial value problem at $x = -M$.

Finally, if

$$\mathrm{Re}(\mu(\lambda)) > \mathrm{Re}(\tilde{\mu}(\lambda)) - \frac{\alpha}{2}$$

for all other eigenvalues, then $P = I$, $Q = 0$, and $V = \mathcal{T}V$ must hold for any $V$ satisfying (3.6), by Duhamel's principle. Further, the only term appearing in (3.12) is the first integral, giving bound (3.13) for $\bar{\alpha} = \alpha$. □

Proposition 3.1 extends also to subspaces of solutions. This can be seen most easily by associating to a $k$-plane of solutions, Span $\{W_1(x), \dots, W_k(x)\}$, the corresponding $k$-form $\eta = W_1 \wedge \cdots \wedge W_k$. The equations (3.1) induce a linear flow

$$\eta' = \mathbb{A}^{(k)}(x, \lambda)\eta, \tag{3.14}$$

on the space of $k$-forms via the Leibnitz rule,

$$\mathbb{A}^{(k)}(W_1 \wedge \cdots \wedge W_k) = (\mathbb{A}W_1 \wedge \cdots \wedge W_k) + \cdots + (W_1 \wedge \cdots \wedge \mathbb{A}W_k). \tag{3.15}$$

The evolution of the $k$-plane of solutions of (3.1) is clearly determined by that of $\eta(x, \lambda)$. It is easily seen that the for a given (constant) matrix $\mathbb{A}$, the eigenvectors of $\mathbb{A}^{(k)}$ are of form $V_1 \wedge \cdots \wedge V_k$, where Span $\{V_1, \cdots, V_k\}$ is an invariant subspace of $\mathbb{A}$, and that the corresponding eigenvalue is the trace of $\mathbb{A}$ on that subspace.

***Definition 3.2.*** *Let $C =$ Span $\{V_{k+1}, \dots, V_N\}$ and $E =$ Span $\{V_1, \dots, V_k\}$ be complementary $\mathbb{A}$-invariant subspaces. We define their spectral gap to be the difference $\beta$ between the real part of the eigenvalue of minimal real part of $\mathbb{A}$ restricted to $C$ and the real part of the eigenvalue of maximal real part of $\mathbb{A}$ restricted to $E$.*

If $\eta$ is a $k$-form associated with an $A$-invariant subspace $E$ as in the definition above, then the spectral gap $\beta$ is the minimum difference between the real part of the eigenvalue $\mu$ of $\mathbb{A}^{(k)}$ associated with $\eta$ and the real part of the eigenvalue associated with any other eigenvector of $\mathbb{A}^{(k)}$. Combining these observations with the result of Proposition 3.1, we obtain a complete version of the Gap Lemma of [GZ]:

***Corollary 3.3 .*** *Let $\mathbb{A}(x, \lambda)$ be $C^{0+\tilde{\alpha}}$ in $x$, analytic in $\lambda$, with $\mathbb{A}(x, \lambda) \to \mathbb{A}_\pm(\lambda)$ as $x \to \pm\infty$ at exponential rate $e^{-\alpha|x|}$, $\alpha > 0$, and let $\eta^-(\lambda)$ and $\zeta^-$ be analytic $k$ and $n-k$-forms associated to complementary $\mathbb{A}_-(\lambda)$-invariant subspaces $C^-$ and $E^-$ as in definition 3.2, with arbitrary spectral gap $\beta$, and let*

$\tau_{C^-}$ *be the trace of* $\mathbb{A}^{(k)}$ *restricted to* $C^-$. *Then, there exists a solution* $\mathcal{W}(\lambda, x)$ *of* (3.14) *of form*

$$\mathcal{W}(\lambda, x) = \eta(\lambda, x)(\lambda, x)e^{\tau_{C^-} x}$$

*where* $\eta$ *(hence* $\mathcal{W}$*) is* $C^{1+\tilde{\alpha}}$ *in* $x$ *and locally analytic in* $\lambda$, *and for each* $j = 0, 1, \dots$ *satisfies*

$$(3.16)\quad \left(\frac{\partial}{\partial\lambda}\right)^j \eta(x, \lambda) = \left(\frac{\partial}{\partial\lambda}\right)^j \eta^-(\lambda) + \mathbf{O}\left(e^{-\bar{\alpha}|x|}\left|\left(\frac{\partial}{\partial\lambda}\right)^j \eta^-(\lambda)\right|\right), \quad x < 0,$$

*for all* $\bar{\alpha} < \alpha$. *Moreover, if* $\beta > -\alpha$, *then* $\eta$ *is uniquely determined by* (3.16) *and* (3.16) *holds for* $\bar{\alpha} = \alpha$.

***Divergence-form operators.*** In the case that $A, B \in C^{1+\tilde{\alpha}}$, the eigenvalue equation (2.1) can be expressed as a first order system (2.2) of form (3.1), with $C^{0+\tilde{\alpha}}(x)$ coefficient matrix (2.3). Hence, from the theory developed above, we immediately obtain solutions $W(\lambda, x) \in C^{2+\tilde{\alpha}}(x)$. Likewise, we can immediately treat the non–divergence-form case $L := Bw'' - Aw' - \lambda w$ with the minimal regularity $A, B \in C^{0+\alpha}$, and reach the same conclusions.

To obtain the same minimal regularity hypotheses also in the divergence- and mixed-form cases, we observe that these cases also can be reduced to the standard form (3.1) by appropriate choice of coordinates. For example, (2.1) can be written as

$$(3.17)\qquad \begin{pmatrix} w \\ Bw' - Aw \end{pmatrix}' = \begin{pmatrix} B^{-1}A & B^{-1} \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} w \\ Bw' - Aw \end{pmatrix},$$

for $A$, $B$ only $C^{0+\tilde{\alpha}}(x)$, while $(Bw)'' - (Aw)' - \lambda w = 0$ can be written as

$$\begin{pmatrix} Bw \\ (Bw)' - Aw \end{pmatrix}' = \begin{pmatrix} AB^{-1} & I \\ \lambda B^{-1} & 0 \end{pmatrix} \begin{pmatrix} Bw \\ (Bw)' - Aw \end{pmatrix}.$$

We obtain in this way $C^{1+\tilde{\alpha}}(x)$ solutions in the new variables, asymptotic to appropriate limiting solutions. Undoing the $C^{0+\tilde{\alpha}}$ coordinate change, we obtain a $C^{1+\tilde{\alpha}}$ solution in the first case, but only a $C^{0+\tilde{\alpha}}$ solution in the second (however, note that both solutions make classical sense). Alternatively, one could carry out directly a divergence-form theory analogous to that of (3.1) for equations $W' = \mathbb{A}_- + (\Theta W)'$ such that $A, \Theta \in C^{0+\tilde{\alpha}}(x)$ and $|\Theta| = \mathbf{O}(e^{-\alpha|x|})$, using an integration by parts to shift the derivative from $(\Theta W)'$ to the parametrix $e^{(\mathbb{A}_- - \mu I)(x-y)}$ in the contraction mapping $\mathcal{T}$ of (3.11). We record these observations as follows.

***Proposition 3.4.*** *For a second order operator $L$ with principal part $(Bw')'$ satisfying* (**C0**)–(**C1**)*, in particular for $L$ as in* (1.6)*, the results of Proposition 3.1 and Corollary 3.3 hold for $W_j^\pm := (w_j^\pm, w_j^{\pm\prime})^t$, where $w_j^\pm$ are solutions of the eigenvalue equation $Lw = \lambda w$, $C^{1+\tilde\alpha}$ in $x$ and locally analytic in $\lambda$.*

***Note.*** In the rest of the paper, we will for convenience work with equations (2.2), even though $A$ and $B$ are only $C^{0+\tilde\alpha}$. This can be interpreted rigorously in the distributional sense (or, for simplicity, one can assume $f \in C^2$, so that $A$, $B \in C^1$). Likewise, the calculation of Proposition 7.3 will for clarity be carried out in the framework of (2.2), with the understanding that (3.17) can be substituted in case $A, B \notin C^{1+\tilde\alpha}$ without affecting the computation.

***Remark 3.5.*** The approach described above clearly extends to nondegenerate ODE of any order $s$ with first term of form $D^{s-q}(A^s D^q w)$ and remaining terms $D^r(A^r D^p w)$ with $r \le s-q$, $p \le q$, giving regularity $w \in C^{s-q+\tilde\alpha}$. This result can be obtained in the non–divergence-form case $q = 0$ by the more straightforward analysis of (3.1) and (3.3).

**4. Construction/Extension of the resolvent kernel.** We now construct an explicit representation for the Resolvent kernel, that is, the Green's function $G_\lambda(x,y)$ associated with the elliptic operator $(L - \lambda I)$, defined by

$$(L - \lambda I)G_\lambda(\cdot, y) = \delta_y I, \tag{4.1}$$

where $\delta_y$ denotes the Dirac delta distribution centered at $y$. A subtle, but crucial issue is the *domain of analyticity* of $G_\lambda$ as a function of $\lambda$, since, according to the usual duality, it is smoothness in the frequency variable $\lambda$ that will later translate to decay in the temporal variable $t$. Let $\Lambda$ be as defined in (2.9). It is a standard fact, see e.g. [He], that both the resolvent $(L - \lambda I)^{-1}$ and the Green's function $G_\lambda(x,y)$ are meromorphic in $\lambda$ on $\Lambda$, with isolated poles of finite order. Using our explicit representation, we will show more, that $G_\lambda$ in fact admits a *meromorphic extension* to a sector

$$\Omega_\theta = \{\lambda : \mathrm{Re}(\lambda) \ge -\theta_1 - \theta_2|\mathrm{Im}\,(\lambda)|\}; \quad \theta_1,\ \theta_2 > 0, \tag{4.2}$$

containing a neighborhood of the origin, and having only isolated poles of finite order.

To avoid repetition, we assume throughout this section that (**C0**)–(**C3**) hold unless otherwise stated. Combining Proposition 2.1, Corollary 3.3, and Proposition 3.4, we have the following result:

***Lemma 4.1.*** *For $\lambda \in B(0,r)$, $r$ sufficiently small, and locally to all except for countably many isolated points $\{\lambda_j\}$ of $\Lambda$, there exist solutions $W_j^\pm(x,\lambda)$ of*

*(2.1), $C^{1+\tilde\alpha}$ in $x$ and analytic in $\lambda$, satisfying*

$$(4.3)\qquad W_j^{\pm}(x,\lambda) = V_j^{\pm}(x,\lambda)e^{\mu_j^{\pm}x}$$

$$\left(\frac{\partial}{\partial\lambda}\right)^k V_j^{\pm}(x,\lambda) = \left(\frac{\partial}{\partial\lambda}\right)^k V_j^{\pm}(\lambda) + \mathbf{O}\left(e^{-\alpha|x|/2}\left|\left(\frac{\partial}{\partial\lambda}\right)^k V_j^{\pm}(\lambda)\right|\right), \quad x \gtrless 0,$$

*for all $k \geq 0$, where $\alpha$ is the rate of decay given in* (**C0**) *and $\mu_j^{\pm}$, $V_j^{\pm}$ are as in Proposition 2.1. Moreover, $W_1^{\pm}\wedge\cdots\wedge W_n^{\pm}$ and $W_{n+1}^{\pm}\wedge\cdots\wedge W_{2n}^{\pm}$ can be chosen to be locally analytic at any point of $\Lambda\cup B(0,r)$.*

On $\Lambda$, the subspaces spanned by

$$(4.4)\qquad \Phi^{+} = (\phi_1^{+},\cdots,\phi_n^{+}) = (W_1^{+},\cdots,W_n^{+}), \quad \text{and}$$

$$(4.5)\qquad \Phi^{-} = (\phi_1^{-},\cdots,\phi_n^{-}) = (W_{n+1}^{-},\cdots,W_{2n}^{-})$$

contain all solutions of (2.2) *decaying* at $x=\pm\infty$, respectively, by Proposition 2.1 and Lemma 4.1. Loosely following the notation of [Sat], we denote the complementary subspaces of *growing modes* by

$$(4.6)\qquad \Psi^{+} = (\psi_1^{+},\cdots,\psi_n^{+}) = (W_{n+1}^{+},\cdots,W_{2n}^{+})$$

and

$$(4.7)\qquad \Psi^{-} = (\psi_1^{-},\cdots,\psi_n^{-}) = (W_1^{-},\cdots,W_n^{-}).$$

Defining $\rho_j^{+} = \mu_j^{+}$, $\rho_j^{-} = \mu_{n+j}^{-}$, $\nu_j^{+} = \mu_{n+j}^{+}$, and $\nu_j^{-} = \mu_j^{-}$, we have

$$(4.8)\qquad \phi_j^{\pm} \sim e^{\rho_j^{\pm}x}; \quad \psi_j^{\pm} \sim e^{\nu_j^{\pm}x}.$$

Eigenfunctions, decaying at both $\pm\infty$, occur precisely when the subspaces $\Phi^{+}$ and $\Phi^{-}$ intersect. This intersection can be detected by the vanishing of their mutual determinant, or equivalently of the *Evans function*,

$$(4.9)\qquad \begin{aligned} D_L(\lambda) &:= \det(\Phi^{+},\Phi^{-})\Big|_{x=0} \\ &= (\phi_1^{+}\wedge\cdots\wedge\phi_n^{+}\wedge\phi_1^{-} no \wedge\cdots\wedge\phi_n^{-})\Big|_{x=0}. \end{aligned}$$

By Lemma 4.1 and (**C3**), we immediately have the following result:

**Lemma 4.2.** *For $\theta_1$, $\theta_2 > 0$ sufficiently small, $D_L$ is locally analytic on the sector $\Omega_\theta$ defined in* (4.2).

(In fact, $D_L$ can be defined in a globally analytic way, with further care [GZ]. However, we need not do so here.)

We now turn to the representation of the Green's function $G_\lambda(x,y)$. We first recall the classical symmetry principle as it applies to non–self-adjoint operators.

**Lemma 4.3.** *Let $H_\lambda(x,y)$ denote the Green's function for the adjoint operator $(L-\lambda I)^*$. Then, $G_\lambda(y,x) = H_\lambda(x,y)^*$. In particular, for $x \neq y$, the matrix $z = G_\lambda(x,\cdot)$ satisfies*

$$(z'B)' = -z'A + z\lambda. \tag{4.10}$$

*Proof.* Letting $\langle\cdot,\cdot\rangle$ denote complex inner product, we have

$$\begin{aligned}
G_\lambda(x_0,y_0) &= \int_{-\infty}^{\infty} \langle \delta_{x_0}, G_\lambda(x,y_0)\rangle\, dx \\
&= \int_{-\infty}^{\infty} \langle (L-\lambda I)^* H_\lambda(x,x_0), G_\lambda(x,y_0)\rangle\, dx \\
&= \int_{-\infty}^{\infty} \langle H_\lambda(x,x_0), (L-\lambda I) G_\lambda(x,y_0)\rangle\, dx \\
&= \int_{-\infty}^{\infty} \langle H_\lambda(x,x_0), \delta_{y_0}\rangle\, dx \\
&= H_\lambda(y_0,x_0)^*.
\end{aligned}$$

□

Consider (4.10) as an ODE for a general row vector $z$. Written as a first order system, it becomes

$$Z' = Z\tilde{\mathbb{A}}(\lambda,x), \tag{4.11}$$

where $Z = (z,z')$ and

$$\tilde{\mathbb{A}} = \begin{pmatrix} 0 & \lambda B^{-1} - A'B^{-1} \\ I & -AB^{-1} - B'B^{-1} \end{pmatrix}. \tag{4.12}$$

The following duality relation, generalizing Lemma 2.1.2 of [LZ.2], will be used repeatedly in our analysis.

**Lemma 4.4.** *$Z$ is a solution of (4.11) iff $Z\mathcal{S}W \equiv$ constant for any solution $W$ of (2.2), where $\mathcal{S} = \begin{pmatrix} -A & B \\ -B & 0 \end{pmatrix}$.*

*Proof.* By direct calculation,

$$\begin{aligned}(Z\mathcal{S}W)' &= (-zAw + zBw' - z'Bw)' \\ &= -z'Aw - z(Aw)' - z''Bw - z'B'w + zBw'' + zB'w' \\ &= z[(Bw')' - (Aw)' - \lambda w] - [(z'B)' + z'A - \lambda z]w \\ &= -[(z'B)' + z'A - \lambda z]w,\end{aligned}$$

where the final equality follows from (2.1). Comparing to (4.10), we are done. ☐

**Remark 4.5.** Note that (4.10) is not the adjoint, but the "transpose" equation of (4.1). Likewise, $Z\mathcal{S}W$ above denotes matrix multiplication, not complex inner product.

Using Lemma 4.4, we can immediately define dual bases $\tilde{W}_j^{\pm}$ of solutions to (4.12) by the relation

$$\tilde{W}_j^{\pm}\mathcal{S}W_k^{\pm} = \delta_k^j, \tag{4.13}$$

where

$$\delta_k^j = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$$

denotes the Kronecker delta function. Note that we have defined the $\tilde{W}_j^{\pm}$ as *row vectors*. By (4.13) combined with (2.13), we find that

$$\tilde{W}_j^{\pm} = \tilde{V}_j^{\pm} e^{\tilde{\mu}_j^{\pm} x}(1 + \mathbf{O}(e^{-(\alpha/2)|x|})), \tag{4.14}$$

where $\tilde{\mu}_j^{\pm}$, $\tilde{V}_j^{\pm}$ are the left eigenvalues and eigenvectors of the asymptotic matrices

$$\tilde{\mathbb{A}}_{\pm} = \begin{pmatrix} 0 & \lambda B_{\pm}^{-1} \\ I & -A_{\pm}B_{\pm}^{-1} \end{pmatrix} \tag{4.15}$$

associated with the adjoint eigenvalue equation (4.11). Note that, by (4.13), we have

$$\begin{cases} \tilde{\mu}_j^{\pm} = -\mu_j^{\pm} \\ \tilde{V}_j^{\pm} V_k^{\pm} = \delta_k^j \end{cases}$$

for all $\lambda$; in particular, by (2.13), we obtain the expansions

$$\tilde{V}_j^{\pm} = \begin{cases} (r_j, 0) + \mathbf{O}(\lambda) & \text{if } a_j^{\pm} > 0, \\ (\tilde{s}_j^{\pm}, -\gamma_j \tilde{s}_j^{\pm}) + \mathbf{O}(\lambda) & \text{if } a_j^{\pm} < 0, \end{cases}$$

$$\tilde{V}_{n+j} = \begin{cases} (\tilde{r}_j^{\pm}, 0) + \mathbf{O}(\lambda) & \text{if } a_j^{\pm} < 0, \\ (\tilde{s}_j^{\pm}, -\gamma_j \tilde{s}_j^{\pm}) + \mathbf{O}(\lambda) & \text{if } a_j^{\pm} > 0, \end{cases}$$

analogous to (2.13)–(2.14), for $j = 1, \cdots, n$ around $\lambda = 0$, where $\tilde{r}_j^{\pm}$, $\tilde{s}_j^{\pm}$ are the left eigenvectors of $A_{\pm}$, $B_{\pm}^{-1} A_{\pm}$, respectively.

In accordance with (4.4)–(4.7), we define the dual subspaces

$$\begin{aligned} \tilde{\Phi}^+ &= (\tilde{\phi}_1^+, \cdots, \tilde{\phi}_n^+) = (\tilde{W}_1^+, \cdots, \tilde{W}_n^+), \\ \tilde{\Phi}^- &= (\tilde{\phi}_1^-, \cdots, \tilde{\phi}_n^-) = (\tilde{W}_{n+1}^-, \cdots, \tilde{W}_{2n}^-), \\ \tilde{\Psi}^+ &= (\tilde{\psi}_1^+, \cdots, \tilde{\psi}_n^+) = (\tilde{W}_{n+1}^+, \cdots, \tilde{W}_{2n}^+), \\ \tilde{\Psi}^- &= (\tilde{\psi}_1^-, \cdots \tilde{\psi}_n^-) = (\tilde{W}_1^-, \cdots, \tilde{W}_n^-), \end{aligned} \tag{4.16}$$

so that, analogously to (4.8),

$$\tilde{\phi}_j^{\pm} \sim e^{-\rho_j^{\pm} x}; \quad \tilde{\psi}_j^{\pm} \sim e^{-\nu_j^{\pm} x}. \tag{4.17}$$

Note that, contrary to the case of (4.4)–(4.7), $\tilde{\Psi}^{\pm}$ denote the *decaying* modes of (4.11), $\tilde{\Phi}^{\pm}$ the *growing* modes.

By (4.13) and (4.16), we have

$$\begin{aligned} \tilde{\phi}_j^{\pm} \mathcal{S} \psi_k^{\pm} &= 0; \quad \tilde{\phi}_j^{\pm} \mathcal{S} \phi_k^{\pm} = \delta_k^j, \\ \tilde{\psi}_j^{\pm} \mathcal{S} \psi_k^{\pm} &= \delta_k^j; \quad \tilde{\psi}_j^{\pm} \mathcal{S} \phi_k^{\pm} = 0. \end{aligned} \tag{4.18}$$

From (4.1) and (4.10), we have that $\begin{pmatrix} G_\lambda(\cdot, y) \\ G_{\lambda_x}(\cdot, y) \end{pmatrix}$ satisfies (2.2) for $x \neq y$, while $(G_\lambda(x, \cdot), G_{\lambda_y}(x, \cdot))$ satisfies (4.11). Further, note that both $G_\lambda(x, \cdot)$ and

$G_\lambda(\cdot, y)$ decay at $\pm\infty$ for $\lambda$ on the resolvent set, since $|(L-\lambda I)^{-1}| < \infty \iff |(L-\lambda I)^*| < \infty$ imply $\|G_\lambda(\cdot, y)\|_{L^1} < \infty$ and $\|G_\lambda(x, \cdot)\|_{L^1} < \infty$ respectively. Combining, we have the representation

$$\begin{pmatrix} G_\lambda & G_{\lambda_y} \\ G_{\lambda_x} & G_{\lambda_{xy}} \end{pmatrix} = \begin{cases} \Phi^+(\lambda, x) M^+(\lambda) \tilde{\Psi}^-(\lambda, y) & \text{for } x > y \\ \Phi^-(\lambda, x) M^-(\lambda) \tilde{\Psi}^+(\lambda, y) & \text{for } x < y, \end{cases} \tag{4.19}$$

where $M^\pm(\lambda)$ are to be determined.

**Lemma 4.6.**

$$\begin{bmatrix} G_\lambda & G_{\lambda_y} \\ G_{\lambda_x} & G_{\lambda_{xy}} \end{bmatrix}_{(y)} = \begin{pmatrix} 0 & -B^{-1} \\ B^{-1} & -B^{-1}AB^{-1} \end{pmatrix} = \mathcal{S}^{-1},$$

*where $[f]_{(y)}$ denotes the jump in $f(x)$ at $x = y$, and $\mathcal{S}$ is as in Lemma 4.4.*

*Proof.* Expanding $\delta_y = (L-\lambda)G_\lambda = (BG_{\lambda_x})_x - (AG_\lambda)_x - \lambda G_\lambda$, and comparing orders of singularity, we find that

$$(AG_\lambda)_x + \lambda G_\lambda = 0 \text{ and } (BG_{\lambda_x})_x = \delta_y,$$

giving, respectively,

$$[G_\lambda]_{(y)} = 0 \text{ and } [G_{\lambda_x}]_{(y)} = B^{-1},$$

Note further that we can expand $[G_\lambda]_{(y)}$ as

$$[G_\lambda]_{(y)} = G_\lambda^{x>y}(y, y) - G_\lambda^{x<y}(y, y), \tag{4.20}$$

where $G_\lambda^{x>y}$ and $G_\lambda^{x<y}$ are the smooth functions denoting the value of $G_\lambda$ on the regions $x > y$ and $x < y$, respectively. Differentiating (4.20) in $y$, we obtain

$$0 = \frac{d}{dy}[G_\lambda]_{(y)} = [G_{\lambda_x}]_{(y)} + [G_{\lambda_y}]_{(y)},$$

hence

$$[G_{\lambda_y}]_{(y)} = -B^{-1}.$$

Differentiating a second time, we find that

$$[G_{\lambda_{xy}}]_{(y)} = -\frac{1}{2}([G_{\lambda_{xx}}]_{(y)} + [G_{\lambda_{yy}}]_{(y)}). \tag{4.21}$$

Finally, we can determine $[G_{\lambda_{xx}}]_{(y)}$ and $[G_{\lambda_{yy}}]_{(y)}$ by solving the ODE (2.1) and (4.10) to express

$$\begin{aligned} G_{\lambda_{xx}} &= B^{-1}((A-B')G_{\lambda_x} + (A'+\lambda)W), \quad \text{and} \\ G_{\lambda_{yy}} &= (G_{\lambda_y}(-A-B') + G_\lambda(-A'+\lambda)B^{-1}). \end{aligned}$$

With (4.21), this gives

$$\left[G_{\lambda_{xy}}\right]_{(y)} = -\frac{1}{2}B^{-1}(A-B')[G_{\lambda_x}]_{(y)} - \frac{1}{2}[G_{\lambda_y}]_{(y)}(-A-B')B^{-1} = -B^{-1}AB^{-1}$$

as claimed. This verifies the first equality asserted; the second follows by direct computation. □

Combining Lemma 4.6 with (4.19), we have

$$(\Phi^+(y), \Phi^-(y)) \begin{pmatrix} M^+(\lambda) & 0 \\ 0 & -M^-(\lambda) \end{pmatrix} \begin{pmatrix} \tilde\Psi^-(y) \\ \tilde\Psi^+(y) \end{pmatrix} = \mathcal{S}^{-1}, \quad \text{or}$$

$$\begin{aligned} \begin{pmatrix} M^+(\lambda) & 0 \\ 0 & -M^-(\lambda) \end{pmatrix} &= (\Phi^+, \Phi^-)^{-1}\mathcal{S}^{-1} \begin{pmatrix} \tilde\Psi^- \\ \tilde\Psi^+ \end{pmatrix}^{-1}(y) \\ &= \left( \begin{pmatrix} \tilde\Psi^- \\ \tilde\Psi^+ \end{pmatrix} \mathcal{S}(\Phi^+, \Phi^-)\right)^{-1}(y) \\ &= \begin{pmatrix} \tilde\Psi^-\mathcal{S}\Phi^+ & 0 \\ 0 & \tilde\Psi^+\mathcal{S}\Phi^- \end{pmatrix}^{-1}(y) \end{aligned} \tag{4.22}$$

Note that the right hand side of (4.22) is indeed independent of $y$, by Lemma 4.4.

From (4.22) together with (4.19), we have

$$\begin{aligned} &\begin{pmatrix} G_\lambda & G_{\lambda_y} \\ G_{\lambda_x} & G_{\lambda_{xy}} \end{pmatrix} \\ &= (\Phi^+(x), 0)(\Phi^+(z), \Phi^-(z))^{-1}\mathcal{S}(z)^{-1} \begin{pmatrix} \tilde\Psi^-(z) \\ \tilde\Psi^+(z) \end{pmatrix}^{-1} \begin{pmatrix} \tilde\Psi^-(y) \\ 0 \end{pmatrix} \end{aligned} \tag{4.23}$$

for $x > y$, any $z$, and similarly for $x < y$. This gives the useful coordinate-free representation of

$$\begin{pmatrix} G_\lambda & G_{\lambda_y} \\ G_{\lambda_x} & G_{\lambda_{xy}} \end{pmatrix} = \mathcal{F}^{z\to x}\,\Pi_+(z)\mathcal{S}^{-1}(z)\,\tilde{\Pi}_-(z)\tilde{\mathcal{F}}^{z\to y}, \tag{4.24}$$

where

$$\Pi_+(y) = (\Phi^+(y), 0)(\Phi^+(y), \Phi^-(y))^{-1}, \quad \text{and} \tag{4.25}$$

$$\tilde{\Pi}_-(y) = \begin{pmatrix} \tilde{\Psi}^-(y) \\ \tilde{\Psi}^+(y) \end{pmatrix}^{-1} \begin{pmatrix} \tilde{\Psi}^-(y) \\ 0 \end{pmatrix} \tag{4.26}$$

denote, respectively, the projection along $\Phi^-(y)$ onto the stable manifold $\Phi^+(y)$ and the dual projection along $\tilde{\Psi}^+(y)$ onto the dual stable manifold $\tilde{\Psi}^-(y)$, and

$$\mathcal{F}^{z\to x} = (\Phi^+(x), \Phi^-(x))(\Phi^+(z), \Phi^-(z))^{-1}, \quad \text{and}$$

$$\tilde{\mathcal{F}}^{z\to y} = \begin{pmatrix} \tilde{\Psi}^-(z) \\ \Phi^+(z) \end{pmatrix}^{-1} \begin{pmatrix} \Psi^-(y) \\ \Psi^+(y) \end{pmatrix}$$

denote the solution operators of (2.1) and (4.11).

Using Lemma 4.4, it can be shown that

$$\Pi_+(z)\mathcal{S}^{-1}(z) = \Pi_+(z)\mathcal{S}^{-1}(z)\,\tilde{\Pi}_-(z) = \mathcal{S}^{-1}(z)\,\tilde{\Pi}_-(z).$$

Thus, taking $z = y$ in (4.24), we obtain a formulation entirely in terms of (2.1),

$$\begin{aligned} \begin{pmatrix} G_\lambda & G_{\lambda_y} \\ G_{\lambda_x} & G_{\lambda_{xy}} \end{pmatrix} &= \mathcal{F}^{y\to x}\,\Pi_+(y)\mathcal{S}^{-1}(y) \\ &= (\Phi^+(x), 0)(\Phi^+(y), \Phi^-(y))^{-1}\mathcal{S}(y)^{-1}, \end{aligned} \tag{4.27}$$

for $x > y$, and similarly for $x < y$. Equation (4.27) (and indeed (4.24) as well) is more easily verified by setting

$$\begin{pmatrix} G_\lambda & G_{\lambda_y} \\ G_{\lambda_x} & G_{\lambda_{xy}} \end{pmatrix} = \Phi^\pm(x)N^\pm(y)$$

and solving directly for $N^\pm$ using the jump conditions. There is of course a symmetric representation,

$$\begin{pmatrix} G_\lambda & G_{\lambda_y} \\ G_{\lambda_x} & G_{\lambda_{xy}} \end{pmatrix} = \mathcal{S}^{-1}(x)\,\tilde{\Pi}_-(x)\,\tilde{\mathcal{F}}^{x\to y}, \tag{4.28}$$

entirely in terms of (4.11). The representations (4.23)–(4.24) and (4.27) are useful in different circumstances.

**Proposition 4.7.** *The function $G_\lambda(x,y)$ defined by* (4.23)–(4.24) (*equivalently, by* (4.27)) *agrees with the Green's function on $\Lambda \setminus \sigma(L)$ and, for $\theta_1$, $\theta_2$ sufficiently small, is meromorphic on the sector $\Omega_\theta$ defined in* (4.2), *with only isolated poles of finite order, each corresponding to zeroes of the Evans function.*

*Proof.* The first claim follows from our derivation, for any local representatives $W_j^\pm$. On any neighborhood where $\Phi^\pm$ are analytic matrices, i.e. at all except the countably many isolated points of Lemma 4.1, it is clear from the representation (4.23) that $G_\lambda$ as a rational matrix function is locally meromorphic, with isolated poles of finite order. Moreover, these poles clearly occur where $(\Phi^+, \Phi^-)$ fails to be invertible, i.e. where $D_L(\lambda) := \det(\Phi^+, \Phi^-) = 0$.

At exceptional points, it is still the case that the *subspaces* Span $\Phi^\pm$ can be chosen analytically, by identification with the analytic $n$-forms $\phi_1^\pm \wedge \cdots \wedge \phi_n^\pm$ guaranteed by Lemma 4.1. Choosing any analytic bases for these spaces, we again obtain local meromorphicity through representation (4.23).

Alternatively, this follows directly from the coordinate free representation (4.24). We omit the details, since these exceptional points only occur on $\Lambda$, where $G_\lambda$ is known by standard theory to be meromorphic.

Finally, global meromorphicity on $\Omega_\theta \subset \Lambda \cup B(0,r)$ follows from meromorphicity on $B(0,r)$, local meromorphicity on $\Lambda$, and uniqueness of the Green's function on $\Lambda$. □

## Part II. Spectral Theory.

**5. The effective spectrum.** In this section, we extend the spectral expansion theory of [Kat, Y] to eigenvalues such as $\lambda = 0$ for the operator $L$ above that lie within or in the closure of the essential spectrum of a differential operator: more generally, to "resonant poles" of the Resolvent kernel $G_\lambda$. For the "effective eigenspace" induced by the residue of $G_\lambda$, we show that the usual Fredholm Theory carries over, modulo certain obviously necessary modifications.

In this and the following section, we drop the usual assumptions (**C0**)–(**C3**). In anticipation that these results will be of general use, we carry out the analysis with minimal assumptions on the operator $L$. Let $C^\infty_{\exp}$ denote the space of $C^\infty$ functions decaying exponentially in all derivatives at some sufficiently high rate.

**Definition 5.1.** *Let $L$ be a linear ordinary differential operator with bounded, $C^\infty$ coefficients* (*so that $L : C^\infty_{\exp} \to C^\infty_{\exp}$* ), *and let $G_\lambda$ denote the Green's function of $L - \lambda I$. Further, let $\Omega$ be an open, simply connected domain intersecting the resolvent set of $L$, on which $G_\lambda$ has a* (*necessarily unique*) *meromorphic extension. Then, for $\lambda_0 \in \Omega$, we define the effective eigenprojection*

$\mathcal{P}_{\lambda_0} : C^\infty_{\exp} \to C^\infty$ *by*

$$\mathcal{P}_{\lambda_0} f(x) = \int_{-\infty}^{+\infty} P_{\lambda_0}(x,y) f(y)\, dy,$$

*where*

$$P_{\lambda_0}(x,y) = \mathrm{Res}_{\lambda_0} G_\lambda(x,y) \tag{5.1}$$

*and* $\mathrm{Res}_{\lambda_0}$ *denotes residue at* $\lambda_0$. *We will refer to* $P_{\lambda_0}(x,y)$ *as the projection kernel. Likewise, we define the effective eigenspace* $\Sigma'_{\lambda_0}(L)$ *by*

$$\Sigma'_{\lambda_0}(L) = \mathrm{Range}\ (\mathcal{P}_{\lambda_0}).$$

The definition above is the natural one from the point of view of the spectral resolution of the identity, $I = \int_\Gamma (L - \lambda I)^{-1} d\lambda$, hence also from the point of view of asymptotic behavior of solutions of PDE (see Section 8). Away from the essential spectrum of $L$, it corresponds with the usual definition [Kat,Y].

It is perhaps not immediately obvious from the definition that $\mathcal{P}$ takes $C^\infty_{\exp}$ to $C^\infty$ as claimed, for $\lambda_0$ in the essential spectrum. However, recall from the explicit calculation of the Green's function that $G_\lambda(x,y)$ can be expressed as

$$G^+_\lambda(x,y) h(x-y) + G^-_\lambda(x,y) h(y-x),$$

where each of $G^\pm_\lambda$ are $C^\infty$ in both $x$ and $y$, and $h(\cdot)$ is the standard Heaviside function. Thus, $(\partial/\partial x)^k \int_{-\infty}^{+\infty} G_\lambda(x,y) f(y)\, dy$ may be split into the sum of terms of form

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \left(\frac{\partial}{\partial x}\right)^j G^\pm_\lambda(x,y) \left(\frac{\partial}{\partial x}\right)^{k-j} h(x-y) f(y)\, dy \\
&= (-1)^{k-j} \int_{-\infty}^{+\infty} h(x-y) \left(\frac{\partial}{\partial y}\right)^{k-j} \left[\left(\frac{\partial}{\partial x}\right)^j G^\pm_\lambda(x,y) f(y)\right] dy \\
&= (-1)^{k-j} \int_{x}^{\pm\infty} \left(\frac{\partial}{\partial y}\right)^{k-j} \left[\left(\frac{\partial}{\partial x}\right)^j G^\pm_\lambda(x,y) f(y)\right] dy,
\end{aligned}$$

which are evidently continuous in $x$, uniformly in $\lambda$ on any compact set. (Here, we have assumed sufficiently rapid decay of $C^\infty_{\exp}$ that boundary terms vanish in the integration by parts). Evaluating the Residue by integrating in $\lambda$ on a closed contour about $\lambda_0$, we thus find that $\mathcal{P}_{\lambda_0} f \in C^k$ for all $k$, as claimed.

**Definition 5.2.** *Let $L$, $\Omega$, $\lambda_0$ be as above, and $K$ be the order of the pole of $(L-\lambda I)^{-1}$ at $\lambda_0$. For $\lambda_0 \in \Omega$, and $k$ any integer, we define $\mathcal{Q}_{\lambda_0,k} : C^\infty_{\exp} \to C^\infty$ by*

$$\mathcal{Q}_{\lambda_0,k} f(x) = \int_{-\infty}^{+\infty} Q_{\lambda_0,k}(x,y) f(y)\, dy,$$

*where*

$$Q_{\lambda_0,k}(x,y) = \mathrm{Res}_{\lambda_0} (\lambda - \lambda_0)^k G_\lambda(x,y).$$

*For $0 \le k \le K$, we define the effective eigenspace of ascent $k$ by*

$$\Sigma'_{\lambda_0,k}(L) = \text{Range } (\mathcal{Q}_{\lambda_0,K-k})$$

With the definitions above, we obtain the following, modified Fredholm Theory.

**Proposition 5.3.** *Let $L$, $\lambda_0$, $\Omega$ be as in Definition 5.1, and $K$ be the order of the pole of $G_\lambda$ at $\lambda_0$. Then,*

(i) *The operators $\mathcal{P}_{\lambda_0}$, $\mathcal{Q}_{\lambda_0,k} : C^\infty_{\exp} \to C^\infty$ are $L$-invariant, with*

$$\mathcal{Q}_{\lambda_0,k+1} = (L - \lambda_0 I)\mathcal{Q}_{\lambda_0,k} = \mathcal{Q}_{\lambda_0,k}(L - \lambda_0 I) \tag{5.2}$$

*for all $k \neq -1$, and*

$$\mathcal{Q}_{\lambda_0,k} = (L - \lambda_0 I)^k \mathcal{P}_{\lambda_0} \tag{5.3}$$

*for $k \ge 0$.*

(ii) *The effective eigenspace of ascent $k$ satisfies*

$$\Sigma'_{\lambda_0,k}(L) = (L - \lambda_0 I)\Sigma'_{\lambda_0,k+1}(L). \tag{5.4}$$

*for all $0 \le k \le K$, with*

$$\{0\} = \Sigma'_{\lambda_0,0}(L) \subset \Sigma'_{\lambda_0,1}(L) \subset \cdots \subset \Sigma'_{\lambda_0,K}(L) = \Sigma'_{\lambda_0}(L). \tag{5.5}$$

*Moreover, each containment in (5.5) is strict.*

(iii) *On* $\mathcal{P}_{\lambda_0}^{-1}(C^\infty_{\text{exp}})$, $\mathcal{P}_{\lambda_0}$, $\mathcal{Q}_{\lambda_0,k}$ *all commute* ($k \geq 0$), *and* $\mathcal{P}_{\lambda_0}$ *is a projection. More generally,* $\mathcal{P}_{\lambda_0} f = f$ *for any* $f \in \Sigma_{\lambda_0}(L : C^\infty_{\text{exp}})$, *hence*

$$\Sigma_{\lambda_0,k}(L : C^\infty_{\text{exp}}) \subset \Sigma'_{\lambda_0,k}(L)$$

*for all* $0 \leq k \leq K$.

(iv) *The multiplicity of the eigenvalue* $\lambda_0$, *defined as* $\dim\ \Sigma'_{\lambda_0}(L)$, *is finite and bounded by* $Kn$. *Moreover, for all* $0 \leq k \leq K$,

$$\dim\ \Sigma'_{\lambda_0,k}(L) = \dim\ \Sigma'_{\lambda_0^*,k}(L^*). \tag{5.6}$$

*Further, the projection kernel can be expanded as*

$$P_{\lambda_0} = \sum_j \varphi_j(x)\pi_j(y), \tag{5.7}$$

*where* $\{\varphi_j\}$, $\{\pi_j\}$ *are bases for* $\Sigma'_{\lambda_0}(L)$, $\Sigma'_{\lambda_0^*}(L^*)$, *respectively.*

(v) (*Restricted Fredholm alternative*) *For* $g \in C^\infty_{\text{exp}}$,

$$(L - \lambda_0 I)f = g$$

*is soluble in* $C^\infty$ *if and only if* $Q_{\lambda_0,K-1}\, g = 0$, *or equivalently*

$$g \in \Sigma'_{\lambda_0^*,1}(L^*)^\perp. \tag{5.8}$$

***Remarks.*** For $\lambda_0$ in the resolvent set, $\mathcal{P}_{\lambda_0}$ agrees with the standard definition, hence $\Sigma'_{\lambda_0,k}(L)$ agrees with the usual $L^p$ eigenspace of generalized eigenfunctions of ascent $\leq k$, for all $p < \infty$, since $C^\infty_{\text{exp}}$ is dense in $L^p$, $p > 1$, and $\Sigma'_{\lambda_0,k}(L)$ is closed. In the context of stability of traveling waves (more generally, whenever the coefficients of $L$ exponentially approach constant values at $\pm\infty$), $\Sigma'_{\lambda_0,k}(L)$ lies *between* the $L^p$ subspace $\Sigma_{\lambda_0,k}(L)$ and the corresponding $L^p_{\text{Loc}}$ subspace.

The modification from standard Fredholm Theory is only that here conclusions (iii) and (v) apply on restricted domains. This restriction is clearly necessary, since the various expressions occurring in their statements would otherwise not be defined.

In this regard, note that $\mathcal{P}_{\lambda_0}$ is, strictly speaking, not a projection in the case that $\lambda \in \sigma_{\text{ess}}(L)$, since its domain does not then match its range. However, it maintains the projective structure (5.7), somewhat justifying our abuse of notation.

*Proof.* The proof is similar in spirit to that of [Kat] for the standard case $\lambda_0 \in \rho(L)$; for example, the proofs of (i)-(ii) and (iv) are essentially the same arguments translated into Green's function notation. However, we have the handicap that compositions of the operators $\mathcal{P}_{\lambda_0}$, $\mathcal{Q}_{\lambda_0,k}$ are no longer defined, since the domain does not match the range. For this reason, the arguments and also the statements of (iii) and (v) must be modified.

(i) That $L$ commutes with the operators $\mathcal{P}_{\lambda_0}$, $\mathcal{Q}_{\lambda_0,k}$ on the domain $C^\infty_{\text{exp}}$ follows from

$$(L_x - \lambda_0)G_\lambda(x,y) = (L_y^* - \lambda_0^*)G_\lambda(x,y) = \delta(x-y),$$

a restatement of Lemma 4.3, and the facts that for $f \in C^\infty_{\text{exp}}$ the order of integration in $\lambda$ and $y$ can be exchanged in evaluating $\mathcal{Q}_{\lambda_0,k}f$, and $(L-\lambda_0 I)$ may be moved inside the integral, by Fubini's Theorem and the Lebesgue Dominated Convergence Theorem, respectively. Here, $L_x$ and $L_y$ denote the operation of the differential operator $L$ on $x$ and $y$ variables, respectively. From now on, $L$ will be understood to denote $L_x$.
Rearranging the definition of the Green's function in (4.1), we obtain the key identity,

$$(\lambda - \lambda_0)G_\lambda(x,y) + \delta(x-y) = (L - \lambda_0 I)G_\lambda(x,y), \tag{5.9}$$

an analog of the basic Resolvent identities in [Kat]. This gives

$$\begin{aligned}(L-\lambda_0 I)Q_{\lambda_0,k}(x,y) &= \text{Res}(L-\lambda_0 I)(\lambda-\lambda_0)^k G_\lambda(x,y)\\ &= \text{Res}(\lambda-\lambda_0)^{k+1}G_\lambda(x,y) + \text{Res}(\lambda-\lambda_0)^k\delta(x-y)\\ &= Q_{\lambda_0,k+1}(x,y),\end{aligned}$$

where in the first equality we have used the fact that for $f \in C^\infty_{\text{exp}}$ the operator $(L-\lambda_0 I)$ can be moved inside the $\lambda$ integral defining the Residue, again by Lebesgue Dominated Convergence, and in the final equality that $\text{Res}(\lambda-\lambda_0)^k = 0$ for $k \neq -1$. From this, and the previous observation that $(L-\lambda_0 I)$ can be moved inside the $y$-integral in evaluating $(L-\lambda_0 I)\mathcal{Q}_{\lambda_0,k}f$, we obtain the first equality in (5.2); the second follows by $L$-invariance. Observing that $\mathcal{Q}_{\lambda_0,K} = \mathcal{P}_{\lambda_0}$, we obtain (5.3) by induction.

(ii) From (i) and Definitions 5.1 and 5.2, we immediately obtain (5.4) and (5.5). Strict inequality follows from the observation that

$$(\Sigma'_{\lambda_0,k} \setminus \Sigma'_{\lambda_0,k-1}) = (L-\lambda_0 I)^{K-k}(\Sigma'_{\lambda_0,K} \setminus \Sigma'_{\lambda_0,K-1})$$

together with the fact that

$$(\Sigma'_{\lambda_0,1} \setminus \Sigma'_{\lambda_0,0}) \neq \emptyset$$

or else $\mathrm{Res}_{\lambda_0}(\lambda-\lambda_0)^{K-1}(L-\lambda I)^{-1}$ would be zero, and the order of the pole of $(L-\lambda I)^{-1}$ at $\lambda_0$ would be only $K-1$, a contradiction.

(iii) Rewriting (5.9) as

$$G_\lambda(x,y) = (L-\lambda_0 I)(\lambda-\lambda_0)^{-1}G_\lambda(x,y) - (\lambda-\lambda_0)^{-1}\delta(x-y) \tag{5.10}$$

and iterating, we obtain

$$\begin{aligned} G_\lambda(x,y) = (L-\lambda_0 I)^K(\lambda-\lambda_0)^{-K}G_\lambda(x,y) \\ - (\lambda-\lambda_0)^{-1}\sum_{j=0}^{K-1}(L-\lambda_0 I)^j(\lambda-\lambda_0)^{-j}\delta(x-y). \end{aligned}$$

Taking the residue at $\lambda_0$ then gives the decomposition

$$P_{\lambda_0}(x,y) = \mathrm{Res}_{\lambda_0}(L-\lambda_0 I)^K(\lambda-\lambda_0)^{-K}G_\lambda(x,y) + \delta(x-y),$$

hence

$$\mathcal{P}_{\lambda_0} f = f + (L-\lambda_0 I)^K \mathcal{Q}_{\lambda_0,2K} f = f + \mathcal{Q}_{\lambda_0,2K}(L-\lambda_0 I)^K f.$$

Applied to $f \in \Sigma_{\lambda_0}(L)$, *i.e.* $f \in C^\infty_{\exp}$ such that $(L-\lambda_0 I)^K f = 0$, this gives the result.

(iv) All $f \in \Sigma'_{\lambda_0}(L)$ are $C^\infty$ solutions of $(L-\lambda_0 I)^K f = 0$, by (ii), of which there are at most dimension $Kn$. The relations (5.6)-(5.7) then follow from the observation that

$$\mathcal{P}_{\lambda_0^*}(L^*) = \mathcal{P}_{\lambda_0}(L)^*, \tag{5.11}$$

together with the fact that $\mathcal{P}_{\lambda_0}(L)$ has finite rank. Observation (5.11) follows in turn from

$$\mathcal{P}^*_{\lambda_0} f = \int_{-\infty}^{+\infty} P^*_{\lambda_0}(y,x) f(y)\,dy$$

and the fact that $G^*_\lambda(y,x)$ is the Green's function for $(L-\lambda_0 I)^* = (L^*-\lambda_0^* I)$, another application of Lemma 4.3.

(v) Motivated by the formal resolvent expansion

$$(L-\lambda_0 I)^{-1} = -\int_\Gamma (\lambda-\lambda_0)^{-1}(L-\lambda I)^{-1} d\lambda,$$

we can expect $-\mathcal{Q}_{\lambda_0,-1}$ to act as some sort of pseudo-inverse, since

$$Q_{\lambda_0,-1} = \text{Res}(\lambda-\lambda_0)^{-1} G_\lambda(x,y).$$

Applying (5.10), we find that

$$(L-\lambda_0 I)(-Q_{\lambda_0,-1})(x,y) = \delta(x-y) - P_{\lambda_0}(L)(x,y),$$

hence, indeed,

$$(L-\lambda_0 I)(-\mathcal{Q}_{\lambda_0,-1}) = I - \mathcal{P}_{\lambda_0}(L).$$

Setting $f = \tilde{f} - \mathcal{Q}_{\lambda_0,-1} g$, we thus find that the equation $(L-\lambda_0 I)f = g$ can be solved if $(L-\lambda_0 I)\tilde{f} = \mathcal{P}_{\lambda_0} g$ can be solved. This is clearly possible if

$$\mathcal{P}_{\lambda_0} g \in \Sigma'_{\lambda_0,1}(L) = (L-\lambda_0 I)\Sigma'_{\lambda_0}(L),$$

or equivalently

$$\mathcal{Q}_{\lambda_0,K-1}(L) g = (L-\lambda_0 I)^{K-1}\mathcal{P}_{\lambda_0}(L) g = 0.$$

The relation

$$0 = \langle h, \mathcal{Q}_{\lambda_0,K-1}(L) g\rangle = \langle \mathcal{Q}_{\lambda_0^*,K-1}(L^*) h, g\rangle$$

for all $h$ shows this to be equivalent to $g \in \Sigma'_{\lambda_0^*,K-1}(L^*)^\perp$. On the other hand, $(L-\lambda_0 I)f = g$ clearly implies that

$$\begin{aligned}\mathcal{Q}_{\lambda_0,K-1}(L) g &= \mathcal{Q}_{\lambda_0,K-1}(L)(L-\lambda_0 I) f\\ &= \mathcal{Q}_{\lambda_0,K}(L) f,\end{aligned}$$

hence $\mathcal{Q}_{\lambda_0,K-1}(L) g \in \Sigma'_{\lambda_0,0}$, and $\mathcal{Q}_{\lambda_0,K-1}(L) g = 0$ by (5.5). This completes the proof. □

**6. The Evans function and effective spectrum.** We next explore the relation between the effective point spectrum and the *Evans function*, showing that the effective eigenspace has dimension equal to the multiplicity of the Evans function. This fact explains the meaning of the Evans function in regions of essential spectrum [PW]. More importantly, it establishes *continuity of spectral dimension* under perturbations of the underlying operator $L$.

We first make two further assumptions. These are relatively mild; in particular, they are satisfied for our application of main interest, the study of linearized operators about traveling wave solutions. Writing $(L-\lambda I)w = 0$ as usual as a first order system

$$W' = \mathcal{A}(\lambda, x)W, \quad W \in C^N, \tag{6.1}$$

we require that:

(**h1**) $L$ is a nondegenerate operator of order $s$, i.e. the coefficient matrix of the principal part is nonsingular for all $x$.
(**h2**) There are solutions $\Phi^+ = (\phi_1^+, \dots, \phi_k^+)$ and $\Phi^- = (\phi_1^-, \dots, \phi_{N-k}^-)$ of (6.1), analytic in $\lambda$ on the set $\Omega$ of Definition 5.1, which on the resolvent set span the manifolds of solutions decaying at $\pm\infty$.

In this case, we can define an analytic Evans function

$$D_L(\lambda) = \det\, (\Phi^+, \Phi^-)|_{x=0},$$

which we assume does not vanish identically. It is easily verified that the order to which $D_L$ vanishes at any $\lambda_0$ is independent of the choice of analytic bases $\Phi^\pm$. To reveal the relation between the effective eigenspace and the Evans function, we will construct special, Jordan bases relative to the operator $L$, in which the calculations considerably simplify. The advantage of such a basis was pointed out by Gardner and Jones, [GJ.1-2], in their treatment of isolated eigenvalues.

**Lemma 6.1.** *Let $L$, $\lambda_0$ be as in Definition 5.1, and satisfying* (**h1**)*–*(**h2**)*. Then, at any zero $\lambda_0 \in \Omega$ of $D_L$, there exist analytic choices of bases $\Phi^\pm$, and indices $p_1 \geq \cdots \geq p_J$ such that*

$$\left(\frac{\partial}{\partial\lambda}\right)^p \phi_j^+ = \left(\frac{\partial}{\partial\lambda}\right)^p \phi_j^-, \quad j = 1, \dots, J;\ p = 0, \dots, p_j, \tag{6.2}$$

*and*

$$\begin{aligned}(\partial/\partial\lambda)^{p_1+1}(\phi_1^+ - \phi_1^-) \wedge \cdots \wedge (\partial/\partial\lambda)^{p_J+1}(\phi_J^+ - \phi_J^-)\\ \wedge\phi_1^+ \wedge \cdots \wedge \phi_k^+ \wedge \phi_{J+1}^- \wedge \cdots \wedge \phi_{N-k}^- \neq 0,\end{aligned} \tag{6.3}$$

*where the multiplicity of the zero of $D_L$ at $\lambda_0$ is $d = \sum_{1\le j\le J}(p_j+1)$.*

*Proof.* It is equivalent to show that

$$\left(\frac{\partial}{\partial\lambda}\right)^p \phi_j^+ = \left(\frac{\partial}{\partial\lambda}\right)^p \phi_j^-, \quad p=0,\dots,P; j=1,\dots,j_p \tag{6.4}$$

and

$$\begin{aligned}\left(\frac{\partial}{\partial\lambda}\right)^p (\phi_{j_p+1}^+ - \phi_{j_p+1}^-)\wedge\cdots\wedge\left(\frac{\partial}{\partial\lambda}\right)^p (\phi_{j_{p-1}}^+ - \phi_{j_{p-1}}^-)\wedge\cdots\\ \wedge\left(\frac{\partial}{\partial\lambda}\right)(\phi_{j_1+1}^+ - \phi_{j_1+1}^-)\wedge\cdots\wedge\left(\frac{\partial}{\partial\lambda}\right)(\phi_{j_0}^+ - \phi_{j_0}^-)\\ \wedge\ \phi_1^+\wedge\cdots\wedge\phi_k^+\wedge\phi_{j_0+1}^-\wedge\cdots\wedge\phi_{N-k}^-\neq 0,\end{aligned} \tag{6.5}$$

for all $0\le p\le P+1$, where $j_p := \max_{p_j\ge p} j$ and $P=p_1$. Note that $j_0=J$ and $j_{P+1}=0$, hence (6.3) is achieved from (6.5) at level $p=P+1$.

Without loss of generality, take $\lambda_0=0$. Since $D_L$ vanishes at $\lambda_0$, there is a $J=j_0\ge 1$ dimensional intersection between Span $\Phi^-$ and Span $\Phi^+$. By a linear change of coordinates, we can thus arrange that (6.4) hold for $p=0$, $1\le j\le j_0$. We proceed by induction on $p$. At each stage, either (6.5) holds with $j_{p+1}=0$, or else there is a nontrivial $m$-dimensional intersection of

$$\text{Span}\left\{\left(\frac{\partial}{\partial\lambda}\right)^{p+1}(\phi_1^+ - \phi_1^-),\dots,\left(\frac{\partial}{\partial\lambda}\right)^{p+1}(\phi_{j_{p+1}}^+ - \phi_{j_{p+1}}^-)\right\}$$

with the manifold spanned by the terms in the expression (6.5) with $\lambda$-derivatives of order $\le p$. Set $j_{p+1}=m$. By making a coordinate change only within the span of $\phi_1^\pm,\dots,\phi_{j_p}^\pm$, we can arrange, without affecting the relations for smaller $p$, that

$$\left(\frac{\partial}{\partial\lambda}\right)^{p+1}\phi_j^- + \sum_{j,q\le p} c_{jq}^-\left(\frac{\partial}{\partial\lambda}\right)^q\phi_j^+ = \left(\frac{\partial}{\partial\lambda}\right)^{p+1}\phi_j^+ + \sum_{j,q\le p} c_{jq}^+\left(\frac{\partial}{\partial\lambda}\right)^q\phi_j^+ \tag{6.6}$$

for $1\le j\le j_{p+1}$, and (6.5) holds up to $p+1$. Redefining

$$\tilde\phi_j^\pm = \phi_j^\pm + \sum_{j,q\le p} c_{jq}^\pm\binom{p+1}{q}^{-1}\lambda^{p+1-q}\phi_j^\pm$$

for $1 \le j \le j_{p+1}$ does not affect the spans of $\Phi^\pm$ for small $\lambda$, for, it clearly does not add to the span, and is the identity transformation at $\lambda = 0$, hence full rank for $\lambda$ small. Thus, it is an allowable, analytic change of coordinates. With this change of coordinates, (6.4) is then satisfied up to $p+1$, by (6.6) together with the elementary fact that

$$\left(\frac{\partial}{\partial\lambda}\right)^n (\lambda^k f)|_{\lambda=0} = \binom{n}{n-k}\left(\frac{\partial}{\partial\lambda}\right)^{n-k} f(0).$$

By induction, we can thus satisfy (6.4)–(6.5) up to any finite order $p$. It is an easy calculation that $D_L$ must therefore vanish up order at least $\sum_{0\le s\le p} j_s$. Since the multiplicity of $D_L$ is finite, by analyticity, the process must terminate at some $P$ such that $j_{P+1} = 0$, establishing the main assertions (6.4)–(6.5).

As remarked above, $D_L$ thus vanishes to order at least

$$d = \sum_{1\le j\le J} (p_j + 1) = \sum_{0\le s\le P} j_s.$$

But a straightforward computation shows that the nonzero quantity in (6.3) is the only term in the $d$th derivative of $D_L$ with respect to $\lambda$, hence $d$ is exactly the multiplicity of the zero of $D_L$ at $\lambda_0$. This completes the proof of the lemma. $\square$

***Remark 6.2.*** Repeated differentiation of the eigenvalue equation with respect to $\lambda$ gives the variational equations

$$\phi_j^\pm = (L - \lambda_0 I)\left(\frac{\partial}{\partial\lambda}\right)\phi_j^\pm = (L-\lambda_0 I)^2\left(\frac{\partial}{\partial\lambda}\right)^2\phi_j^\pm = \cdots. \tag{6.7}$$

Thus, the basis constructed above consists, formally, of Jordan chains with respect to $L$.

***Theorem 6.3.*** *Let $L$, $\lambda_0$ be as above. Then:*

(i) *The functions $(\partial/\partial\lambda)^p\phi_j^-$,*[5] *$1 \le j \le J$, $0 \le p \le p_j$ described in the previous lemma, projected onto their first $n$ coordinates, are a basis for $\Sigma'_{\lambda_0}$. Moreover, the projection of $(\partial/\partial\lambda)^p\phi_j^-$ is an eigenfunction of ascent $p+1$.*

(ii) *dim $\Sigma'_{\lambda_0}(L)$ is equal to the order $d$ to which the Evans function $D_L$ vanishes at $\lambda_0$.*

(iii) *$P_{\lambda_0}(L) = \sum_j \varphi_j(x)\pi_j(y)$, where $\varphi_j$ and $\pi_j$ are in $\Sigma'_{\lambda_0}(L)$ and $\Sigma'_{\lambda_0}(L^*)$, respectively, with ascents summing to $\le K+1$, where $K$ is the order of the pole of $G_\lambda$ at $\lambda_0$.*

[5] Alternatively, $(\partial/\partial\lambda)^p\phi_j^+$, see (6.2).

*Proof.* (i) Let $\Phi^\pm$ be the Jordan bases described in Lemma 6.1. The Green's function $G_\lambda(x,y)$ associated with $L$, by a calculation analogous to that used to prove (4.27), satisfies

$$\begin{pmatrix} G_\lambda & \cdots & (\partial/\partial y)^s G_\lambda \\ (\partial/\partial x)G_\lambda & \cdots & (\partial/\partial y)^s(\partial/\partial x)G_\lambda \\ \vdots & \vdots & \vdots \\ (\partial/\partial x)^s G_\lambda & \cdots & (\partial/\partial y)^s(\partial/\partial x)^s G_\lambda \end{pmatrix} = (\Phi^+(x),0)(\Phi^+(y),\Phi^-(y))^{-1}\mathcal{S}(y)^{-1}, \tag{6.8}$$

for $x > y$, where $s$ is the order of the operator $L$ and $\mathcal{S}^{-1}(y)$ is an invertible, block lower cross-triangular matrix, with cross-diagonal blocks given by $\pm$ the coefficient matrix of the principal part of $L$ (nonsingular, by (**h1**)).

Thus, the projection kernel $P_{\lambda_0}(x,y)$ is given by

$$\begin{pmatrix} P_{\lambda_0} & \cdots & (\partial/\partial y)^s P_{\lambda_0} \\ (\partial/\partial x)P_{\lambda_0} & \cdots & (\partial/\partial y)^s(\partial/\partial x)P_{\lambda_0} \\ \vdots & \vdots & \vdots \\ (\partial/\partial x)^s P_{\lambda_0} & \cdots & (\partial/\partial y)^s(\partial/\partial x)^s P_{\lambda_0} \end{pmatrix} = \mathrm{Res}_{\lambda_0}(\Phi^+(x),0)(\Phi^+(y),\Phi^-(y))^{-1}\mathcal{S}(y)^{-1},$$

or, by Kramer's rule:

$$\mathrm{Res}_{\lambda_0} D_L(\lambda)^{-1}(\Phi^+(x),0)(\Phi^+(y),\Phi^-(y))^{\mathrm{adj}}\mathcal{S}(y)^{-1}, \tag{6.9}$$

where *adj* denotes adjugate matrix, or transposed matrix of minors. Expression (6.9) can be expanded as

$$\left(\frac{d}{d\lambda}\right)^d D_L(\lambda_0)^{-1} \sum_{j,p\le d-1} \left(\frac{\partial}{\partial\lambda}\right)^p \phi_j(\lambda_0,x)\left(\frac{\partial}{\partial\lambda}\right)^{d-1-p} \Theta_j(\lambda_0,y)\mathcal{S}(y)^{-1}, \tag{6.10}$$

where $\Theta_j(\lambda,\cdot)$ denotes the $j$th row of $(\Phi^+(y),\Phi^-(y))^{adj}$, or the $j$th column of the matrix of minors.

It is clear from expansion (6.10) that the range of $\mathcal{P}_{\lambda_0}$ is the span of all $(\partial/\partial\lambda)^p\phi_j^+(\lambda_0,\cdot)$ for which $(\partial/\partial\lambda)^{d-1-p}\Theta_j(\lambda,\cdot)\not\equiv 0$, projected onto their first $n$ coordinates (recall that $\phi_j$ is a phase vector, the physical vector augmented by its first $s-1$ derivatives). From property (6.2) of the bases $\Phi^\pm$, we easily find that the minor $(\partial/\partial\lambda)^{d-1-p}\Theta_j(\lambda,\cdot)$ vanishes whenever $p > p_j$. For, with less than

$d-1-p_j$ $\lambda$-derivatives distributed among the columns $r \neq j$ of $(\Phi^+, \Phi^-)$, there will always be a linear dependence. Indeed, this is the same calculation that shows that the determinant $D_L(\lambda)$ of the whole matrix vanishes up to derivative at least $d-1$.

Likewise, we can show that $(\partial/\partial\lambda)^{d-1-p_j}\Theta_j(\lambda,\cdot)$ *does not* vanish by multiplying on the right by $(\partial/\partial\lambda)^j\phi_j^+$. The result is the single nonvanishing term in the Leibnitz expansion for $(d/d\lambda)D_L(\lambda_0)$, hence $(\partial/\partial\lambda)^{p_j}\phi_j^+(\lambda_0,\cdot) \in \Sigma'_{\lambda_0}$ for all $j \le J$.

The factors $(\partial/\partial\lambda)^{d-1-p}\Theta_j(\lambda,\cdot)$ for $p < p_j$ are more difficult to evaluate. Fortunately, we do not have to. For, we already know from (6.7) and the $L-\lambda_0 I$ invariance of $\Sigma'_{\lambda_0}$ that, for $p < p_j$,

$$\left(\frac{\partial}{\partial\lambda}\right)^p \phi_j^+ = (L-\lambda I)^{p_j-p}\left(\frac{\partial}{\partial\lambda}\right)^{p_j}\phi_j^+$$

are all contained in $\Sigma'_{\lambda_0}$. Furthermore, no $(\partial/\partial\lambda)^p\phi_j^+$ is identically zero, since each solves an inhomogeneous generalized eigenvalue equation, Remark 6.2. This completes the verification of claim (i).

(ii) As observed in Lemma 6.1, $d = \sum_{j\le J}(p_j+1)$, which is precisely the number of basis elements for $\Sigma'_{\lambda_0}$.

(iii) Immediate, from expansion (6.10) and duality (cf. Lemmas 4.3–4.4). □

***Remark.*** Proposition 5.3 implies continuity of the spectral projection $\mathcal{P}_{\lambda_0}$ under perturbations of $L$ preserving (**h1**)–(**h2**). From an abstract point of view, the main import of Theorem 6.4 is perhaps that it implies continuity of the *dimension* of the effective eigenspace, hence of the eigenspace itself. Thus, we recover by direct calculation much of the standard spectral perturbation theory, [Kat], for eigenvalues separated from the essential spectrum.

Note, in contrast to the standard (isolated eigenvalue) case, that continuity of dimension does not follow immediately from continuity of the spectral projection, for the reason that $\mathcal{P}_{\lambda_0}$ is not a true projection in the case of embedded spectrum. The beautiful lemma, [Kat], that nearby projections have ranges of equal dimension therefore does not apply. (However, we suspect that a direct proof is possible using the structure $P_{\lambda_0} = \sum \varphi_j \pi_k$ to reduce to a finite-dimensional calculation, an approach suggested by H. Freistühler. This would have the advantage of greater generality, since it does not rely on (**h1**)–(**h2**) or the introduction of the Evans function.).

**Part III. Pointwise Bounds.**

**7. Pointwise Bounds on $G_\lambda$.** Using the representation formulae of section 4, we now derive pointwise bounds on the elliptic Green's function $G_\lambda$, for small, medium, and large values of $|\lambda|$. An immediate corollary is the spectral resolution formula (1.20).

The critical estimate is near $\lambda = 0$. Here, we decompose $G_\lambda$ into products of different scalar modes, encoding interactions of the shock layer with the far field.

***Proposition 7.1*** (***Small*** $\lambda \sim$ ***large time***)**.** *Assuming* (**C0**)–(**C3**), *let* $K$ *be the order of the pole of* $G_\lambda$ *at* $\lambda = 0$, *and* $r$ *be sufficiently small that there are no other poles in* $B(0,r)$. *Then, for* $\lambda \in \Omega_\theta$ *such that* $|\lambda| \le r$, *we have:*

(i) *For* $y < 0 < x$,

$$\begin{pmatrix} G_\lambda & G_{\lambda_y} \\ G_{\lambda_x} & G_{\lambda_{xy}} \end{pmatrix} = \sum_{j,k} \phi_j^+(x) d_{jk}(\lambda) \tilde{\psi}_k^-(y), \tag{7.1}$$

*where* $d_{jk}(\lambda) = \mathbf{O}(\lambda^{-K})$ *is a meromorphic, scalar function;*

(ii) *For* $y < x < 0$:

$$\begin{pmatrix} G_\lambda & G_{\lambda_y} \\ G_{\lambda_x} & G_{\lambda_{xy}} \end{pmatrix} = \sum_{j,k} \phi_j^-(x) d_{jk}(\lambda) \tilde{\psi}_k^-(y) + \sum_k \psi_k^-(x) e_k(\lambda) \tilde{\psi}_k^-(y) \tag{7.2}$$

*where* $d_{jk}(\lambda) = \mathbf{O}(\lambda^{-K})$ *and* $e_k(\lambda) = \mathbf{O}(\lambda^{1-K})$ *are scalar, meromorphic functions;*

(iii) *For* $x < y < 0$:

$$\begin{pmatrix} G_\lambda & G_{\lambda_y} \\ G_{\lambda_x} & G_{\lambda_{xy}} \end{pmatrix} = \sum_{j,k} \phi_j^-(x) d_{jk}(\lambda) \tilde{\psi}_k^-(y) + \sum_k \phi_k^-(x) e_k(\lambda) \tilde{\phi}_k^-(y) \tag{7.3}$$

*where* $d_{jk}(\lambda)$ *and* $e_k(\lambda)$ *are as above.*

*Symmetric representations hold in case* $y > 0$.

Note that Proposition 7.1 includes detailed pointwise information on $G_\lambda$ through the bounds (4.3) and (4.14). The fact that no mixed, $j$, $k$ summands occur in the second terms of (ii)–(iii) is crucial for our later analysis in Section 7, and is the main point in this calculation.

*Proof.* For $y < x$, the representation (7.1) follows by direct expansion of representation (4.19); a symmetric expansion holds for $x < y$. Since $(\Phi^+(y), \Psi^+(y))$, $(\Phi^-(y), \Psi^-(y))$, are by definition bases of solutions of (2.1), we can further expand these representations to obtain

$$\begin{pmatrix} G_\lambda & G_{\lambda_y} \\ G_{\lambda_x} & G_{\lambda_{xy}} \end{pmatrix} = \sum_{j,k} \phi_j^-(x) d_{jk}(\lambda) \tilde{\psi}_k^-(y) + \sum_{j,k} \psi_j^-(x) e_{jk}(\lambda) \tilde{\psi}_k^-(y) \tag{7.4}$$

in case (ii) and

$$\begin{pmatrix} G_\lambda & G_{\lambda_y} \\ G_{\lambda_x} & G_{\lambda_{xy}} \end{pmatrix} = \sum_{j,k} \phi_j^-(x) d_{jk}(\lambda) \tilde{\psi}_k^-(y) + \sum_{j,k} \phi_j^-(x) e_{jk}(\lambda) \tilde{\phi}_k^-(y). \tag{7.5}$$

in case (iii).

To verify that (7.2) and (7.3) hold, i.e., $e_{jk} = 0$ for $j \neq k$, requires a slight further computation. We carry this out for $x < y < 0$; the case $y < x < 0$ is symmetric (in fact, dual). Recall the alternate representation

$$\begin{pmatrix} G_\lambda & G_{\lambda_y} \\ G_{\lambda_x} & G_{\lambda_{xy}} \end{pmatrix} = (0, \Phi^-(x))(\Phi^+(y), \Phi^-(y))^{-1} \mathcal{S}(y)^{-1}, \tag{7.6}$$

coming from (4.27). Expanding the right hand side of (7.6) gives

$$\sum_j \phi_j^-(x) E_{n+j}^t (\Phi^+(y), \Phi^-(y))^{-1} \mathcal{S}(y)^{-1},$$

where $E_{n+j}$ denotes the $(n+j)$th standard basis element in $\mathbb{C}^{2n}$ and $^t$ denotes transpose. Comparing with (7.5), we find that

$$\sum_k d_{jk}(\lambda) \tilde{\psi}_k^-(y) + \sum_k e_{jk} \tilde{\phi}_k^-(y) = E_{n+j}^t (\Phi^+(y), \Phi^-(y))^{-1} \mathcal{S}(y)^{-1}.$$

From (4.18), it follows that

$$\begin{aligned} e_{jk} &= \Big(\sum_k e_{jk} \tilde{\phi}_k^-\Big) \mathcal{S} \phi_k^- \\ &= E_{n+j}^t (\Phi^+(y), \Phi^-(y))^{-1} \phi_k^-, \end{aligned}$$

which by Kramer's rule is

$$\det\, (\Phi^+(y), \Phi^-(y))^{-1} \det\, (\Phi^+, \cdots, \phi_{j-1}^-, \phi_k^-, \phi_{j+1}^-, \cdots),$$

and clearly vanishes when $j \neq k$, det $(\Phi^+(y), \Phi^-(y)) \neq 0$. Since $e_{jk}$ is meromorphic, this implies that indeed $e_{jk} \equiv 0$ for $j \neq k$.

That $d_{jk}, e_k = \mathbf{O}(\lambda^{-K})$ follows by our assumption that the pole of $G_\lambda$ at $\lambda = 0$ is of order $K$. This in turn implies that

$$\mathrm{Res}_0 \lambda^{K-1} \phi_k^-(\lambda, x) e_k(\lambda) \tilde{\phi} -_k (\lambda, y) = \phi_k^-(0, x)(\mathrm{Res}_0 \lambda^{K-1} e_k(\lambda)) \tilde{\phi}_k^-(0, y).$$

It follows that

$$\Sigma'_{0,1}(L^*) = \text{Range } \langle \cdot, \mathrm{Res}_0 \lambda^{K-1} G_\lambda \rangle$$

must contain all $\tilde{\phi}_k^-(0, x)$ for which $\mathrm{Res}_0 \lambda^{K-1} e_k$ is nonzero. But, $\Sigma'_{0,1}(L^*) \subset$ Span $\tilde{\Psi}^\pm$ is transverse to Span $\tilde{\Phi}^\pm$, hence $\mathrm{Res}_0 \lambda^{K-1} e_k \equiv 0$ and $e_k = \mathbf{O}(\lambda^{1-K})$ as claimed. □

For $|\lambda|$ of medium range, we require only the rather weak result that $G_\lambda$ is bounded in the $L^\infty$ norm on compact subsets of the resolvent set. This reflects, but does not immediately follow from the fact that the *Resolvent* by definition is bounded on the resolvent set, in the operator norm.

***Proposition 7.2 (Medium $\lambda \sim$ intermediate time).*** *Assuming* (**C0**)–(**C1**), *then on any compact subset of the resolvent set intersect $\Lambda$, in particular on $\Omega_\theta \cap \{\lambda : r \leq |\lambda| \leq R\}$ minus any open neighborhood of $\sigma_p(L)$, with $r$ as in Lemma 4.2, it holds that*

$$|G_\lambda|,\ |G_{\lambda_x}|,\ |G_{\lambda_y}|,\ |G_{\lambda_{xy}}| \leq C \tag{7.7}$$

*for all $x, y$, where $C$ is independent of $x$, $y$.*

*Proof.* On the resolvent set, we have that $\Phi^-(x)$ and $\Phi^+(x)$ are tangent as $x \to +\infty$ to the stable and unstable invariant subspaces of $\mathbb{A}_+(\lambda)$, respectively, and thus the projection $\Pi_+(y)$ in (4.23) is bounded as $y \to +\infty$, for any fixed $\lambda$. By symmetric argument, $\Pi_+(y)$ is bounded also as $y \to -\infty$. Similarly, $\Pi_-(y)$, $\tilde{\Pi}_\pm(y)$ are bounded as $y \to \pm\infty$, hence bounded everywhere. It follows by continuity that all projections are uniformly bounded on compact subsets of the resolvent set.

Likewise, we obtain from the basic bounds (4.3) and (4.14) (or the more fundamental (3.4) and its adjoint analog, in the case that $\lambda$ is at or near an exceptional point $\lambda_j$) that the flow $\mathcal{F}_+^{y \to x}$ has $|\mathcal{F}_+^{y \to x}| < 1$ for $x, y \geq 0$ sufficiently large (uniform in $\lambda$). By continuous dependence, it follows that

$$|\mathcal{F}_+^{y \to x}| < C \tag{7.8}$$

for $x$, $y \geq 0$, uniformly on compact subsets of the resolvent set. Combining, and appealing to (4.27), we have

$$|G_\lambda|, |G_{\lambda_x}|, |G_{\lambda_y}|, |G_{\lambda_{xy}}| \leq C \tag{7.9}$$

for $x$, $y \geq 0$, and, by symmetric argument, for $x$, $y \leq 0$.

For $x$, $y$ of opposite signs, the result follows from the analogous decomposition (4.28). □

For large $|\lambda|$, we require slightly sharper, but still rather crude bounds, on any sector contained in the resolvent set.

***Proposition 7.3 (Large $\lambda \sim$ short time).*** *Assuming* (**C0**)–(**C1**), *it follows that for some $C$, $\beta$, $R > 0$, and $\theta_1$, $\theta_2 > 0$ sufficiently small.*

$$|G_\lambda(x,y)| \leq C|\lambda^{-1/2}|e^{-\beta^{-1/2}|\lambda^{1/2}||x-y|}; \tag{7.10}$$

$$|G_{\lambda_x}(x,y)|, \; |G_{\lambda_x}(x,y)| \leq Ce^{-\beta^{-1/2}|\lambda^{1/2}||x-y|},$$

*for all $\lambda \in \Omega_\theta \setminus B(0,R)$.*

(*Here, we may choose any $\beta^{-1/2} < \min_{j,\lambda\in\Omega_\theta\cap\{|\lambda|\geq R\}} \mathrm{Re}(\sqrt{\lambda/|\lambda| b_j})$ where $b_j$ are the eigenvalues of $B$*).

The crude bound (7.10) is roughly equivalent to the short–time estimates of standard parabolic theory, and could probably be obtained by similar techniques, e.g. the parametrix method of Levi [Fr,Le]. It is closely related to the fact that $-(L-\lambda I)$ is *coercive* on $\Omega_\theta \cap B(0,R)^c$, with

$$\frac{1}{2}((L-\lambda I) + (L-\lambda I)^*) \geq \beta^{-1/2}|\lambda^{1/2}|I \tag{7.11}$$

with respect to the $L^2$ inner product

$$\langle\langle f, g\rangle\rangle := \int_{-\infty}^{\infty} \langle f, g\rangle \, dx.$$

Indeed, we suspect that a shorter and more general proof, valid in any spatial dimension, should be possible from this point of view. (see [Ag] for related analysis on decay of eigenfunctions for the Schrödinger operator). Here, we will give a proof by a rescaling argument in the spirit of Proposition 4.1 of [GZ] and Proposition 2 of [AGJ].

*Proof.* For simplicity, we carry out the argument for $A$, $B \in C^{1+\alpha}$, allowing us to work in the original coordinates $(w, w')$. The case $A$, $B \in C^{0+\alpha}$ can be treated by first mollifying $B$, then using the coordinate change described above Proposition 3.4. This yields

$$\mathbb{B} = \begin{pmatrix} 0 & \bar{B}^{-1} \\ \bar{\lambda} & 0 \end{pmatrix}$$

in place of (7.14) below, and $\mathbf{O}(\varepsilon) + \mathbf{O}(|\lambda^{-1/2}|)$ in place of $\mathbf{O}(|\lambda^{-1/2}|)$ wherever it appears, where $\mathbf{O}(\varepsilon)$ is independent of $\lambda$, depending only on the degree of mollification. With these changes, the argument then goes through as before.

Setting $\bar{x} = |\lambda^{1/2}|x$, $\bar{\lambda} = \lambda/|\lambda|$, $\bar{B}(\bar{x}) = B(\bar{x}/|\lambda^{1/2}|)$, $\bar{w}(\bar{x}) = w(x/|\lambda^{1/2}|)$, in (2.1), we obtain

$$\bar{w}'' = \bar{\lambda} \cdot \bar{B}^{-1}\bar{w} + \mathbf{O}(|\lambda^{-1/2}|)(\bar{w} + \bar{w}'), \tag{7.12}$$

or

$$\overline{W}' = \bar{\mathbb{B}}\overline{W} + \mathbf{O}(|\lambda^{-1/2}|)\overline{W}, \tag{7.13}$$

where $\overline{W} = (\bar{w}, \bar{w}')$, and

$$\bar{\mathbb{B}} := \begin{pmatrix} 0 & I \\ \bar{\lambda}\bar{B}^{-1} & 0 \end{pmatrix}, \quad \bar{\mathbb{B}}' = \mathbf{O}(|\lambda^{-1/2}|), \quad |\bar{\lambda}| = 1. \tag{7.14}$$

It is easily computed that the eigenvalues of $\bar{\mathbb{B}}$ are

$$\bar{\mu} = \mp\sqrt{\bar{\lambda}/b_j}, \tag{7.15}$$

where $b_j$ are the eigenvalues of $B$. By (H1),

$$\min_j \mathrm{Re}\sqrt{\bar{\lambda}/b_j} > \beta^{-1/2} \tag{7.16}$$

for all $\lambda \in \Omega_\theta$, for some $\beta > 0$, hence the stable and unstable subspaces of each $\bar{\mathbb{B}}(\bar{x})$ are both of dimension $n$, and separated by a spectral gap of more than $2\beta^{-1/2}$. Since $\bar{\mathbb{B}}(\lambda, \bar{x})$ varies within a compact set, it follows that there are continuous eigenprojections $P_\pm(\mathbb{B})$ taking $\overline{W}$ onto the stable and unstable subspaces, respectively, of $\bar{\mathbb{B}}$, with $|P'_\pm| = \mathbf{O}(|\lambda^{-1/2}|)$.

Introducing new coordinates $y_\pm = P_\pm \bar{w}$, we thus obtain a block diagonal system

$$\begin{pmatrix} y_+ \\ y_- \end{pmatrix}' = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} \begin{pmatrix} y_+ \\ y_- \end{pmatrix} + \mathbf{O}(|\lambda^{-1/2}|) \begin{pmatrix} y \\ y \end{pmatrix}, \tag{7.17}$$

where the eigenvalues of $A_\pm$ have strictly negative/positive real parts, respectively (note: as usual, the $\pm$ indices are associated with decay at $x = \pm\infty$, hence the apparently contrary sign convention). Equivalently, there exist continuous invertible transformations $Q_\pm$ such that $E_\pm = Q_\pm A_\pm Q_\pm^{-1}$ are "real positive definite" in the sense that

$$\text{Re}(E) := \frac{1}{2}(E_\pm + E_\pm^*) \lesseqgtr \mp\beta^{-1/2} I. \tag{7.18}$$

By a third coordinate change $z_\pm = Q_\pm y_\pm$, we obtain, finally,

$$\begin{pmatrix} z_+ \\ z_- \end{pmatrix}' = \begin{pmatrix} E_+ & 0 \\ 0 & E_- \end{pmatrix} \begin{pmatrix} z_+ \\ z_- \end{pmatrix} + \mathbf{O}(|\lambda^{-1/2}|) \begin{pmatrix} z \\ z \end{pmatrix}, \tag{7.19}$$

where (by good conditioning of $P_\pm$, $Q_\pm$, following from compactness),

$$|\bar{w}|/C \le |z| \le C|\bar{w}|. \tag{7.20}$$

From (7.19), we obtain the "energy estimates"

$$\begin{aligned} \langle z_\pm, z_\pm \rangle' &= \langle z_\pm, (E_\pm + E_\pm^*) z_\pm \rangle + \mathbf{O}(|\lambda^{-1/2}|)(\langle z_+, z_+ \rangle + \langle z_-, z_- \rangle) \\ &\lesseqgtr \mp 2\beta^{-1/2} \langle z_\pm, z_\pm \rangle + \mathbf{O}(|\lambda^{-1/2}|)(\langle z_+, z_+ \rangle + \langle z_-, z_- \rangle) \end{aligned} \tag{7.21}$$

In consequence, the ratios $r_+ := \langle z_-, z_- \rangle / \langle z_+, z_+ \rangle$ and $r_- := \langle z_+, z_+ \rangle / \langle z_-, z_- \rangle$ satisfy

$$r_\pm' \lesseqgtr \mp 4\beta^{-1/2} r_\pm \pm C|\lambda^{-1/2}|(1 + r_\pm + r_\pm^2) \tag{7.22}$$

for some $C > 0$.

From (7.22) it follows easily that the cones $\mathbb{K}_\mp = \{r_\mp < C|\lambda^{-1/2}|\beta^{1/2}\}$ are invariant under forward and backward flow, respectively, of (7.19), provided that

$$C|\lambda^{-1/2}|\beta^{1/2} < 4/3.$$

Since the stable/unstable subspaces of $E = \begin{pmatrix} E_+ & 0 \\ 0 & E_- \end{pmatrix}$ at $x = \pm\infty$ are precisely $\{z_\pm = 0\}$, we have that the stable/unstable subspaces of $E + \mathbf{O}(\lambda^{-1/2})$ at $x = \pm\infty$ lie within the respective cones $\mathbb{K}^\pm$, provided $|\lambda|$ is sufficiently large. It follows that the stable/unstable manifolds of solutions of (7.19) lie within $\mathbb{K}^\pm$ *for all* $x$.

Plugging this information back into (7.21), we find that

$$(|z_+|^2)' \lessgtr \mp 2\tilde{\beta}^{-1/2}|z_+|^2 \tag{7.23}$$

for any solution $(z_+, z_-)^t$ decaying at $x = \pm\infty$, hence

$$\frac{|z_+(x)|}{|z_+(y)|} \le e^{-\tilde{\beta}^{-1/2}|(x-y)|},$$

$\tilde{\beta} < \beta$, and thus

$$\frac{|z(x)|}{|z(y)|} \le C_1 e^{-\tilde{\beta}^{-1/2}|x-y|} \tag{7.24}$$

for any $x \lessgtr y$, provided $|\lambda|$ is sufficiently large. This gives

$$\frac{|\overline{W}(x)|}{|\overline{W}(y)|} \le C_1 C^2 e^{-\tilde{\beta}^{-1/2}|x-y|}, \tag{7.25}$$

where $C$ is as in (7.20). Further, untangling intermediate coordinate changes, we find that

($\mathbb{K}$) *the stable/unstable manifolds of solutions of* (7.13) *lie within angle* $\mathbf{O}(|\lambda^{-1/2}|)$ *of the stable/unstable subspaces of* $\bar{\mathbb{B}}(x)$.

Now, recall the coordinate–free representation (4.27) of the Green's function as

$$\begin{pmatrix} G_\lambda \\ G_{\lambda_x} \end{pmatrix} = \mathcal{F}^{y\to x}\,\Pi_+(y)\begin{pmatrix} 0 \\ B^{-1}(y) \end{pmatrix}.$$

Translating the bound (7.25) back to the original system (2.1), we obtain

$$|\mathcal{F}^{y\to x}| \le C_1 C^2 e^{-\tilde{\beta}|\lambda^{1/2}|\,|x-y|}, \tag{7.26}$$

Likewise, the projection $\Pi_+$ can be related to its counterparts $\bar{\Pi}_+$ for the rescaled system by the factorization

$$\Pi_+ = \begin{pmatrix} I & 0 \\ 0 & |\lambda^{1/2}|I \end{pmatrix} \bar{\Pi}_+ \begin{pmatrix} I & 0 \\ 0 & |\lambda^{-1/2}|I \end{pmatrix},$$

and similarly for $\tilde{\Pi}_-$. Since the stable and unstable manifolds stay separated, by $(\mathbb{K})$, and $\bar{\lambda}$ varies within a compact set, the projections $\bar{\Pi}_+$ and $\bar{\tilde{\Pi}}_-$ are uniformly bounded. Thus, we have

$$\begin{aligned}\Pi_+(y)\begin{pmatrix} 0 \\ B^{-1}(y) \end{pmatrix} I &= \begin{pmatrix} I & 0 \\ 0 & |\lambda^{1/2}|I \end{pmatrix} \mathbf{O}(1) \begin{pmatrix} I & 0 \\ 0 & |\lambda^{-1/2}|I \end{pmatrix} \begin{pmatrix} 0 \\ B^{-1}(y) \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{O}(\lambda^{-1/2}) \\ \mathbf{O}(1) \end{pmatrix}.\end{aligned}$$

Combining with (7.26), and recalling that $\beta > \tilde{\beta}$ was arbitrary in (7.16), we obtain the claimed bounds on $|G_\lambda|$ and $|G_{\lambda_x}|$. The bound on $|G_{\lambda_y}|$ follows by symmetric argument applied to the adjoint operator $L^*$, or, equivalently, using the symmetric representation

$$(G_\lambda \quad G_{\lambda_y}) = (0 \quad B^{-1}(x))\tilde{\Pi}_-(x)\tilde{\mathcal{F}}^{x\to y},$$

where $\tilde{\mathcal{F}}^{x\to y}$ denotes the flow of the adjoint eigenvalue equation. □

***Remark.*** The result $(\mathbb{K})$ is equivalent to Proposition 4.1 of [GZ], itself a slight extension of Proposition 2.1 in [AGJ]. Indeed, our argument is equivalent to those given in the references, though presented in different language. We have chosen to present the analysis in a manner highlighting the connection with coercivity.

***Corollary 7.4.*** *Given* (**C1**), *the parabolic operator $\partial/\partial t - L$ has a Green's function $G(x,t;y) \in C^{(1,0)+(\tilde{\alpha},\tilde{\alpha}/2)}(x,t)$, $\tilde{\alpha} > 0$, for each fixed $y$ and $(x,t) \neq (y,0)$, given by*

$$G(x,t;y) = \frac{1}{2\pi i}\int_{\partial(\Omega_\theta\setminus B(0,R))} e^{\lambda t} G_\lambda(x,y)\,d\lambda, \tag{7.27}$$

*for $R > 0$ sufficiently large and $\theta_1, \theta_2 > 0$ sufficiently small.*

*Proof.* Using the spatial decay of $G_\lambda$ given in the previous proposition together with the $\lambda$-decay of $e^{\lambda t}$, we have, for $y$ fixed, that $e^{\lambda t}G_\lambda(x,y)$ is in $L^1(\lambda, x, t)$, where $\lambda$ is restricted to lie in $\partial(\Omega_\theta \setminus B(0,R))$. An application of Fubini's Theorem then gives that distributional $x$- and $t$-derivatives commute with the $\lambda$-integration in (7.27), since orders of $x$,$t$, and $\lambda$ integration against a test function can be exchanged. Thus,

$$\begin{aligned}(\partial/\partial t - L)G &= \frac{1}{2\pi i}\int \big(\lambda e^{\lambda t}G_\lambda - (e^{\lambda t}\lambda G_\lambda + \delta_y(x))\big)\,d\lambda \\ &= \frac{-1}{2\pi i}\int e^{\lambda t}\delta_y(x)\,d\lambda \\ &= \delta_y(x)\delta_0(t),\end{aligned}$$

the final step following by standard Laplace Transform facts.

Likewise, the stated regularity in $x$ and $t$ follows from the known bounds on $G_\lambda(x,y)$, $G_{\lambda x}(x,y)$, and $e^{\lambda t}$, and their Hölder quotients in $x$ and $t$ respectively, through direct estimation of their integrals assuming $(x,t) \neq (y,0)$. (Note: the integral estimates involved are the same as for the heat equation, since we have comparable bounds on the integrand). The Hölder bounds on $G_{\lambda x}$ are the only new bounds required. These follow for $x \neq y$ from the corresponding bounds on $\phi_j^\pm$ (see Remark 3.5, representation (4.27)), and can be extended to all $x$, $y$ by subtracting out the known jump, $B^{-1}(y)h(x-y)$, where $h(\cdot)$ denotes the Heaviside function. Observing that

$$\frac{1}{2\pi i}\int e^{\lambda t}B^{-1}(y)h(x-y)\,d\lambda = B^{-1}(y)h(x-y)\delta_0(t)$$

is zero for $t \neq 0$, we are done. □

***Remark 7.5.*** The observation of Remark 3.5 shows, more generally, that our construction gives a Green's function with optimal regularity $C^{(s-q,0)+(\tilde\alpha,\tilde\alpha/s)}$ for an operator $\partial/\partial t - L$ with $L$ a sectorial operator of the general form $Lw = D^{s-q}(A^sD^qw) + \cdots + A^0w$ described there, $A^j \in C^{0+\tilde\alpha}$ (cf. Proposition 11.3). In the non–divergence-form case, $q = 0$, we obtain the classical regularity result $G \in C^{(s,1)+(\tilde\alpha,\tilde\alpha/s)}$.

## 8. Pointwise Bounds on $G$.

We are now ready to carry out the central calculation of this paper, the estimation of the parabolic Green's function $G(x,t;y)$. We first carry out the relatively simple calculation hinted at in the introduction, of behavior in the inner shock layer, verifying *necessity* of the stability condition $(\mathcal{D})$. Next, we motivate our strategy for more general pointwise bounds through a brief discussion of the scalar, constant-coefficient case. Finally, we determine bounds on $G(x,t;y)$ in all temporo-spatial regimes.

### 8.1. Dynamics of the inner shock layer.

**Proposition 8.1.** *Let* (**C0**)–(**C3**) *hold. Then, there exists* $\eta > 0$ *such that, for* $x$, $y$ *restricted to any bounded set, and* $t$ *sufficiently large,*

$$G(x,y;t) = \sum_{\lambda \in \sigma_p'(L) \cap \{\mathrm{Re}(\lambda) \geq 0\}} e^{\lambda t} \sum_{k \geq 0} t^k (L - \lambda I)^k P_\lambda(x,y) + \mathbf{O}(e^{-\eta t}), \tag{8.1}$$

*where* $P_\lambda(x,y)$ *is the projection kernel described in Definition 5.1.*

*Proof.* By Corollary 7.4, we have

$$G(x,t;y) = \frac{1}{2\pi i} \int_{\partial(\Omega_\theta \setminus B(0,R))} e^{\lambda t} G_\lambda(x,y)\, d\lambda, \tag{8.2}$$

for $R$ sufficiently large, $\theta_1$, $\theta_2 > 0$ sufficiently small. By taking $\theta$ still smaller if necessary, we can arrange further that

$$\sigma_p'(L) \cap \Omega_\theta \subset \{\mathrm{Re}\lambda \geq 0\}, \tag{8.3}$$

since the effective point spectrum is isolated, and restricted to a bounded region of $\Omega_\theta$ (Proposition 7.3).

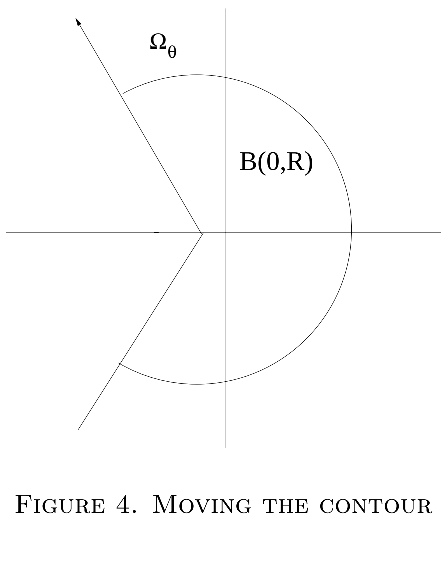


FIGURE 4. MOVING THE CONTOUR

Recall that $G_\lambda$ is meromorphic on $\Omega_\theta$. Thus, expanding the right hand side of (8.2) as

$$\int_{\partial(\Omega_\theta \setminus B(0,R))} = \int_{\partial\Omega_\theta} + \int_{\partial(\Omega_\theta \cap B(0,R))}$$

(see Figure 4) and using Cauchy's Theorem, we find that

$$G(x,t;y) = \frac{1}{2\pi i}\int_{\partial\Omega_\theta} e^{\lambda t} G_\lambda(x,y)d\lambda + \mathrm{Res}_{\lambda\in\Omega_\theta\cap B(0,R)} e^{\lambda t} G_\lambda(x,y). \tag{8.4}$$

By Definition 5.1 and Proposition 5.3,

$$\begin{aligned}
&\mathrm{Res}_{\lambda\in\Omega_\theta\cap B(0,R)} e^{\lambda t} G_\lambda(x,y) \\
&\qquad = \sum_{\lambda_0\in\sigma_p'(L)\cap\Omega_\theta} e^{\lambda_0 t}\mathrm{Res}_{\lambda_0} e^{(\lambda-\lambda_0)t} G_\lambda(x,y) \\
&\qquad = \sum_{\lambda_0\in\sigma_p'(L)\cap\Omega_\theta} e^{\lambda_0 t}\sum_{k\geq 0}(t^k/k!)\mathrm{Res}_{\lambda_0}(\lambda-\lambda_0)^k G_\lambda(x,y) \\
&\qquad = \sum_{\lambda_0\in\sigma_p'(L)\cap\Omega_\theta} e^{\lambda_0 t}\sum_{k\geq 0}(t^k/k!)Q_{\lambda_0,k}(x,y) \\
&\qquad = \sum_{\lambda_0\in\sigma_p'(L)\cap\Omega_\theta} e^{\lambda_0 t}\sum_{k\geq 0}(t^k/k!)(L-\lambda_0 I)^k P_{\lambda_0}(x,y)).
\end{aligned} \tag{8.5}$$

On the other hand, for $x$, $y$ bounded and $t$ sufficiently large, $t$ dominates $|x|$ and $|y|$ and we obtain from Propositions 7.1–7.3 that

$$|e^{\lambda t} G_\lambda(x,t;y)| \leq C e^{\gamma \mathrm{Re}\lambda t}$$

for all $\lambda \in \partial\Omega_\theta$, for some $\gamma > 0$ (recall that we have chosen $\theta$ so that $\sigma_p'(L)\cap \partial\Omega_\theta = \varnothing$), hence

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\partial\Omega_\theta} e^{\lambda t} G_\lambda(x,y)\,d\lambda &\leq C e^{-\eta t}\int_{\partial\Omega_\theta} e^{-\eta_1|\mathrm{Im}\lambda|t}\,d\lambda \\
&\leq C_2 e^{-\eta t},
\end{aligned} \tag{8.6}$$

for some $\eta$, $\eta_1 > 0$. Combining (8.4), (8.5), and (8.6), we obtain the result. □

***Corollary 8.2.*** *Let* (**C0**)–(**C3**) *hold. Then,* ($\mathcal{D}$) *is necessary for linearized orbital stability of* (1.6) *with respect to* $C^\infty_{\text{exp}}$ *perturbations* (*recall: perturbations decaying exponentially rapidly in all derivatives*).

*Proof.* From (8.1), we find that (1.6) is linearly orbitally stable only if

$$P_\lambda = 0 \quad \text{for all } \lambda \in \{\mathrm{Re}\lambda \geq 0\} \setminus \{0\}, \text{ and}$$

$$\text{Range } \mathcal{P}_0 = \text{Span}\left\{ \frac{\partial \bar{u}^\delta}{\partial \delta_j} \right\}.$$

By Theorems 5.3 and 6.4, this is equivalent to ($\mathcal{D}$). □

***8.2. A motivating example.*** Before presenting our main analysis, we indicate some of the general features of the argument in the simplest case of a scalar, constant-coefficient equation,

$$v_t = Lv := -av_x + bv_{xx}. \tag{8.7}$$

This is most easily treated by Fourier analysis. Taking Fourier transforms, we find that the Green's function of (8.7) is given by

$$G(x, t; y) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{ik(x-y)} e^{\lambda(k)t} dk, \tag{8.8}$$

where

$$\lambda(k) = -iak - bk^2 \tag{8.9}$$

is the dispersion relation for $L$.

Spatial decay as $|x| \to \infty$ is a consequence of the cancellation induced by the combination of rapid oscillation in $e^{ikx}$ and smoothness in the term $e^{\lambda(k)t}$. This cancellation can be revealed by shifting the original contour of integration,

$$\mathrm{Re}(ik) \equiv 0, \tag{8.10}$$

using Cauchy's Theorem, to

$$\mathrm{Re}(ik) \equiv \bar{\alpha} := \frac{-(x - y - at)}{2bt}. \tag{8.11}$$

Writing $k = \tilde{k} + i\bar{\alpha}$, we obtain after some rearrangement the sharp decay estimate

$$\begin{aligned}(8.12)\qquad |G(x,t;y)| \leq& \frac{1}{2\pi}\int_{-\infty}^{+\infty} |e^{ik(x-y)}e^{(-ika-k^2b)t}|\,dk\\ \leq& \frac{e^{-b\bar{\alpha}^2 t}}{2\pi}\int_{-\infty}^{+\infty} e^{-\tilde{k}^2 t}\,d\tilde{k}\\ \leq& Ct^{-1/2}e^{(-|x-y-at|^2)/(4bt)}.\end{aligned}$$

In fact, (8.12) gives an exact inversion of the Fourier Transform in this case. However, the point is that we have obtained this bound using only *modulus information*; thus, the method applies in much greater generality. This type of "Paley–Wiener" estimate has been used successfully in, e.g. [Jo, Z, LZ, HoZ.1-2]; for further description, we refer the reader to [HoZ.1].

If we take instead the Laplace Transform, we obtain in place of (8.8) the spectral resolution formula

$$(8.13)\qquad G(x,t;y) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}G_\lambda(x,y)\,d\lambda,$$

where $G_\lambda(x,y)$ as usual is the elliptic Green's function of the operator $(L - \lambda I)$. This can be computed by the methods of Section 4 as

$$(8.14)\qquad G_\lambda(x,y) = \frac{e^{\mu(\lambda)(x-y)}}{\Delta\mu},$$

for $x > y$, where

$$(8.15)\qquad \mu(\lambda) = \frac{a - \sqrt{a^2 + 4b\lambda}}{2b}; \quad \Delta\mu = \sqrt{a^2 + 4b\lambda}.$$

Thus, we can write (8.13) equivalently as

$$(8.16)\qquad G(x,t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}e^{\mu(\lambda)(x-y)}\frac{d\lambda}{\Delta\mu(\lambda)}.$$

The relation to (8.8) is now clear. Taking $\Gamma = \{\lambda : \mu(\lambda) = ik\}$ in (8.13), or

$$(8.17)\qquad \mathrm{Re}(\mu) \equiv 0,$$

and solving for $\mu(\lambda) = ik$, we obtain the dispersion relation (8.9). Likewise, it is easily checked that

$$\frac{1}{\Delta\mu(\lambda)} = \frac{dk}{d\lambda}.$$

To reveal the cancellation in (8.13), we can therefore mimic the Fourier case, choosing the parabolic contour

$$\text{Re}(\mu(\lambda)) \equiv \bar{\alpha} \tag{8.18}$$

as in (8.11), to obtain *exactly the same estimates.*

Like the exact inversion in (8.12), however, this is somewhat accidental. A more direct, and general approach to the problem of detecting cancellation in an analytic integral $\int_\Gamma e^{f(z)}dz$, using only modulus bounds, is given by the *Riemann saddlepoint method* [R, CH, DeB]. This method *prescribes* the optimal path of integration as a mountainpass, or minimax, contour minimizing $\max_{z\in\Gamma}|e^{f(z)}|$, or equivalently $\max_{z\in\Gamma}\text{Re}\, f(z)$. In the present case, the argument $\lambda t + \mu(\lambda)(x-y)$ of the integrand in (8.16) respects conjugation, hence its real part is symmetric about the real axis. Further, restricting to the real axis, we find that $\text{Re}(\lambda t + \mu(\lambda)(x-y))$ has a global minimum of $-\bar{\alpha}^2 t$ occurring at the point

$$\lambda_{\min} = \bar{\alpha}(\bar{\alpha}b - a), \tag{8.19}$$

where $\mu(\lambda) = \bar{\alpha}$, $\bar{\alpha}$ defined as in (8.11). It follows that $\lambda_{\min}$ is a *stationary point* of $\text{Re}(\lambda t + \mu(\lambda)(x-y))$, in fact a *saddle*. The choice of $\Gamma = \{\lambda : \text{Re}(\mu(\lambda)) = \bar{\alpha}\}$, likewise, is a *mountain pass*, or minimax contour for the integral (8.13), justifying our previous choice (8.18). This approach will guide us also in the general case.

It is worth noting the behavior of (8.19) for $|\bar{\alpha}|$ near 0 and $\infty$; this scaling will serve as a useful guideline in the analyis to come. In the large-$|\bar{\alpha}|$, or equivalently the large-$|x-y|/t$ regime, (8.19) becomes

$$\lambda_{\min} = b\bar{\alpha}^2 + \mathbf{O}(\bar{\alpha}); \quad \bar{\alpha} = -(x-y)/2bt + \mathbf{O}(1), \tag{8.20}$$

and the contour (8.18) becomes

$$\text{Re}\lambda^{1/2} = b^{1/2}\bar{\alpha} + \mathbf{O}(1). \tag{8.21}$$

That is, both representations correspond to the classical large $\lambda$/short time behavior predicted by the heat equation ($a = 0$). In the small-$\bar{\alpha}$ regime, lying near the characteristic path $x = y + at$, we find instead

$$\lambda_{\min} = -a\bar{\alpha} + \mathbf{O}(\bar{\alpha}^2) \sim a(x-y-at)/2bt. \tag{8.22}$$

This corresponds to critical small $\lambda$/large time behavior *within a single scalar mode* (note: (8.22) must be modified to treat scattering, which necessarily involves more than one scalar mode).

It is worth noting also that the optimal contours even for this simple scalar example enter the left complex half-plane for small $a\bar{\alpha} > 0$, that is, into the essential spectrum. This shows the importance of the analytic extension of $G_\lambda$ into this region.

***8.3. Main calculation.*** We now establish the following detailed description of the parabolic Green's function and its derivatives.

***Theorem 8.3.*** *Let* $(\mathcal{D})$ *hold, as well as* (**C0**)–(**C3**). *Then, the Green's function* $G(x,t,y)$ *of* (1.6) *can be decomposed as*

$$G(x,t;y) = S(x,t;y) + E(x,t;y) + R(x,t;y), \tag{8.23}$$

*where*

$$\begin{aligned} S(x,t;y) = &\sum_{k,\pm} \mathbf{O}(t^{-1/2} e^{-(x-y-a_k^\pm t)^2/Mt}) \\ &\times\ (r_k^\pm \chi_{\{x\gtrless 0\}} + \mathbf{O}(e^{-\eta|x|}))(l_k^\pm \chi_{\{y\gtrless 0\}} + \mathbf{O}(e^{-\eta|y|})) \\ &+ \sum_{a_k^\pm \lessgtr 0, a_j^\pm \gtrless 0} \chi_{\{t\geq |y/a_k^\pm|\}} \mathbf{O}(t^{-1/2} e^{-(x-a_j^\pm(t-|y/a_k^\pm|))^2/Mt}) \\ &\times\ (r_j^\pm \chi_{\{x\gtrless 0\}} + \mathbf{O}(e^{-\eta|x|}))(l_k^\pm \chi_{\{y\gtrless 0\}} + \mathbf{O}(e^{-\eta|y|})) \end{aligned} \tag{8.24}$$

*comprises the scattering modes,*

$$\begin{aligned} E(x,t;y) = &\sum_{k,y\gtrless 0} \chi_{\{|x-y|\leq |a_k \pm t|\}} \varphi_k^\pm(x) \pi_k^\pm(y) \\ &+\ \mathbf{O}(e^{-(x-y-a_k\pm t)^2/Mt} + e^{-(x-y+a_k\pm t)^2/Mt}) e^{-\eta|x|} \pi_k^\pm(y) \end{aligned} \tag{8.25}$$

*comprises the excited modes, and* $R = R_S + R_E$, *with*

$$
\begin{aligned}
(8.26)\quad R_S(x,t;y) &= \sum_{k,\pm} \mathbf{O}(t+1)^{-1/2}t^{-1/2}e^{-(x-y-a_k^{\pm}t)^2/Mt}e^{-\eta x^{\pm}}e^{-\eta y^{\pm}} \\
&+ \sum_{a_k^{\pm}\lessgtr 0,\, a_j^{\pm}\gtrless 0} \chi_{\{|a_k^{\pm}t|\ge|y|\}}\mathbf{O}(t+1)^{-1/2}t^{-1/2}e^{-(x-a_j^{\pm}(t-|y/a_k^{\pm}|))^2/Mt} \\
&\qquad \times e^{-\eta x^{\pm}}e^{-\eta y^{\pm}},
\end{aligned}
$$

$$
R_E = \sum_{k,y\gtrless 0} \mathbf{O}(t^{-1/2}e^{-\eta|x|}e^{-(x-y-a_k^{\pm}t)^2/Mt}).
$$

*Here, $\eta > 0$, $M > 0$ is a suitably large constant, $x^{\pm}$ denotes the positive/negative part of $x$, $a_j^{\pm}$ denote the eigenvalues of $A_{\pm}$ and $r_j^{\pm}$ and $l_j^{\pm}$ the corresponding right and left eigenvectors, and $\varphi_k^{\pm} \in \Sigma_0'(L)$, $\pi^{\pm} \in \Sigma_0'(L^*)$. If $\Sigma_0'(L) = \varnothing$, then $E \equiv R_E \equiv 0$.*

*Likewise, we have $G_x = S^1 + E^1 + R_S^1$ and $G_y = S^2 + E^2 + R_S^2$, where*

$$
\begin{aligned}
S^1 &= \sum_{k,\pm} \mathbf{O}(t^{-1}e^{-(x-y-a_k^{\pm}t)^2/Mt}) \\
&\quad \times (r_k^{\pm}\chi_{\{x\gtrless 0\}} + \mathbf{O}(e^{-\eta|x|}))(l_k^{\pm}\chi_{\{y\gtrless 0\}} + \mathbf{O}(e^{-\eta|y|})) \\
&\quad + \sum_{k,\pm} \mathbf{O}(t^{-1/2}e^{-(x-y-a_k^{\pm}t)^2/Mt}e^{-\eta|x|})e^{-\eta y^{\mp}} \\
&\quad + \sum_{a_k^{\pm}\lessgtr 0,\, a_j^{\pm}\gtrless 0} \chi_{\{|a_k^{\pm}t|\ge|y|\}}\mathbf{O}(t^{-1}e^{-(x-a_j^{\pm}(t-|y/a_k^{\pm}|))^2/Mt}) \\
&\quad \times (r_k^{\pm}\chi_{\{x\gtrless 0\}} + \mathbf{O}(e^{-\eta|x|}))(l_k^{\pm}\chi_{\{y\gtrless 0\}} + \mathbf{O}(e^{-\eta|y|})) \\
&\quad + \sum_{a_k^{\pm}\lessgtr 0,\, a_j^{\pm}\gtrless 0} \chi_{\{|a_k^{\pm}t|\ge|y|\}}\mathbf{O}(t^{-1/2}e^{-(x-a_j^{\pm}(t-|y/a_k^{\pm}|))^2/Mt}e^{-\eta|x|})e^{-\eta y^{\mp}},
\end{aligned}
$$

$$
\begin{aligned}
E^1 &= \sum_{k,y\gtrless 0} \chi_{\{|x-y|\le|a_k^{\pm}t|\}}\varphi_k^{\pm\prime}(x)\pi_k^{\pm}(y) \\
&\quad + \sum_{k,y\gtrless 0} \mathbf{O}(e^{-(x-y-a_k^{\pm}t)^2/Mt} + e^{-(x-y+a_k^{\pm}t)^2/Mt})e^{-\eta|x|}\pi_k^{\pm}(y) \\
&\quad + \sum_{k,y\gtrless 0} \mathbf{O}(t^{-1/2}e^{-(x-y-a_k^{\pm}t)^2/Mt} + t^{-1/2}e^{-(x-y+a_k^{\pm}t)^2/Mt})e^{-\eta|x|}\pi_k^{\pm}(y),
\end{aligned}
$$

$$R_S^1 = \sum_{k,\pm} \mathbf{O}(t+1)^{-1/2} t^{-1} e^{-(x-y-a_k^{\pm}t)^2/Mt}) e^{-\eta x^{\pm}} e^{-\eta y^{\pm}}$$

$$+ \sum_{a_k^{\pm} \lessgtr 0, a_j^{\pm} \gtrless 0} \chi_{\{|a_k^{\pm} t| \geq |y|\}} \mathbf{O}(t+1)^{-1/2} t^{-1} e^{-(x-a_j^{\pm}(t-|y/a_k^{\pm}|))^2/Mt} e^{-\eta x^{\pm}} e^{-\eta y^{\pm}},$$

*and*

$$S^2 = \sum_{k,\pm} \mathbf{O}(t^{-1} e^{-(x-y-a_k^{\pm}t)^2/Mt})$$

$$\times \; (r_k^{\pm} \chi_{\{x \gtrless 0\}} + \mathbf{O}(e^{-\eta|x|}))(l_k^{\pm} \chi_{\{y \gtrless 0\}} + \mathbf{O}(e^{-\eta|y|}))$$

$$+ \sum_{k,\pm} \mathbf{O}(t^{-1/2} e^{-(x-y-a_k^{\pm}t)^2/Mt} e^{-\eta|y|}) e^{-\eta x^{\mp}}$$

$$+ \sum_{a_k^{\pm} \lessgtr 0, a_j^{\pm} \gtrless 0} \chi_{\{|a_k^{\pm} t| \geq |y|\}} \mathbf{O}(t^{-1} e^{-(x-a_j^{\pm}(t-|y/a_k^{\pm}|))^2/Mt})$$

$$\times \; (r_k^{\pm} \chi_{\{x \gtrless 0\}} + \mathbf{O}(e^{-\eta|x|}))(l_k^{\pm} \chi_{\{y \gtrless 0\}} + \mathbf{O}(e^{-\eta|y|}))$$

$$+ \sum_{a_k^{\pm} \lessgtr 0, a_j^{\pm} \gtrless 0} \chi_{\{|a_k^{\pm} t| \geq |y|\}} \mathbf{O}(t^{-1/2} e^{-(x-a_j^{\pm}(t-|y/a_k^{\pm}|))^2/Mt} e^{-\eta|y|}) e^{-\eta x^{\mp}},$$

$$E^2 = \sum_{k, y \gtrless 0} \chi_{\{|x-y| \leq |a_k^{\pm} t|\}} \varphi_k^{\pm}(x) \pi_k^{\pm\prime}(y)$$

$$+ \sum_{k, y \gtrless 0} \mathbf{O}(e^{-(x-y-a_k^{\pm}t)^2/Mt} + e^{-(x-y+a_k^{\pm}t)^2/Mt}) e^{-\eta|x|} \pi_k^{\pm\prime}(y)$$

$$+ \sum_{k, y \gtrless 0} \mathbf{O}(t^{-1/2} e^{-(x-y-a_k^{\pm}t)^2/Mt} + t^{-1/2} e^{-(x-y+a_k^{\pm}t)^2/Mt}) e^{-\eta|x|} \pi_k^{\pm}(y),$$

$$R_S^2 = \sum_{k,\pm} \mathbf{O}(t+1)^{-1/2} t^{-1} e^{-(x-y-a_k^{\pm}t)^2/Mt}) e^{-\eta x^{\pm}} e^{-\eta y^{\pm}}$$

$$+ \sum_{a_k^{\pm} \lessgtr 0, a_j^{\pm} \gtrless 0} \chi_{\{|a_k^{\pm} t| \geq |y|\}} \mathbf{O}(t+1)^{-1/2} t^{-1} e^{-(x-a_j^{\pm}(t-|y/a_k^{\pm}|))^2/Mt} e^{-\eta x^{\pm}} e^{-\eta y^{\pm}}.$$

*(Recall that $\varphi'(\xi)$, $\pi'(\xi) = \mathbf{O}(e^{-\eta|\xi|})$ for any $\phi \in \Sigma_0'(L)$, $\pi \in \Sigma_0'(L^*)$.)*

***Remark 8.4.*** For $t$ sufficiently small, the $\mathbf{O}(1)$ term $E$ is negligible compared to the $\mathbf{O}(t^{-1/2})$ scattering terms. Moreover, it is easily checked that

$$2|x-y|^2/t - 5 \le (x-y-at)^2/t \le |x-y|^2/2t + 5$$

for any bounded $a$. Thus, combining all terms $S$, $E$, and $R$, we obtain for $t$ sufficiently small, that

$$(1/C_1)t^{-1/2}e^{|x-y|^2/M_1 t} \le |G(x,t;y)| \le t^{-1/2}C_2 e^{|x-y|^2/M_2 t}, \tag{8.27}$$

where the inequalities hold *in each entry* of $G$. Similarly,

$$(1/C_1)t^{-1}e^{|x-y|^2/M_1 t} \le |G_x(x,t;y)|, |G_y(x,t;y)| \le t^{-1}C_2 e^{|x-y|^2/M_2 t}, \tag{8.28}$$

That is, our bounds for short time reduce to the classical bounds given by the heat kernel.

For bounded $x$, $y$, our large time bounds of necessity reduce to $P_0(x,y) + \mathbf{O}(e^{-\eta t})$ as predicted by Proposition 8.1. The new information contained in (8.23)–(8.26) is the detailed scattering picture of *intermediate-time* behavior. in terms of distinct signals interacting with the shock.

***Remark 8.5.*** The bounds above hold also for non–divergence-form and mixed-type operators $L := -Av_x + Bv_{xx}$ and $L := -Av_x + (Bv_x)_x$; for non–divergence-form operators, there hold also analogous bounds on second order spatial derivatives of $G$. For fully divergence-form operators $L := -(Av)_x + (Bv)_{xx}$, $G$ satisfies the bounds above, but $G_x$ and $G_y$ may not exist.

***Example 8.6.*** In the case of a *Burgers Shock*, $\overline{u}(x) = -\tanh(\frac{x}{2})$, of the scalar Burgers equation $u_t + (u^2/2)_x = u_{xx}$, the linearized equation (1.6) can be solved exactly by linearized Hopf–Cole transformation, [Sat, Z.1, LZ.1, GSZ], to give an explicit formula for the Green's function of:

$$\begin{aligned} G(x,t;y) = &\Bigg[\left(\frac{e^{-x/2}}{e^{x/2}+e^{-x/2}}\right)(4\pi t)^{-1/2}e^{-(x-y-t)^2/(4t)} \\ &\qquad + \left(\frac{e^{x/2}}{e^{x/2}+e^{-x/2}}\right)(4\pi t)^{-1/2}e^{-(x-y+t)^2/(4t)}\Bigg] \\ &+ \frac{1}{2}\frac{\partial \overline{u}}{\partial x}\left[\operatorname{errfn}\left(\frac{x-y-t}{\sqrt{4t}}\right) - \operatorname{errfn}\left(\frac{x-y+t}{\sqrt{4t}}\right)\right]. \end{aligned} \tag{8.29}$$

For the various choices of $x \lessgtr 0$, $y \lessgtr 0$, formula (8.29) can be rearranged as in (8.23)–(8.26); for example, in case $x, y < 0$, we have

$$S = (4\pi t)^{-1/2} e^{-(x-y-t)^2/(4t)},$$

$$E = \frac{1}{2}\frac{\partial \bar{u}}{\partial x}\left[\text{errfn}\left(\frac{x-y-t}{\sqrt{4t}}\right) - \text{errfn}\left(\frac{x-y+t}{\sqrt{4t}}\right)\right],$$

and

$$\begin{aligned} R &= \left(\frac{e^{x/2}}{e^{x/2}+e^{-x/2}}\right)(4\pi t)^{-1/2} e^{-(x-y+t)^2/(4t)} \\ &\quad + \left(\frac{e^{-x/2}}{e^{x/2}+e^{-x/2}} - 1\right)(4\pi t)^{-1/2} e^{-(x-y-t)^2/(4t)} \\ &= \mathbf{O}(e^{-|x|/2})\left[t^{-1/2} e^{-(x-y+t)^2/(4t)} + t^{-1/2} e^{-(x-y-t)^2/(4t)}\right]. \end{aligned}$$

Here, $\phi_1 = \bar{u}_x \in \Sigma_0'(L)$ and $\pi_1 = 1/2 \in \Sigma_0'(L^*)$. This verifies that the bounds of Theorem 8.3 are sharp, for all $t \geq 0$.

Likewise,

$$\begin{aligned} G_y(x,t;y) &= \left(\frac{e^{-x/2}}{e^{x/2}+e^{-x/2}}\right)\mathbf{O}(t^{-1} e^{-(x-y-t)^2/(8t)}) \\ &\quad + \left(\frac{e^{x/2}}{e^{x/2}+e^{-x/2}}\right)\mathbf{O}(t^{-1} e^{-(x-y+t)^2/(8t)}) \\ &\quad + \frac{1}{2}\frac{\partial \bar{u}}{\partial x}\left[(4\pi t)^{-1/2} e^{-(x-y-t)^2/(4t)} - (4\pi t)^{-1/2} e^{-(x-y+t)^2/(4t)}\right] \end{aligned} \tag{8.30}$$

is described sharply by (8.26), with $\pi' \equiv 0$; this observation is quite relevant to our treatment of Lax and overcompressive shocks in Section 11. Differentiation with respect to $x$ shows that our $G_x$ bounds are sharp as well.

***Proof of Theorem 8.3.*** Let $\theta_1 > 0$, $\theta_2 > 0$ be chosen sufficiently small, in particular so small as to satisfy the hypotheses of all previous assertions. It follows from assumption $(\mathcal{D})$, the large $|\lambda|$ bounds of Proposition 7.3, and analyticity of the Evans function $D_L(\lambda)$ that the effective point spectrum of $L$ in $\Omega_\theta$ (corresponding to roots of $D_L$) consists of the origin, $\lambda = 0$, plus a bounded set of finitely many, isolated eigenvalues, each with strictly negative real part. Choosing $\theta_1, \theta_2$ still smaller, if necessary, we can thus arrange that $\lambda = 0$ *is the unique effective eigenvalue of* $L$ *contained in* $\Omega_\theta$.

It follows from Corollary 7.4 and Cauchy's Theorem that

$$G(x,t;y) = \int_\Gamma e^{\lambda t} G_\lambda(x,y)\, d\lambda, \tag{8.31}$$

for any contour $\Gamma$ that can be expressed as

$$\Gamma = \partial(\Omega_\theta \setminus \mathcal{S})$$

for some set $\mathcal{S}$ containing $\lambda = 0$.

***Case I.*** $|x-y|/t$ ***large.*** We first treat the trivial case that $|x-y|/t \geq S$, $S$ sufficiently large, the regime in which standard short-time parabolic theory applies. Loosely following (8.20)–(8.21), set

$$\bar\alpha := \frac{|x-y|}{2\beta t}, \quad R := \beta \bar\alpha^2, \tag{8.32}$$

where $\beta$ is as in Lemma 7.3, and consider again the representation (8.2) of $G$, that is

$$G(x,t;y) = \int_{\Gamma_1 \cup \Gamma_2} e^{\lambda t} G_\lambda(x,y)\, d\lambda, \tag{8.33}$$

where $\Gamma_1 := \partial B(0,R) \cap \bar\Omega_\theta$ and $\Gamma_2 := \partial\Omega_\theta \setminus B(0,R)$ (see Figure 5). Note that the intersection of $\Gamma$ with the real axis is $\lambda_{\min} = R = \beta\bar\alpha^2$, in agreement with (8.20).

By the large $|\lambda|$ estimates of Proposition 7.3, we have for all $\lambda \in \Gamma_1 \cup \Gamma_2$ that

$$|G_\lambda(x,y)| \leq C|\lambda^{-1/2}| e^{-\beta^{-1/2}|\lambda^{1/2}|\, |x-y|} \tag{8.34}$$

Further, we have

$$\begin{aligned} &\mathrm{Re}\lambda \leq R(1-\eta\omega^2), \quad \lambda \in \Gamma_1, \\ &\mathrm{Re}\lambda \leq \mathrm{Re}\lambda_0 - \eta(|\mathrm{Im}\lambda| - |\mathrm{Im}\lambda_0|), \quad \lambda \in \Gamma_2 \end{aligned} \tag{8.35}$$

for $R$ sufficiently large, where $\omega$ is the argument of $\lambda$ and $\lambda_0$ and $\lambda_0^*$ are the two points of intersection of $\Gamma_1$ and $\Gamma_2$, for some $\eta > 0$ independent of $\bar\alpha$.

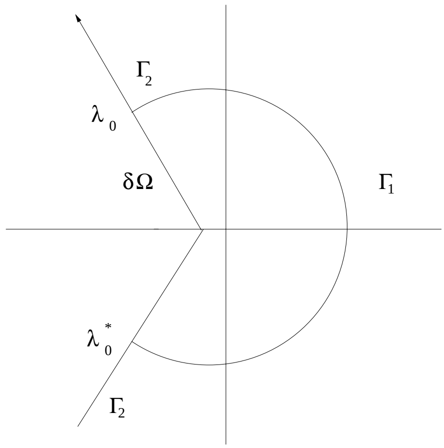


FIGURE 5. CONTOUR FOR $\frac{|x-y|}{t} \geq S$.

Combining (8.34), (8.35), and (8.32), we obtain

$$\begin{aligned}\left|\int_{\Gamma_1} e^{\lambda t} G_\lambda \, d\lambda\right| &\leq \int_{\Gamma_1} C|\lambda^{-1/2}| e^{\mathrm{Re}\lambda t - \beta^{-1/2}|\lambda^{1/2}|\ |x-y|} \, d\lambda \\ &\leq C e^{-\beta\bar{\alpha}^2 t} \int_{-L}^{+L} R^{-1/2} e^{-\beta R\eta\omega^2 t} R \, d\omega \\ &\leq C t^{-1/2} e^{-\beta\bar{\alpha}^2 t}.\end{aligned}$$

Likewise,

$$\begin{aligned}\left|\int_{\Gamma_2} e^{\lambda t} G_\lambda \, d\lambda\right| & \\ &\leq \int_{\Gamma_2} C|\lambda^{-1/2}| C e^{\mathrm{Re}\lambda t - \beta^{-1/2}|\lambda^{1/2}|\ |x-y|\, d\lambda} \\ &\leq C e^{\mathrm{Re}(\lambda_0)t - |\beta^{-1/2}|\lambda_0^{1/2}||x-y|} \int_{\Gamma_2} |\lambda^{-1/2}| e^{(\mathrm{Re}\lambda) - \mathrm{Re}\lambda_0)t} \ |d\lambda| \\ &\leq C e^{-\beta\bar{\alpha}^2 t} \int_{\Gamma_2} |\mathrm{Im}\,\lambda|^{-1/2} e^{-\eta|\mathrm{Im}\,\lambda - \mathrm{Im}\,\lambda_0|t} \ |d\mathrm{Im}\lambda| \\ &\leq C t^{-1/2} e^{-\beta\bar{\alpha}^2 t}.\end{aligned} \tag{8.36}$$

Combining these last two estimates, and recalling (8.32), we have

$$|G(x,t;y)| \le Ct^{-1/2}e^{-\beta\bar\alpha^2 t/2}e^{-(x-y)^2/8\beta t} \le Ct^{-1/2}e^{-\eta t}e^{-(x-y)^2/8\beta t}, \tag{8.37}$$

for $\eta > 0$ independent of $\bar\alpha$. Observing that

$$\frac{|x-at|}{2t} \le \frac{|x-y|}{t} \le \frac{2|x-at|}{t}$$

for any bounded $a$, for $|x-y|/t$ sufficiently large, we find that $|G|$ can be absorbed in the residual term $R_S$ for $t \ge \varepsilon$, any $\varepsilon > 0$, by any summand

$$\mathbf{O}(t^{-1/2}(t+1)^{-1/2}e^{-(x-y-a_k^{\pm}t)^2/Mt})e^{-\eta x\pm}e^{-\eta y\pm}.$$

For $t$ small, on the other hand, (8.37) is equivalent to the short-time bound (8.27). Likewise, $|G_x|$ and $|G_y|$ can be absorbed in $R_S^1$ and $R_S^2$ for $t \ge \varepsilon$, and reduce to (8.28) for $t \le \varepsilon$.

***Case II.*** $|x-y|/t$ ***bounded.*** We now turn to the critical case that $|x-y|/t \le S$. A few remarks are in order at the outset. Our goal is to bound $|G|$ by its excited modes plus terms of form $Ct^{-1/2}e^{-\bar\alpha^2 t/M}$, where $\bar\alpha := (x-a_j^{\pm}(t-|y/a_k^{\pm}|)/2t$ or $\bar\alpha := (x-y-a_k^{\pm}t)/2t$ are now *uniformly bounded*, by

$$|x-y|/2t + \max_j\{|a_j^{\pm}|\}/2 \le S/2 + \max\{|a_j^{\pm}|\}/2.$$

Thus, in particular, *contributions of order* $t^{-1/2}e^{-\eta t}$, $\eta > 0$, *can be absorbed by the residual term* $R_S$, in any summand

$$\mathbf{O}(t^{-1/2}(t+1)^{-1/2}e^{-(x-y-a_k^{\pm}t)^2/Mt})e^{-\eta x\pm}e^{-\eta y\pm},$$

if we take $M$ sufficiently large. Likewise, for $G_x$ and $G_y$, contributions of order $t^{-1}e^{-\eta t}$ can be absorbed in $R_S^1$ and $R_S^2$. We will use this observation repeatedly.

In contrast to the previous case of large characteristic speed $|x-y|/t \ge S$, we are not trying to show rapid time-exponential decay. Rather, we are trying to show that the rate of exponential decay of the solution does not degrade too rapidly as $\bar\alpha \to 0$: precisely, that it vanishes to order $\bar\alpha^2$ and no more. Thus, the crucial part of our analysis will be for small $\bar\alpha$. All other situations can be estimated crudely as described just above.

Let $r$ be sufficiently small that the small-$|\lambda|$ bounds of Proposition 7.1 hold on $B(0,r)$. Next, choose $\theta_1$ and $\theta_2$ still smaller than before, if necessary, so that $\Omega_\theta \setminus B(0,r) \subset \Lambda$ and therefore the medium-$|\lambda|$ bounds of Proposition 7.2 hold on $\partial\Omega_\theta \setminus B(0,r)$. This implies that $\partial\Omega_\theta \cap B(0,r) \neq \emptyset$, giving the configuration pictured in Figure 6. Similarly as in the previous case, define $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1$ is the portion of the circle $\partial B(0,r)$ contained in $\bar{\Omega}_\theta$, and $\Gamma_2$ is the portion of $\partial\Omega_\theta$ outside $B(0,r)$. Writing

$$G(x,t;y) = \int_{\Gamma_1} e^{\lambda t} G_\lambda(x,y)\, d\lambda + \int_{\Gamma_2} e^{\lambda t} G_\lambda(x,y)\, d\lambda,$$

we separately estimate the terms $\int_{\Gamma_1}$ and $\int_{\Gamma_2}$.

***Large and medium $\lambda$ estimates.*** The $\int_{\Gamma_2}$ term is straightforward. The points $\lambda_0$, $\lambda_0^*$ where $\Gamma_1$ meets $\Gamma_2$ satisfy $\mathrm{Re}(\lambda_0) = -\eta < 0$. Moreover, combining the results of Propositions 7.2 and 7.3, we have the crude bound

$$|G_\lambda| \le C|\lambda|^{-1/2} \quad \text{for } \lambda \in \Gamma_2. \tag{8.38}$$

Thus, similarly as in (8.36), we have

$$\begin{aligned} \left| \int_{\Gamma_2} e^{\lambda t} G_\lambda \, d\lambda \right| &\le C e^{-\mathrm{Re}\,\lambda_0 t} \int_{\Gamma_2} |\mathrm{Im}\,\lambda|^{-1/2} e^{-\eta |\mathrm{Im}\,\lambda - \mathrm{Im}\,\lambda_0|} \, |d\mathrm{Im}\lambda| \\ &\le C t^{-1/2} e^{-\eta t}. \end{aligned} \tag{8.39}$$

This contribution can be absorbed into the residual term $R$ as described above. An analogous computation using $|G_{\lambda_x}|, |G_{\lambda_x}| \le C|\lambda|^{-1}$ shows that the $\Gamma_2$ contribution to $G_x$ and $G_y$ is $\mathbf{O}(t^{-1}e^{-\eta t})$, and can likewise be absorbed.

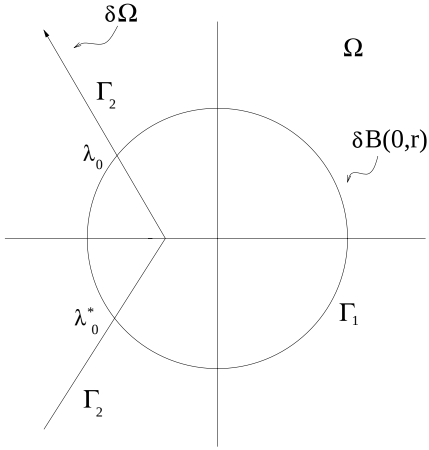


FIGURE 6. CONTOUR FOR $|x-y|/t \le S$.

***Small*** $|\lambda|$ ***estimates.*** It remains to estimate the critical term $\int_{\Gamma_1} e^{\lambda t} G_\lambda d\lambda$. This we will estimate in different ways, depending on the size of $t$.

***Bounded time.*** For $t$ bounded, we can use the medium-$\lambda$ bounds $|G_\lambda|$, $|G_{\lambda_x}|$, $|G_{\lambda_y}| \le C$ to obtain

$$\left|\int_{\Gamma_1} e^{\lambda t} G_\lambda \, d\lambda\right| \le C_2 |\Gamma_1|.$$

This contribution is order $Ce^{-\eta t}$ for bounded time, hence can be absorbed.

***Large time.*** For $t$ large, we must instead estimate $\int_{\Gamma_1} e^{\lambda t} G_\lambda \, d\lambda$ using the small-$|\lambda|$ expansions given in Proposition 7.1. First, observe that, by assumption $(\mathcal{D})$, all eigenfunctions in $\Sigma_0'$ are of ascent one, i.e. $G_\lambda$ has a *simple pole* at $\lambda = 0$. Thus, $K = 1$ in Proposition 7.1, and the coefficient functions $e_k(\lambda) = \mathbf{O}(\lambda^{1-K})$ are *analytic*, while the coefficient functions $d_{jk}(\lambda) = \mathbf{O}(\lambda^{-K})$ have at most a pole of order one at $\lambda = 0$.

Moreover, any *singular* term $\phi_j^{\pm} d_{jk} \tilde{\psi}_k^{\pm}$,

$$d_{jk} = d_{jk,-1}\lambda^{-1} + d_{jk,0} + d_{jk,1}\lambda + \cdots, \tag{8.40}$$

contributes

$$\begin{aligned} &\mathrm{Res}_0(d_{jk})(I,0)\phi_j^{\pm}(x,0)\tilde{\psi}_k^{\pm}(0,y)(I,0)^t \\ &\qquad = -2\pi i d_{jk,-1}(I,0)\phi_j^{\pm}(x,0)\tilde{\psi}_k^{\pm}(0,y)(I,0)^t \end{aligned}$$

to $P_0(x,y) = \mathrm{Res}_0 G_\lambda(x,y)$, from which it is easily seen that $(I,0)\phi_j^{\pm}(0,x) \in \Sigma_0'(L)$, $\tilde{\psi}_k^{\pm}(0,y)(I,0)^t \in \Sigma_0'(L^*)$. But assumption $(\mathcal{D})$ implies that all eigenfunctions in $\Sigma_0'(L)$ decay exponentially at $\pm\infty$, whereas $\phi_j^{\pm} \sim e^{\rho_j^{\pm} x}$. It follows that $\rho_j^{\pm} \lessgtr 0$, or, comparing with results of Proposition 2.1, that $a_j \pm \lessgtr 0$. We record this important fact as

***Observation 8.7.*** The coefficient function $d_{jk}$ is singular only if $a_j^{\pm} \lessgtr 0$ (i.e. $j^{\pm}$ is an incoming mode) and $\phi_j^{\pm}(0,x) \in \Sigma_0'(L)$, $\tilde{\psi}_k^{\pm}(0,y) \in \Sigma_0'(L^*)$, in which case $\phi_j^{\pm} \sim e^{\rho_j^{\pm} x}$ is *rapidly decaying*, with

$$\rho_j^{\pm} \lessgtr \mp\eta \tag{8.41}$$

on $B(0,r)$, $\eta > 0$.

Without loss of generality, take $y \leq 0$. There are three cases to analyze, corresponding to (i)–(iii) of Proposition 7.1. As the analysis of each case is very similar. we will carry out one case in complete detail, only sketching the other two.

Expanding (8.31) as

$$\begin{pmatrix} G & G_x \\ G_y & G_{xy} \end{pmatrix} = \int_\Gamma e^{\lambda t} \begin{pmatrix} G_\lambda & G_{\lambda_x} \\ G_{\lambda_y} & G_{\lambda_{xy}} \end{pmatrix} d\lambda, \tag{8.42}$$

we estimate the $\int_{\Gamma_1}$ contributions to $G$, $G_x$ and $G_y$ simultaneously.

***Definition 8.8.*** For $y \leq 0$, set

$$\begin{aligned} (\varphi_k^-(x), {\varphi_k^-}'(x))^t &:= \sum_j \mathrm{Res}_0(d_{jk})\phi_j^\pm(0,x) \\ &= -2\pi i \sum_j d_{jk,-1}\phi_j^\pm(0,x), \end{aligned} \tag{8.43}$$

$$(\pi_k^-(y), {\pi_k^-}'(y)) := \begin{cases} \tilde{\psi}_k^-(0,y) & \text{if } \varphi_k^- \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where the $d_{jk}$ are taken from expansion (i) or (ii) of Proposition 7.1, according as $x \gtrless 0$. Note that $\varphi_k^- \in \Sigma_0'(L)$, $\pi_k^- \in \Sigma_0'(L^*)$, by the discussion preceding Observation 8.7; indeed,

$$\sum_k \varphi_k^-(x)\pi_k^-(y) = \begin{pmatrix} P_0(x,y) & P_{0_y}(x,y) \\ P_{0_x}(x,y) & P_{0_{xy}}(x,y) \end{pmatrix}.$$

For $y \geq 0$, define $\varphi_k^+(x)$ and $\pi_k^+(y)$ in symmetric fashion.

***Case II(i).*** $(y < 0 < x)$. By Proposition 7.1 (i), we have

$$\int_\Gamma e^{\lambda t} \begin{pmatrix} G_\lambda & G_{\lambda_x} \\ G_{\lambda_y} & G_{\lambda_{xy}} \end{pmatrix} d\lambda = \int_\Gamma \sum_{j,k} e^{\lambda t}\phi_j^+(x) d_{jk} \tilde{\psi}_k^-(y)\, d\lambda, \tag{8.44}$$

where $d_{jk}$ has a simple pole at $\lambda = 0$ if $a_j^+ < 0$, and otherwise is analytic. We estimate separately each of the terms

$$\int_{\Gamma_1} e^{\lambda t}\phi_j^+(x) d_{jk} \tilde{\psi}_k^-(y)\, d\lambda \tag{8.45}$$

on the righthand side of (8.44).

***Case II(ia).*** ***($a_j^+ < 0$, $a_k^- < 0$).*** First consider the case $a_j^+ < 0$, $a_k^- < 0$. By (8.41), $\rho_j^+ < -\eta < 0$ on $B(0,r)$, and $-\nu_k^- > \eta > 0$ as well. Recalling that $|d_{jk}| \le C|\lambda|^{-1}$, we thus have

$$|\phi^+(x) d_{jk} \tilde{\psi}_k^-(y)| \le C|\lambda|^{-1} e^{-\eta(|x|+|y|)}. \tag{8.46}$$

Choose any $0 < \varepsilon < \eta |a_k^-|/2$. For $t < |x-y|/|a_k^-|$, define $\Gamma_1'$ to be the contour from $\lambda_0^*$ to $\varepsilon$ to $\lambda_0$, as pictured in Figure 7a. Since the integrand of (8.45) is analytic on the region enclosed by $\Gamma_1 \Gamma_1'$, we have by Cauchy's Theorem, that

$$\int_{\Gamma_1} e^{\lambda t} \phi_j^+(x) d_{jk} \tilde{\psi}_k^-(y) d\lambda = \int_{\Gamma_1'} e^{\lambda t} \phi_j^+(x) d_{jk} \tilde{\psi}_k^-(y) d\lambda. \tag{8.47}$$

Noting that $|\lambda| > \varepsilon/C$, $\mathrm{Re}(\lambda) \le \varepsilon$ on $\Gamma_1'$, we have

$$\begin{aligned} \left| \int_{\Gamma_1'} e^{\lambda t} \phi_j^+(x)\, d_{jk} \tilde{\psi}_k^-(y)\, d\lambda \right| &\le \int_{\Gamma_1'} C|\lambda|^{-1} e^{\mathrm{Re}(\lambda)t - \eta|x-y|}\, d\lambda \\ &\le C_2 e^{\varepsilon t - \eta|y|} \\ &\le C_3 e^{-\varepsilon t/2}. \end{aligned} \tag{8.48}$$

This exponentially decaying term can be absorbed into the residual terms $R_S$, $R_S^1$, and $R_S^2$, as noted previously.

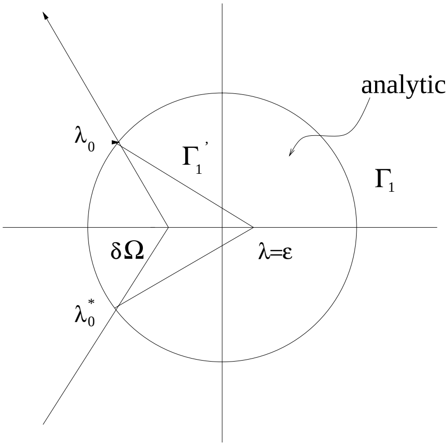


FIGURE 7A. $a_k^- < 0, \quad t < |y/a_k^-|$.

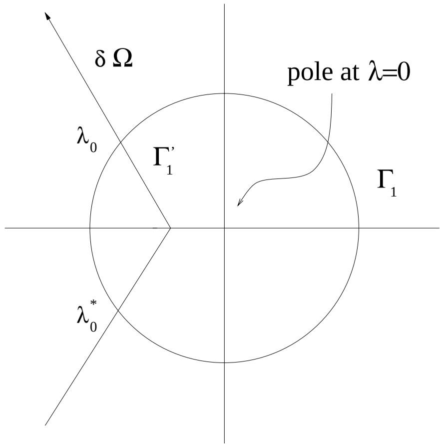


FIGURE 7B. $a_k^- < 0, \quad t \ge |ya_k^-|$.

For $t \geq |x-y|/|a_k^-|$, define $\Gamma_1'$ instead to simply follow $\partial\Omega_\theta$, as in Figure 7b. Now the integrand has a single pole at $\lambda = 0$ contained in the region enclosed by $\Gamma_1\Gamma_1'$. Thus, by Cauchy's Theorem, we have

$$\begin{aligned}(8.49)\quad \int_{\Gamma_1} e^{\lambda t}\phi_j^+(x)d_{jk}\tilde{\psi}_k^-(y)\,d\lambda &= \int_{\Gamma_1'} e^{\lambda t}\phi_j^+(x)d_{jk}\tilde{\psi}_k^-(y)\,d\lambda \\ &\quad + \chi_{\{|a_k^- t|\geq|x-y|\}}(\mathrm{Res}_{\lambda=0}d_{jk})\phi_j^+(0,x)\tilde{\psi}_k^-(0,y).\end{aligned}$$

The first term on the righthand side is bounded by

$$C\int e^{-\theta_1 t-\eta|x-y|}\,d\lambda \leq Ce^{-\theta_1 t},$$

and can be absorbed in $R_S$ terms as before. The second term contributes to the excited modes $\chi_{\{|a_k^- t|\geq|x-y|\}}(\varphi_k^-,\varphi_k^{-\prime})^t(\pi_k^-,\pi_k^{-\prime})$ appearing in $\begin{pmatrix} E & E^1 \\ E^2 & * \end{pmatrix}$, with $\varphi_k^-$, $\pi_k^-$ as in Definition 8.8.

***Case II(ib). ( $a_j^+ < 0$, $a_k^- > 0$).*** Next, consider the critical case that $a_j^+ < 0$ and $a_k^- > 0$. In this case,

$$(8.50)\qquad \mathrm{Re}(\rho_j^+) \leq \mathrm{Re}(\nu_k^-) - \eta$$

$\eta > 0$, and (8.46) becomes

$$(8.51)\qquad \left|\varphi_j^+(x)d_{jk}\tilde{\psi}_k^-(y)\right| \leq C|\lambda|^{-1}e^{-\eta|x|}e^{\mathrm{Re}(\nu_k^-)(x-y)}$$

where, by Proposition 2.1, we have

$$(8.52)\qquad \nu_k^-(\lambda) = -\frac{\lambda}{a_k^-} + \frac{\lambda^2\beta_k^-}{(a_k^-)^3} + \mathbf{O}(\lambda^3).$$

Setting

$$(8.53)\qquad \bar{\alpha} := \frac{x-y-a_k^- t}{2t}, \quad p := \frac{\beta_k^-(x-y)}{(a_k^-)^2 t} > 0,$$

define $\Gamma'_{1a}$ to be the portion contained in $\Omega_\theta$ of the hyperbola

$$
\begin{aligned}
\operatorname{Re}(v_k^-) + \mathcal{O}(\lambda^3) &= -(1/a_k^-)\operatorname{Re}(\lambda - \lambda^2\beta_k^-/(a_k^-)^2) \\
&\equiv \text{constant} \\
&= (\lambda_{\min}/a_k^- - \lambda_{\min}^2\beta_k^-/(a_k^-)^3),
\end{aligned}
\tag{8.54}
$$

or equivalently

$$
\operatorname{Re}(\lambda) - (\operatorname{Re}(\lambda)^2 - \operatorname{Im}(\lambda)^2)\beta_k^-/(a_k^-)^2 = \text{constant},
\tag{8.55}
$$

intersecting the real axis at

$$
\lambda_{\min} := \begin{cases}
\dfrac{\bar\alpha}{p} & \text{if } t^{-1/2} < \left|\dfrac{\bar\alpha}{p}\right| \le \varepsilon \\[2ex]
-t^{-1/2} & \text{if } -t^{-1/2} \le \dfrac{\bar\alpha}{p} < 0, \\[2ex]
t^{-1/2} & \text{if } 0 \le \dfrac{\bar\alpha}{p} \le t^{-1/2}, \\[2ex]
\pm\varepsilon & \text{if } \dfrac{\bar\alpha}{p} \gtrless \varepsilon,
\end{cases}
\tag{8.56}
$$

where $\varepsilon > 0$ is chosen sufficiently small. Denoting by $\lambda_1$, $\lambda_1^*$ the intersections of this hyperbola with $\partial\Omega_\theta$, define $\Gamma'_{1_b}$ to be the union of the two segments $\lambda_1$, $\lambda_0$ and $\lambda_0^*\lambda_1^*$, and define $\Gamma'_1 = \Gamma'_{1_a} \cup \Gamma'_{1_b}$. Choosing first $\theta_1$ and $\theta_2$, then $\varepsilon$ sufficiently small, we obtain that $\lambda_1$, $\lambda_1^* \in B(0,r)$, with $\operatorname{Im}\lambda_1 \ge \eta > 0$. The complete configuration is shown in Figure 8. Our choice of contour is motivated by the Riemann saddlepoint method, as we will explain further in a moment.

Note that

$$
\lambda_{\min} = \bar\alpha/p = -a_k^-(x - y - a_k^- t)/2\beta_k^- t + \mathbf{O}(\bar\alpha^2),
$$

in agreement with (8.22). Moreover, for $\bar\alpha$ sufficiently small, $p \sim 2\beta_k^-/a_k^-$ is bounded above and below:

$$
1/C \le p \le C.
\tag{8.57}
$$

Finally, Taylor expansion of (8.55) about $\bar\alpha$ gives

$$
\operatorname{Re}(\lambda) = \lambda_{\min} - \eta\operatorname{Im}(\lambda)^2 + \mathbf{O}(\operatorname{Im}(\lambda)^3)
\tag{8.58}
$$

for some $\eta > 0$, uniformly in $\bar\alpha$ for $\bar\alpha$ bounded, on $\Gamma'_{1a}$. In particular, this implies

$$(|\lambda_{\min}| + |\mathrm{Im}(\lambda)|)/C \le |\lambda| \le C(|\lambda_{\min}| + |\mathrm{Im}(\lambda)|). \tag{8.59}$$

Similarly as in the previous case, we have by Cauchy's Theorem that

$$\int_{\Gamma_1} e^{\lambda t}\varphi_j^+(x) d_{jk}\tilde\psi_k^-(y)d\lambda = \int_{\Gamma_1'} e^{\lambda t}\varphi_j^+(x) d_{jk}\tilde\psi_k^-(y)d\lambda + \chi_{\{\bar\alpha<0\}}(\mathrm{Res}_{\lambda=0} d_{jk})\varphi_j^+(0,x)\tilde\psi_k^-(0,y). \tag{8.60}$$

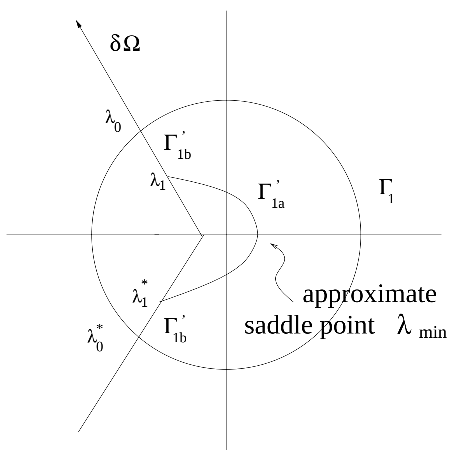


Figure 8.

Observing that $\chi_{\{\bar\alpha<0\}} = \chi_{\{|a_k^- t|>|x-y|\}}$, we find as in the previous case that the second term contributes to excited modes $\chi_{\{|a_k^- t|\ge|x-y|\}}(\varphi_k^-, {\varphi_k^-}')^t(\pi_k^-, {\pi_k^-}')$ in $\begin{pmatrix} E & E^2 \\ E^1 & * \end{pmatrix}$, with $\varphi_k^-$, $\pi_k^-$ as in Definition 8.8.

It remains to estimate the first term,

$$\int_{\Gamma_1'} e^{\lambda t}\varphi_j^+(x) d_{jk}\tilde\psi_k^-(y)d\lambda = \int_{\Gamma'_{1_a}} + \int_{\Gamma'_{1_b}}. \tag{8.61}$$

Note that the modulus of the integrand is bounded by

$$
\begin{aligned}
(8.62)\qquad & C|\lambda|^{-1}e^{\mathrm{Re}(\lambda)t-\mathrm{Re}(\nu_k^-)y+\mathrm{Re}(\rho_k^+)x} \\
&\le C_2 e^{-\eta(x)}|\lambda|^{-1}e^{t\mathrm{Re}(\lambda+\nu_k^-(x-y)/t)} \\
&\le C_2 e^{-\eta|x|}|\lambda|^{-1}e^{(t/a_k^-)\mathrm{Re}(-\lambda(2\bar\alpha-p\lambda+\mathbf{O}(\lambda^2))}.
\end{aligned}
$$

The significance of the point $\lambda=\bar\alpha/p$ can now be seen. It is the unique saddle point, or minimax, of the principle part

$$\mathrm{Re}(-\lambda(2\bar\alpha-p\lambda))$$

of the argument of the exponential term. Likewise, the hyperbola $\Gamma'_{1a}$ through the stationary point $\bar\alpha/p$ is an approximate mountain pass for the function $Re(-\lambda(2\bar\alpha-p\lambda))$. Thus, we have chosen our contour approximately in accordance with the Riemann *Saddlepoint Method*. The lower cutoff of $\pm t^{-1/2}$ is in order to avoid the singularity $\lambda^{-1}$, the upper cutoff of $\pm\varepsilon$ to avoid leaving $B(0,r)$.

From the development of Proposition 2.1, we have

$$(8.63)\qquad V_j^+(\lambda)=\left(r_j^+ +\mathbf{O}(\lambda)\ ,\ -\frac{\lambda r_j^+}{a_j^+}+\mathbf{O}(\lambda^2)\right)^t,$$

and, by the same argument applied to the adjoint operator $L^*$,

$$(8.64)\qquad \tilde V_k^-(\lambda)=\left(\ell_k^- +\mathbf{O}(\lambda)\ ,\ \frac{-\lambda\ell_k^-}{a_k^-}+\mathbf{O}(\lambda^2)\right),$$

where $\tilde V_k^-$ denote the asymptotic modes of the adjoint eigenvalue equation, and $r_j^+$ and $\ell_k^-$ are right and left eigenvectors of $A_+$ and $A_-$, respectively. Combining with the basic estimates (4.3) and (4.14), we thus have

$$
\begin{aligned}
(8.65)\qquad \varphi_j^+(x) &= (V_j^+(\lambda)+\mathbf{O}(e^{-\alpha|x|}))e^{\rho_j^+x} \\
&= e^{\rho_j^+x}\left(\begin{pmatrix} r_j^+ \\ -\lambda r_j^+/a_j^+\end{pmatrix}+\mathbf{O}(e^{-\alpha|x|})\right)+e^{\rho_j^+x}\begin{pmatrix}\mathbf{O}(\lambda)\\ \mathbf{O}(\lambda^2)\end{pmatrix},
\end{aligned}
$$

and likewise

$$(8.66)\qquad \tilde\psi_k^-(y)=e^{-\nu_k^-y}\left((\ell_k^-,-\lambda\ell_k^-/a_k^-)+\mathbf{O}(e^{-\alpha|y|})\right)+e^{-\nu_k^-y}(\mathbf{O}(\lambda),\mathbf{O}(\lambda^2)).$$

Thus, we can separate the integrand of (8.45) into two parts,

$$
\begin{aligned}
(8.67)\qquad & e^{\lambda t}\varphi_j^+(x)\, d_{jk}\tilde{\psi}_k^-(y) \\
&= d_{jk,-1}\lambda^{-1}e^{\lambda t+\rho_j^+ x-\nu_k^- y} \\
&\qquad \times \left(\begin{pmatrix} r_j^+ \\ -\lambda r_j^+/a_j^+ \end{pmatrix} + \mathbf{O}(e^{-\alpha|x|})\right)\left((l_j, -\lambda\ell_k^-/a_k^-) + \mathbf{O}(e^{-\alpha|y|})\right) \\
&\qquad + e^{\lambda t+\rho_j^+ x-\nu_k^- y}\begin{pmatrix} \mathbf{O}(1) & \mathbf{O}(\lambda) \\ \mathbf{O}(\lambda & \mathbf{O}(\lambda^2) \end{pmatrix} \\
&= I + II.
\end{aligned}
$$

The second term, $II$, lacking the factor $\lambda^{-1}$, will lead to faster decay by factor $t^{-1/2}$ than will term $I$ (recall that $t^{-1/2} = \min_{\lambda\in\Gamma_1'}|\lambda|$); we begin by estimating this error term. By (8.50), (8.67), we have

$$
(8.68)\qquad |II| \le e^{-\eta|x|}\begin{pmatrix} C & C\lambda \\ C\lambda & C\lambda^2 \end{pmatrix} e^{\mathrm{Re}(\lambda t+\nu_k^-(x-y))}.
$$

On $\Gamma_{1_a}'$, we have by (8.54), (8.58) that

$$
\begin{aligned}
(8.69)\qquad & \mathrm{Re}(\lambda t + \nu_k^-(x-y)) \\
&= (\mathrm{Re}(\lambda) - \lambda_{\min}) + \mathrm{Re}(\lambda_{\min} t + \nu_k^-(\lambda_{\min})(x-y)) \\
&= (\mathrm{Re}(\lambda) - \lambda_{\min}) - (t/a_k^-)(2\bar{\alpha}\lambda_{\min} - p\lambda_{\min}^2 + \mathbf{O}(\lambda_{\min}^3)).
\end{aligned}
$$

In case $t^{-1/2} < |\bar{\alpha}/p| < \varepsilon$, we have $\lambda_{\min} = \bar{\alpha}/p$, and we obtain by direct evaluation that

$$
(8.70)\qquad 2\bar{\alpha}\lambda_{\min} - p\lambda_{\min}^2 = \bar{\alpha}^2/p,
$$

hence

$$
(8.71)\qquad \mathrm{Re}(\lambda t + \nu_k^-(x-y)) \le -\bar{\alpha}^2 t/M - \eta\mathrm{Im}(\lambda)^2 t,
$$

for some $\eta$, $M > 0$, provided $r$ and $\varepsilon$ are chosen sufficiently small, for all $\lambda \in \Gamma_{1_a}'$.

On $\Gamma'_{1b}$, a similar analysis shows that

$$\begin{aligned}\text{(8.72)}\qquad \operatorname{Re}(\lambda t+\nu_k^-(x-y)) &\le \operatorname{Re}(\lambda_1 t+\nu_k^-(\lambda_1)(x-y))\\ &\le \eta \operatorname{Im}(\lambda_1)^2 t\\ &\le -\eta_2 t,\end{aligned}$$

where $\eta_2>0$. As in previous analyses, the contribution from the resulting time-exponentially decaying term $\int_{\Gamma'_{1b}} d\lambda$ can be absorbed in the error terms $R$, $R^1$, and $R^2$ as usual.

On the other hand,

$$\text{(8.73)}\qquad \int_{\Gamma'_{1_a}} |II| \le Ce^{-\eta|x|}\int_{\Gamma'_{1_a}} e^{\operatorname{Re}(\lambda t+\nu_k^-(x-y))}\begin{pmatrix}1 & |\lambda|\\ |\lambda| & |\lambda|^2\end{pmatrix} d\lambda,$$

From the bound (8.71), together with (8.59), we have for $q\ge 0$ that

$$\begin{aligned}\text{(8.74)}\quad &\int_{\Gamma'_{1_a}} |\lambda|^q e^{\operatorname{Re}(\lambda t+\nu_k^-(x-y))} d\lambda\\ &\le Ce^{-\bar\alpha^2 t/M}\int_{\Gamma'_{1_a}} (|\lambda_{\min}|^q+\operatorname{Im}(\lambda)^q)e^{-\eta \operatorname{Im}(\lambda)^2 t}\, d\lambda\\ &\le C\left(\bar\alpha^q e^{-\bar\alpha^2 t/M}\int_{\Gamma'_{1_a}} e^{-\eta\operatorname{Im}(\lambda)^2 t}\, d\lambda\right)\\ &+e^{-\bar\alpha^2 t/M}\int_{\Gamma'_{1_a}} \operatorname{Im}(\lambda)^q e^{-\eta\operatorname{Im}(\lambda)^2 t}\, d\lambda\\ &\le C(\bar\alpha^q e^{-\bar\alpha^2 t/M}\int_{-\infty}^{\infty} e^{-\eta k^2 t}dk+e^{-\bar\alpha^2 t/M}\int_{-\infty}^{\infty} k^q e^{-\eta k^2 t}dk)\\ &\le C_2 t^{-1/2-q/2}e^{-\bar\alpha^2 t/2M}.\end{aligned}$$

Applying (8.74) in (8.73), we obtain

$$\left|\int_{\Gamma'_{1_a}} II\right| \le \begin{pmatrix} t^{-1/2}e^{-\eta|x|}e^{-\bar\alpha^2 t/C} & t^{-1}e^{-\eta|x|}e^{-\bar\alpha^2 t/C}\\ t^{-1}e^{-\eta|x|}e^{-\bar\alpha^2 t/C} & t^{-3/2}e^{-\eta|x|}e^{-\bar\alpha^2 t/C}\end{pmatrix}.$$

Recalling that $\bar\alpha=(x-y-a_k^- t)/2t$, we find that this contribution absorbs in terms $S$, $S^1$, and $S^2$.

It remains to treat the cases for which $|\bar{\alpha}/p| \geq \varepsilon$ and $|\bar{\alpha}/p| \leq t^{-1/2}$. The $\bar{\alpha}/p \geq \varepsilon$ case is trivial, since on the fixed curves $\Gamma_1'$ corresponding to $\bar{\alpha}/p = \pm\varepsilon$, the quantity $\mathrm{Re}(\lambda t + \nu_k^-(x-y))$ clearly decreases as $\bar{\alpha}$ (equivalently, $(x-y)$) decreases (resp. increases). Thus, we obtain the bound

$$\int_{\Gamma_1'} |II| \leq \mathbf{O}(e^{-\eta\varepsilon^2 t})$$

in this case, which, as a time-exponentially decaying term, can be absorbed in $R$, $R^1$, and $R^2$. In the case $|\bar{\alpha}/p| \leq t^{-1/2}$, relation (8.69) still holds; however, instead of (8.70), we have by Taylor expansion about the critical point (minimax) $\bar{\alpha}/p$ that

$$\begin{aligned} 2\bar{\alpha}\lambda_{\min} - p\lambda_{\min}^2 &= \bar{\alpha}^2/p + 2p(\lambda_{\min} - \bar{\alpha}/p)^2 \\ &= \bar{\alpha}^2/p + \mathbf{O}(t^{-1}), \end{aligned} \tag{8.75}$$

where in the last line we have used the fact that $p$ is bounded above for $\bar{\alpha}$ small.

Carrying through the remaining analysis (8.74) as before, we obtain the same bounds multiplied by a factor $e^{t\mathbf{O}(t^{-1})} = \mathbf{O}(1)$, giving the same results.

The analysis of the term $I$ follows in exactly the same way. Note that the bound (8.74) holds also when $q < 0$. For, since $\min_{\lambda \in \Gamma_1'} |\lambda| \geq t^{-1/2}$, we have

$$\begin{aligned} \int |\lambda|^q e^{\mathrm{Re}(\lambda t + \nu_k^-(x-y))\, d\lambda} &\leq t^{-q/2} \int e^{\mathrm{Re}(\lambda t + \nu_k^-(x-y)) d\lambda} \\ &\leq t^{-q/2-1/2} e^{-\bar{\alpha}^2 t/2M} \end{aligned}$$

as before. In place of expansion (8.66), we use now

$$\begin{aligned} \tilde{\psi}_k^-(\lambda, y) &= e^{-\nu_k^- y}\Big(\tilde{V}_k^-(0,y) + \lambda)\mathbf{O}(1),\ \mathbf{O}(e^{-\eta|y|}))\Big) \\ &= e^{-\nu_k^- y}\Big((\pi_k^-(y),\ \pi_k^{-\prime}(y)) + \lambda(\mathbf{O}(1),\ \mathbf{O}(e^{-\eta|y|}))\Big), \end{aligned}$$

where the final equality follows by Definition 8.8 and the fact that $\nu_k^-(0) = 0$.

The rest of the analysis carries through as before to give

$$\begin{aligned}\int_{\Gamma_1'} |I|\, d\lambda \le e^{-\eta|x|}e^{-((x-y)-a_k^- t)^2/Mt} & (8.76)\\ &\times \begin{pmatrix} \mathbf{O}(1)\pi_k^{\pm}(y) & \mathbf{O}(t^{-1/2})\pi_k^{-\prime}(y) \\ \mathbf{O}(t^{-1/2})\pi_k^{\pm}(y) & \mathbf{O}(t^{-1})\pi_k^{-\prime}(y) \end{pmatrix}\\ &+\mathbf{O}(e^{-\eta|x|}e^{-((x-y)-a_k^- t)^2/Mt})\\ &\times \begin{pmatrix} t^{-1/2} & (t+1)^{-1/2}t^{-1/2}e^{-\eta|y|} \\ (t+1)^{-1/2}t^{-1/2} & (t+1)^{-1}t^{-1/2}e^{-\eta|y|} \end{pmatrix}.\end{aligned}$$

Comparing with the statement of Proposition 8.3, we find that the first term on the righthand side can be absorbed in $E$, $E^1$, and $E^2$, and the second in $R_S$, $R_S^1$, and $R_S^2$.

This completes our treatment of the case $a_j^+ < 0$, and thus of the possible singular terms $d_{jk} \sim \lambda^{-1}$. Note that the sum over $j$, $k$ of all terms $\chi_{\{|a_k^- t|\ge|x-y|\}}\mathrm{Res}_0 d_{jk}(\phi_j^-, \phi_j^{-\prime})^t(\tilde\psi_k^-, \tilde\psi_k^{-\prime})$ exactly gives

$$\sum_k \chi_{\{|a_k^- t|\ge|x-y|\}}(\varphi_k^-, \varphi_k^{-\prime})^t(\pi_k^-, \pi_k^{-\prime}),$$

accounting for the first terms in $E$, $E^1$, and $E^2$.

We now turn to the case $a_j^+ > 0$, corresponding to scattering terms.

***Case II(ic).*** ***($a_j^+$, $a_k^- > 0$).*** First, consider the critical case $a_j^+, a_k^- > 0$. For this case,

$$|\varphi_{j(x)}^+ d_{jk}\tilde\psi_k^-(y)| \le Ce^{\mathrm{Re}(\rho_j^+ x - \nu_k^- y)}, \tag{8.77}$$

where, by Proposition 2.1,

$$\begin{cases} \nu_k^-(\lambda) = -\dfrac{\lambda}{a_k^-} + \dfrac{\lambda^2\beta_k^-}{(a_k^-)^3} + \mathbf{O}(\lambda^3) \\[2ex] \rho_j^+(\lambda) = -\dfrac{\lambda}{a_j^+} + \dfrac{\lambda^2\beta_j^+}{(a_j^+)^3} + \mathbf{O}(\lambda^3). \end{cases} \tag{8.78}$$

Similarly as in (8.53) set

$$\bar\alpha := \frac{a_k^- x/a_j^+ - y - a_k^- t}{2t}, \quad p := \frac{\beta_j^+ a_k^- x/(a_j^+)^3 - \beta_k^- y/(a_k^-)^2}{t} > 0. \tag{8.79}$$

Define $\Gamma'_{1a}$ to be the portion contained in $\Omega_\theta$ of the hyperbola

$$
\begin{aligned}
(8.80)\quad & \mathrm{Re}(\rho_j^+ x - \nu_k^- y) + \mathcal{O}(\lambda^3)(|x|+|y|) \\
&= (1/a_k^-) Re[\lambda(-a_k^- x/a_j^+ + y) + \lambda^2(x\beta_j^+ a_k^-/(a_j^+)^3 - y\beta_k^-/(a_k^-)^2)] \\
&\equiv \text{constant} \\
&= (1/a_k^-)[(\lambda_{\min}(-a_k^- x/a_j^+ + y) + \lambda_{\min}^2(x\beta_j^+ a_k^-/(a_j^+)^3 - y\beta_k^-/(a_k^-)^2)],
\end{aligned}
$$

where

$$
\lambda_{\min} := \begin{cases} \dfrac{\bar\alpha}{p} & \text{if } t^{-1/2} < \left|\dfrac{\bar\alpha}{p}\right| \le \varepsilon, \\ -t^{-1/2} & \text{if } -t^{-1/2} \le \dfrac{\bar\alpha}{p} < 0, \\ t^{-1/2} & \text{if } 0 \le \dfrac{\bar\alpha}{p} \le t^{-1/2}, \\ \pm\varepsilon & \text{if } \frac{\bar\alpha}{p} \gtrless \varepsilon, \end{cases} \tag{8.81}
$$

Denoting by $\lambda_1$, $\lambda_1^*$, the intersections of this hyperbola with $\partial\Omega_\theta$, define $\Gamma'_{1_b}$ to be the union of $\lambda_1\lambda_0$ and $\lambda_0^*\lambda_1^*$, and define $\Gamma'_1 = \Gamma'_{1_a} \cup \Gamma'_{1_b}$, as in Figure 6. Note that $\lambda = \bar\alpha/p$, similarly, as before, minimizes the left hand side of (8.80) for $\lambda$ real. Likewise, we again have the crucial property that $p$ is bounded for $\bar\alpha$ sufficiently small, since $\bar\alpha \le \varepsilon$ implies that

$$
(|a_k^- x/a_j^+| + |y|)/t \le 2|a_k^-| + 2\varepsilon, \tag{8.82}
$$

i.e. $(|x|+|y|)/t$ is controlled by $\bar\alpha$.

With these definitions, the calculations of (8.69)–(8.71) carry through, modulo obvious modifications, to give the analogous result

$$
\begin{aligned}
\mathrm{Re}(\lambda t + \rho_j^+ x - \nu_k^+ y) &\le -(t/a_k^-)(\bar\alpha^2/4p) - \eta\,\mathrm{Im}(\lambda)^2 t \\
&\le -\bar\alpha^2 t/M - \eta\,\mathrm{Im}(\lambda)^2 t,
\end{aligned} \tag{8.83}
$$

for $\lambda \in \Gamma'_{1_a}$ (note: here, we have again used the crucial fact that $\bar\alpha$ controls $(|x|+|y|)/t$, in bounding the error term $\mathcal{O}(\lambda^3)(|x|+|y|)/t$ arising from expansion (8.80)). Likewise, analogously to (8.74), we obtain for any $q$ that

$$
\int_{\Gamma'_{1_a}} |\lambda|^q e^{\mathrm{Re}(\lambda t + \rho_j^+ x - \nu_k^- y)}\, d\lambda \le C t^{-1/2-q/2} e^{-\bar\alpha^2 t/M}, \tag{8.84}
$$

for suitably large $C, M > 0$ (depending on $q$). Observing that

$$\bar{\alpha} = (a_k^-/a_j^+)(x - a_j^+(t - |y/a_k^-|))/2t, \tag{8.85}$$

we find that the contribution of (8.84) can be absorbed in the scattering terms $S$, $S^1$, and $S^2$ for $t \geq |y/a_k^-|$. At the same time, we find that $\bar{\alpha} \geq x > 0$ for $t \leq |y/a_k^-|$, whence

$$\bar{\alpha} \geq \frac{x - y - a_j^+ t}{Mt} + \frac{|x|}{M},$$

for some $\varepsilon > 0$ sufficiently small and $M > 0$ sufficiently large. This gives

$$e^{-\bar{\alpha}^2/p} \leq e^{-(x-y-a_k^- t)^2/Mt} e^{-\eta|x|}$$

provided $|x|/t > a_j^+$, a contribution which can again be absorbed in $S$, $S^1$, and $S^2$. On the other hand, if $t \leq |x/a_j^+|$, we can use the dual estimate

$$\begin{aligned}\bar{\alpha} &= \frac{-y - a_k^-(t - |x/a_j^+|)}{2t}\\ &\geq \frac{x - y - a_k^- t}{Mt} + \frac{|y|}{M},\end{aligned}$$

together with $|y| \geq |a_k^- t|$, to obtain

$$e^{-\bar{\alpha}^2/p} \leq e^{-(x-y-a_j^+ t)^2/Mt} e^{-\eta|y|},$$

a contribution that can likewise be absorbed. Together with the (still valid) expansion (8.67), these results comprise all the essential ingredients in the argument of the previous case, with the difference that, in place of the former $e^{-\eta|x|}$ factor, we have an extra factor of $\lambda$. This difference results (see (8.74)) after integration in the substitution of a factor $t^{-1/2}$ for the factor $e^{-\eta|x|}$, giving the claimed bounds.

***Case II(id).*** ***($a_j^+ > 0$, $a_k^- < 0$).*** The final remaining possibility is the case $a_j^+ > 0$, $a_k^- < 0$, for which

$$\mathrm{Re}(\lambda t + \rho_j^+ x - \nu_k^- y) \leq \mathrm{Re}(\lambda t + \rho_j^+ x - \rho_j^+ y) - \eta|y|$$

for all $\lambda \in B(0,r)$. Applying expansion (8.63) as in previous cases, we obtain a contribution to $\begin{pmatrix} G & G_y \\ G_x & G_{xy} \end{pmatrix}$ of

$$
\begin{aligned}
(8.86)\qquad & e^{-(x-y-a_j^+ t)^2/Mt} \quad \times \\
& \times \begin{pmatrix} t^{-1/2}(r_j^+ + \mathbf{O}(e^{-\eta|y|}))^t\mathbf{O}(e^{-\eta|y|}) & t^{-1}(r_j^+ + \mathbf{O}(e^{-\eta|y|}))^t\mathbf{O}(e^{-\eta|y|}) \\ t^{-1}(r_j^+ + \mathbf{O}(e^{-\eta|y|}))^t\mathbf{O}(e^{-\eta|y|}) & t^{-\frac{3}{2}}(r_j^+ + \mathbf{O}(e^{-\eta|y|}))^t\mathbf{O}(e^{-\eta|y|}) \end{pmatrix},
\end{aligned}
$$

which can be absorbed in the first terms of $S$, $S^1$ and $S^2$. This completes the proof for Case II(i), $(y < 0 < x)$.

***Case II(ii)*** $(y < x < 0)$**.** The case $y < x < 0$ can be treated very similarly to the previous one. In place of (8.44), we now have the analogous expansion

$$
\begin{aligned}
(8.87)\qquad & \int_{\Gamma_1} e^{\lambda t} \begin{pmatrix} G_\lambda & G_{\lambda_x} \\ G_{\lambda_y} & G_{\lambda_{xy}} \end{pmatrix} \\
& \qquad = \int_\Gamma \Big( \sum_{j,k} \varphi_j^-(x) d_{jk} \tilde{\psi}_k^-(y) + \sum_k \psi_k^-(x) e_k \tilde{\psi}_k^-(y) \Big)\, d\lambda.
\end{aligned}
$$

The singular terms $\varphi_j^-(x) d_{jk} \tilde{\psi}_k^-(y)$, $a_j^- > 0$, can be treated exactly as in the previous case. For example, consider the critical case $a_j^- > 0$, $a_k^+ > 0$. Just as in (8.50), we have

$$
(8.88)\qquad \mathrm{Re}(\rho_j^-) \geq \mathrm{Re}(\nu_k^-) + \eta,
$$

since $\varphi_j^-(x)$ is fast-decaying and $\nu_k^-$ is slow growing. Thus, (8.51)–(8.52) hold, and we can carry out the entire analysis of (8.53)–(8.76) as before. Likewise, the trivial case $a_k^- < 0$, $a_j^- > 0$ follows as in (8.46)–(8.49).

Also similarly as in the previous case, the critical nonsingular terms, in this case $\varphi_j^- d_{jk} \tilde{\psi}_k^-(y)$ with $a_j^- < 0$, $a_k^- > 0$, represent *scattering terms*. Similarly as in (8.77), we have

$$
(8.89)\qquad |\varphi_j^-(x) d_{jk} \tilde{\psi}_k^-(y)| \leq C e^{Re(\rho_j^- x - \nu_k^- y)},
$$

where

$$
(8.90)\qquad \begin{cases} \nu_k^-(\lambda) = -\lambda/a_k^- + \lambda^2 \beta_k^-/(a_k^-)^2 + \mathbf{O}(\lambda^3), \\ \rho_j^-(\lambda) = -\lambda/a_j^- + \lambda^2 \beta_j^2/(a_j^-)^2 + \mathbf{O}(\lambda^3). \end{cases}
$$

Setting

$$(8.91)\qquad \bar\alpha = \frac{a_k^- x/a_j^- - y - a_k^- t}{t}, \quad p = \frac{\beta_j^- a_k^- x/(a_j^-)^3 - \beta_k^- y/(a_k^-)^2}{t} > 0,$$

as in (8.79), define $\Gamma'_{1_a}$, $\Gamma'_{1_b}$ and $\Gamma'_1 = \Gamma'_{1_a} \cup \Gamma'_{1_b}$ as in (8.80)–(8.81). Again, $\lambda = \bar\alpha/p$ is a global minimum of the left hand side of (8.80) for $\lambda$ real. Likewise, $p$ is bounded for $\bar\alpha$ small, since, as in (8.82), $\bar\alpha$ again controls $(|x|+|y|)/t$. Thus, all the analyses of (8.53)–(8.76) goes through as before. The noncritical terms $a_j^- < 0$, $a_k^- < 0$ introduce a factor of $e^{-\eta|y|}$ as before, and can be absorbed in the residual terms $R$, $R^1$, and $R^2$.

It remains to estimate the new term $\psi_k^-(x)e_k\tilde\psi_k^-(y)$, $a_k^- > 0$. Here, we have

$$(8.92)\qquad |\psi_k^-(x)e_k\tilde\psi_k^-(y)| \le Ce^{-\nu_k^-(x-y)},$$

and we reduce to essentially the same calculations done in (8.52)– (8.76), but with an extra factor of $\lambda$ replacing the $e^{-\eta|x|}$ term. The resulting contribution is

$$\chi_{\{x<0\}} e^{-(x-y-a_k^- t)^2/Ct} \begin{pmatrix} \mathbf{O}(t^{-1/2}) & \mathbf{O}(t^{-1}) \\ \mathbf{O}(t^{-1}) & \mathbf{O}(t^{-3/2}) \end{pmatrix},$$

which can be absorbed in the scattering terms $S$, $S^1$, and $S^2$, similarly as in case II(i). We omit the details.

***Case II(iii).*** $(x < y < 0)$**.** This case is entirely similar to the previous one.

This completes the proof of Case II, and the theorem. □

***Remark.*** The fact that there occur no *mixed terms* $\psi_k^-(x)e_{jk}\tilde\psi_j^-(y)$ in the decomposition of $G_\lambda$ is critical in the calculations of cases (ii) and (iii). Comparison with the calculation using bounds (8.92) shows that such terms would not obey our estimates.

***Remark.*** In the case that $(\mathcal{D})$ does not hold, it is straightforward but tedious to derive similar pointwise bounds on the Green's function incorporating also the *unstable* excited modes identified in Proposition 8.1. The excitation of these modes occurs with large but approximately finite speed of propagation. Except for neutrally stable modes belonging to $\Sigma_0'(L)$ but not $\Sigma_0(L)$, this speed of propagation is greater than the maximum characteristic speed.

***Remark.*** In the stable case $(\mathcal{D})$, the proof of Theorem 8.3 shows that $\Sigma_0'(L^*) \subset \mathrm{Span}\tilde\Psi^-(0,y) \cap \mathrm{Span}\tilde\Psi^+(0,y)$, as can also be seen directly by the construction of $\Sigma_0'(L^*)$ given in Section 6. By the bifurcation analysis analogous to that of Proposition 2.1, each $\tilde{\psi}_k^{\pm}(0, y)$ is asymptotic as $y \to \pm\infty$ either to zero or to a left eigenvector $l_k^{\pm}$ of $A_{\pm}$ for which the corresponding eigenvalue $a_k^{\pm}$ is $\lessgtr 0$, that is to an *incoming characteristic mode*. This confirms the intuitive observation made in [LZ.2] (Section IV, (4.3)) that only incoming signals contribute to the asymptotic shift or deformation of a stable shock.

## Part IV. Stability.

**9. Linearized Stability.** In this section, we establish that the Evans function criterion $(\mathcal{D})$ identified in Result 3 of the introduction is both necessary and *sufficient* for linearized stability. We show this condition to be equivalent to the standard spectral requirements $\sigma(L) \setminus \{0\} \subset \{\mathrm{Re}\lambda < 0\}$ and $\mathrm{Ker}(L) = \{\partial \bar{u}^\delta / \partial \delta_j\}$, augmented with a single, computable *transversality condition* at $\lambda = 0$.

Using the Green's function bounds developed in the previous section, the study of linearized orbital stability is relatively straightforward. Indeed, we can considerably refine past analyses (e.g. [LZ.2, L.3]) by tracking the instantaneous location of the shock, and not only its time-asymptotic position. In the context of the linearized problem, this amounts to calculating an *instantaneous projection* of the solution $v(x,t)$ onto $\Sigma_0'(L)$, refining the time-asymptotic projection $\mathcal{P}_0 v(0, \cdot)$.

***Definition 9.1 .*** *Assuming the hypotheses of Theorem 8.3, define the instantaneous projection $\varphi(\cdot, t) \in \Sigma_0'(L)$ of the solution at time $t$ to be*

$$\varphi(x,t) = \sum_k \varphi_k^+(x) \int_0^{|a_k^+|t} \pi_k^+(y) v(y,0)\, dy \qquad (9.1)$$

$$+ \sum_k \varphi_k^-(x) \int_{-|a_k^-|t}^0 \pi_k^-(y) v(y,0)\, dy.$$

The instantaneous projection $\varphi(\cdot, t)$ has a simple interpretation as the superposition of all time-asymptotic states that have been excited by the arrival at $x = 0$ of a signal from the far field at $y$. Signals travel with finite characteristic speed $|a_k^{\pm}|$, emanating in both directions. This reflects a certain symmetry in the situation; the propagation of a shock adjustment in one direction can equally well be thought of as the propagation of an inverse adjustment in the opposite direction.

***Proposition 9.2.*** *Given* (**C0**)–(**C3**), *linearized $L^p$ orbital stability, $p > 1$, of* (1.6) *with respect to perturbations in $\mathcal{A} = L^1$ is equivalent to the stability condition* $(\mathcal{D})$. *In the case of stability,*

$$(v(\cdot, t) - \phi(\cdot, t)) \to 0 \qquad (9.2)$$

*and*

$$v(\cdot,t),\,\phi(\cdot,t)\to\mathcal{P}_0(v(0,y)), \tag{9.3}$$

*with no rate given. For* $\mathcal{A}=L^1\cap\{f:|f|\le(1+|x|)^{-r}\}$, *we obtain the rates of convergence*

$$\|v(\cdot,t)-\phi(\cdot,t))\|_p\le Ct^{\max\{1/2-r,1/2p-1/2\}} \tag{9.4}$$

*for* $r>\frac{1}{2}$, *and*

$$\|v(\cdot,t)-\mathcal{P}_0(v(0,y))\|_p,\ \|\phi(\cdot,t)-\mathcal{P}_0(v(0,y))\|_p\le Ct^{\max\{1-r,1/2p-1/2\}} \tag{9.5}$$

*for* $r>1$.

*Proof.* Necessity of $(\mathcal{D})$ was established in Corollary 8.2, while sufficiency will follow once we establish (9.2)–(9.3). The convergence of $\phi$ in (9.3) is clear from comparison of (9.1) and (8.25) and the Lebesgue dominated convergence Theorem, and the convergence of $v$ will follow once we establish (9.2). Likewise, (9.5) is immediate once we establish (9.4). Thus, it is only (9.2) and (9.4) we must demonstrate.

Expanding, we have

$$\begin{aligned}|v(x,t)-\phi(x,t)|\le\left|\int S(x,t;y)v(0,y)\,dy\right|+\left|\int R(x,t;y)v(0,y)\,dy\right|\\+\left|\int T(x,t;y)v(0,y)\,dy\right|+\left|\int N(x,t;y)v(0,y)\,dy\right|,\end{aligned}$$

where

$$T(x,t;y):=\sum_{k,\pm}\mathbf{O}(e^{-(x-y-a_k^\pm t)^2/Mt}+e^{-(x-y+a_k^\pm t)^2/Mt})e^{-\eta|x|}\pi_k^\pm(y)$$

is the *transient* part of $E$, and

$$N(x,t;y):=-\sum_{k,\pm}(\chi_{\{|x-y|>|a_k^\pm t|\text{ and }|y|\le|a_k^\pm t|\}}\varphi_k^\pm(x)\pi_k^\pm(y)$$

is a negligible tail term, correcting (9.1) in the far field.

The bound (9.2) then follows from the observations that:

$$\begin{aligned}\left\|\int R(x,y,t)v(0,y)\,dy\right\|_p, &\left\|\int S(x,y,t)v(0,y)\,dy\right\|_p\\ &\leq C\|t^{-1/2}e^{(x-y)^2/Mt}\|_p\|v(0,\cdot)\|_1\\ &\leq t^{1/2p-1/2};\end{aligned}$$

$$\begin{aligned}\left\|\int N(x,y,t)v(0,y)\,dy\right\|_p &\leq C\left\|\sum_{k,\pm}\int_{-|a_k^\pm t|}^{+|a_k^\pm t|}\chi_{\{|x|\geq t-|y/a_k^\pm|\}}\,e^{-\eta|x|}\,v_0(y)\,dy\right\|_p\\ &\leq \sum_{k,\pm}C\int_{-|a_k^\pm t|}^{+|a_k^\pm t|}e^{-\eta(t-|y/a_k^\pm|)}\,v_0(y)\,dy;\end{aligned}$$

and

$$\begin{aligned}\left\|\int T(x,y,t)v(0,y)\,dy\right\|_p &\leq C\|e^{-\eta x}\|_p\max_{-C\log t\leq x\leq C\log t}\left|\int T(x,y,t)v(0,y)\,dy\right|\\ &\qquad +Ct^{-N}\\ &\leq C\sum_{k,\pm}\left|\int_{||a_k^\pm t|\mp y|\leq C\log t}v(0,y)\,dy\right|+Ct^{-N},\end{aligned}$$

where $N$ is as large as desired. For $v(\cdot,0)\in\{f:|f|\leq(1+|x|)^{-r}\}$, this gives the bound $Ct^{1/2-r}\log t$. By using the bound $v(\cdot,0)\in\{f:|f|\leq(1+|x|)^{-r}\}$ from the start, we can remove the $\log t$ factor, giving (9.4). □

***Remark.*** The bounds (9.5) and (9.3) correspond to those of [LZ.2,L.3]. The refined bounds (9.2) and (9.4) are of use in treating stability of wave patterns (see [SZ]) and of undercompressive shocks [Z.2]. The transient adjustment of the shock location associated with term $T$ is analogous to the resonant wave observed in [GSZ]. The error associated with scattering terms $S$ corresponds to the diffusion waves of [L.1]. These two dominant components in the solution determine the rate of convergence to the stationary manifold given in (9.4).

***Remark.*** So long as there is some outgoing mode $a_j^\pm \gtrless 0$, the far field behavior in that mode is dominated by the solution to a convected heat equation. This justifies the discussion surrounding the example in Section 1.1.4 of the introduction.

***Remark.*** In the case that there are $L^\infty$ stationary modes, essentially the same description of the Green's function holds. A difference is, however, that the instantaneous projection no longer accurately describes behavior; indeed, the description of the wave as tracking along $\Sigma_0'(L)$ is no longer appropriate, and the tail term $N$ becomes the dominant effect. See Section 12 for a discussion of this case.

The Evans function criterion of Proposition 8.2 can be used directly to study stability. Indeed, Brin has used this as the basis of a numerical algorithm to check stability [Br], explicitly calculating the winding number of $D$ around the positive complex half-plane. This method can in principle be used to verify stability of shocks of any type or strength.

Alternatively, condition $(\mathcal{D})$ can be related to standard spectral stability conditions, augmented by a single piece of extra information. From Theorem 6.4(ii) and the fact that $\Sigma'_{\lambda_0}(L) = \Sigma_{\lambda_0}(L)$ for $\mathrm{Re}\lambda_0 \geq 0, \lambda_0 \neq 0$, together with the fact that $\mathrm{Ker}(L) \subset \Sigma_0'(L)$, we immediately obtain the following result:

***Lemma 9.3.*** *The Evans function criterion* $(\mathcal{D})$ *is equivalent to* $\sigma(L) \setminus \{0\} \subset \{\mathrm{Re}\lambda < 0\}$, *together with the transversality condition*

$$\left(\frac{d}{dt}\right)^{\ell} D_L(0) \neq 0. \tag{9.6}$$

The quantity $(d/dt)^\ell D_L(0)$ has been explicitly evaluated in [GZ] for waves of all types. There was observed a suggestive connection between (9.6) and either linearized stability of the associated inviscid shock, or the property that asymptotic state is determined by perturbation mass, depending on the type of the wave. In the following section, we explore this connection further, showing that it is in fact an *equivalence*. Our analysis depends, not on direct calculation, but on the following abstract characterization.

***Lemma 9.4.*** *Let* $\mathcal{R}^\pm = \mathrm{Span}\,\{r_j^\pm : a_j^\pm \lessgtr 0\}$. *Then, condition* (9.6) *is equivalent to* $z \in \mathrm{Span}\,\{\partial\bar{u}^\delta/\partial\delta_j\}$ *for all bounded solutions of*

$$(Bz')' = (Az)' + \phi \tag{9.7}$$

*such that* $\phi \in \mathrm{Span}\,\{\partial\bar{u}^\delta/\partial\delta_j\}$ *and* $z(\pm\infty) \in \mathcal{R}^\pm$.

*Proof.* The transversality condition (9.6) can be restated as

$$\Sigma'_{0,1}(L) = \Sigma'_{0,2}(L) = \mathrm{Span}\,\{\partial\bar{u}^\delta/\partial\delta_j\},$$

where, recall, $\Sigma'_{0,1}(L)$ denotes the space of effective eigenfunctions of ascent one at $\lambda = 0$ ("genuine eigenfunctions") and $\Sigma'_{0,2}(L)$ the space of effective eigenfunctions of ascent two.

Following the procedure described in Lemmas 6.1 and 6.4, these spaces can be constructed from the sets of solutions $\Phi^\pm$ of the eigenvalue equation. Solutions $\Phi^\pm$ in turn are determined by maximal rate of approach to the asymptotic solutions $e^{\mu_j x}V_j$ described in Lemma 2.1. At $\lambda = 0$, this implies that Span $\Phi^\pm$ are exactly the sets of solutions of the zero eigenvalue equation

$$(Bw')' = (Aw') \tag{9.8}$$

such that $w \to r^\pm$ as $x \to \pm\infty$, with $r^\pm \in \text{Span}\,\{r_j^\pm : a_j^\pm \lessgtr 0\}$.

Thus, $\Sigma'_{0,1}(L) = \Phi^+ \cap \text{Span}\,\Phi^-$ is exactly the set of bounded solutions of (9.8) such that $\lim_{x\to\pm\infty} w \in \mathcal{R}^\pm$ and $\Sigma'_{0,1}(L) = \text{Span}\,\{\partial\bar{u}^\delta/\partial\delta_j\}$ if and only if all such $w \in \text{Span}\,\{\partial\bar{u}^\delta/\partial\delta_j\}$. This is the special case of (9.7) with $\phi = 0$.

Next, supposing that $\Sigma'_{0,1}(L) = \text{Span}\,\{\partial\bar{u}^\delta/\partial\delta_j\}$, we investigate conditions for $\Sigma'_{0,2}(L) = \text{Span}\,\{\partial\bar{u}^\delta/\partial\delta_j\}$. In this case, the construction of $\Sigma'_{0,2}(L)$ is considerably simplified. For, $(d/d\lambda)\phi \to 0$ as $x \to \pm\infty$ whenever $\phi \in \Sigma'_{0,1}(L)$, so that $\Sigma'_{0,2}(L) \setminus \Sigma'_{0,1}(L)$ can be characterized simply as the set of bounded solutions $z$ of the generalized eigenvalue equation

$$(Bz')' = (Az)' + \phi, \tag{9.9}$$

where $\phi \in \text{Span}\,\{\partial\bar{u}^\delta/\partial\delta_j\}$, such that $z \to \mathcal{R}^\pm$ as $x \to \pm\infty$. Comparing with (9.7), we are done. □

**10. Alternative stability criteria.** We now investigate the meaning of the transversality condition (9.6) for each of the three major types of shock wave, Lax, overcompressive, and undercompressive, showing in each case that it reduces to a condition that is already familiar. For example, in the Lax case, (9.6) reduces simply to the Liu–Majda condition, [L.1, MZPM, Fre.2], an algebraic criterion equivalent to *linearized stability of the associated inviscid shock* (equivalently, linearized well-posedness of the associated Riemann problem), or alternatively the property that *asymptotic shock location is (formally) determined by perturbation mass* ([M], [L.1] respectively; see also [ZPM, Fre.2]). More generally, for Lax and undercompressive shocks (9.6) is equivalent to linearized well-posedness, while for Lax and overcompressive shocks it is equivalent to determinability of the asymptotic state.

Pursuing the latter observation a bit further, we establish the alternative criterion for Lax and overcompressive shocks that linearized orbital stability is equivalent to linearized *asymptotic* stability with respect to zero mass perturbations, plus the easily checked condition that the "outgoing" eigenvectors of $A_\pm$ be independent. The latter condition is quite useful, since zero-mass results exist for many cases in which the general problem remains open [G.1, MN, KMN, Fri.1-2]. A second, equivalent condition is that the Evans function for the "integrated equations" have no zeroes in the nonnegative complex half-plane, $\{\text{Re}\lambda \geq 0\}$.

This is useful in numerical calculation of stability, being less sensitive than the ordinary Evans function criterion [Br].

***10.1. Classification of shock waves.*** For a given wave $\bar{u}(x)$, let $i^+$ denote the number of negative $a_j^+$ and $i^-$ the number of positive $a_j^-$, $a_j^\pm$ defined as above to be the eigenvalues of $A_\pm = f'(u_\pm)$, and set $i = i^+ + i^-$. By Lemma 1.1, $i^\pm$ are the dimensions of the stable/unstable manifolds $\mathcal{S}^+$ and $\mathcal{U}^+$ of $u_\pm$ with respect to the traveling wave ODE (1.4) (whose intersection gives the manifold $\{\bar{u}^\delta\}$ of viscous profiles), and likewise of their tangent manifolds along $\bar{u}$. Note that the intersection of these tangent manifolds is precisely $\mathrm{Ker}(L)$, since they necessarily satisfy the linearized equations about $\bar{u}$.

***Definition 10.1.*** *A viscous shock $\bar{u}(x)$ is of "standard", or "pure" type if the intersection of $\mathcal{S}^+$ and $\mathcal{U}^+$ along $\bar{u}$ is maximally transverse consistent with the existence of a traveling wave, i.e.*

$$\dim \mathrm{Ker}(L) = \ell = \dim\{\bar{u}^\delta\} = \dim \mathrm{Span}\{\partial\bar{u}^\delta/\partial\delta_j\} \tag{10.1}$$

*and*

$$\ell = \begin{cases} i-n & i \geq n+1, \\ 1 & i \leq n, \end{cases} \tag{10.2}$$

***Remark.*** Note that conditions (10.1)–(10.2) are satisfied automatically for *extreme shocks*, i.e. for $i^-$ or $i^+$ equal to $n$ or 1. This includes many interesting cases: all shocks for $2\times 2$ systems, all overcompressive shocks for $3\times 3$ systems, and, notably, all shocks for the equations of gas dynamics. Under reasonable hypotheses, (10.1)–(10.2) are generically satisfied for *weak shocks* of arbitrary $n\times n$ systems, as shown by the bifurcation analyses of [MP,AMPZ.2]. Indeed, we do not know of any physical example in which nonstandard shocks appear.

***Definition 10.2.*** *A standard viscous shock is of (pure) Lax type if $i = n+1$, undercompressive type if $i < n+1$, and overcompressive type if $i > n+1$.*

*More generally, we define the degree of undercompressivity of a (possibly nonstandard ) viscous shock to be $n + \ell - i$. and the degree of overcompressivity to be $\ell - 1$.*

Recall, Theorem 8.3, that the $a_j^\pm$ represent speeds of propagation in different modes of perturbations relative to the shock, hence $i$ is the number of incoming modes, a measure of "compressivity" of the convection field induced by the shock. Equivalently, $i$ is the number of incoming characteristics of the corresponding inviscid, or ideal shock, in the hyperbolic theory [Sm]. Thus, our classification of *standard* shocks agrees with standard, hyperbolic convention.

Shocks failing (10.1) are linearly unstable by Theorem 8.2. Any stable nonstandard shock must therefore fail (10.2), giving $n+\ell-i$, $\ell-1>0$; that is, it must be of *mixed* over- and undercompressive type. Examples of such "mixed type" shocks are described in [LZ.2], section 4.4, and indeed exhibit behaviors characteristic of both over- and undercompressive waves. The definition above simplifies an equivalent classification given in [LZ.2]. From now on, we will refer to pure, or standard Lax, over- and undercompressive shocks simply as Lax, over- and undercompressive waves, and nonstandard waves as mixed type.

***10.2. Lax and Overcompressive shocks.***

***Proposition 10.3.*** *For (pure) Lax and overcompressive shocks, the transversality condition* (9.6) *is equivalent to either of:*

(i)
$$\{r_j^{\pm} : a_j^{\pm} \lessgtr 0\} \cup \left\{ \int_{-\infty}^{+\infty} \frac{\partial \bar{u}^\delta}{\partial \delta_j} \, dx : \; 1 \le j \le \ell \right\}$$

*is a basis for $R^n$.*

(ii) $\Sigma_0'(L^*)$ *consists entirely of constant functions, and the vectors* $\{r_j^{\pm} : a_j^{\pm} \lessgtr 0\}$ *are independent.*

*Proof.*

(i) This follows easily from Lemma 9.4. Given any $\phi$ that is contained in Span $\{\partial\bar{u}^\delta/\partial\delta_j\}$, there is a unique bounded solution of

$$(Bz')' = (Az)' + \phi, \tag{10.3}$$

modulo Span $\{\partial\bar{u}^\delta/\partial\delta_j\}$, such that $z$ approaches any desired value $r^- = z(-\infty)$ as $x \to -\infty$. For, the intersection of the $1+i^-$ dimensional manifold of solutions approaching Span $\{r_-\}$ with the $n+i^+$ dimensional manifold of solutions bounded at $+\infty$ is of dimension at least $(1+i^-)+(n+i^+)-2n = 1+\ell$; that is, there is at least one such solution modulo the $\ell$-dimensional manifold of solutions decaying at both infinities.

Integrating (9.8) from $-\infty$ to $+\infty$, and recalling that $z' \to 0$ as $e^{-\alpha|x|}$ whenever $w \to$ *constant*, we find that

$$A^+ z(+\infty) = A^- z(-\infty) + \int_{-\infty}^{+\infty} \phi \, dx \tag{10.4}$$

for all such solutions, i.e. the value $r^+ = z(+\infty)$ is determined. This implies that in fact there is a *unique* solution modulo the $\ell$-dimensional manifold of solutions decaying at both infinities.

Noting that multiplication by $A_\pm$ preserves the invariant subspaces $\mathcal{R}^\pm = \text{Span}\,\{r_j^\pm : a_j^\pm \lessgtr 0\}$, we thus find that (i) is satisfied if and only if, for all solutions of (9.8) such that $z(\pm\infty) \in \mathcal{R}^\pm$, $z(\pm\infty) = 0$ and $\int \phi\, dx = 0$, so that $z \in \text{Ker}(L)$. The claim then follows by (10.1).

(ii) The adjoint eigenvalue equation at $\lambda = 0$, $(B^* z')' = -A^* z'$, has no zero-order term, hence $z \equiv \ell_j^\pm$ are solutions. Indeed, since $\{\ell_j^\pm : a_j^\pm \lessgtr 0\}$ clearly approach values $(\ell_j^\pm, 0)$ at maximal rate as $x \to \pm\infty$, these are contained in the sets of solutions $\tilde{\Psi}^\pm$ from which $\Sigma'_{0,1}(L^*)$ is constructed. In particular,

$$\mathcal{L} := \mathcal{L}^- \cap \mathcal{L}^+ \subset \text{Span}\,\tilde{\Psi}^- \cap \text{Span}\,\tilde{\Psi}^+ \subset \Sigma'_{0,1}(L^*),$$

where $\mathcal{L}^\pm := \text{Span}\,\{\ell_j^\pm : a_j^\pm \lessgtr 0\}$. Indeed $\mathcal{L}$ is precisely the intersection of $\Sigma'_{0,1}(L^*)$ with the set of constant solutions.

Next, observe that $\dim \mathcal{L}^\pm = i^\pm$, hence

$$\begin{aligned} \dim \mathcal{L} &\geq i^- + i^+ - n \\ &= i - n = \ell, \end{aligned} \tag{10.5}$$

with equality if and only if $\mathcal{L}^- \cup \mathcal{L}^+ = \mathbb{R}^n$, or equivalently

$$\emptyset = (\mathcal{L}^-)^\perp \cap (\mathcal{L}^+)^\perp = \mathcal{R}^- \cap \mathcal{R}^+. \tag{10.6}$$

Therefore, the set of constant solutions contained in $\Sigma'_{0,1}(L^*)$ has dimension $\ell = \dim \text{Ker}(L)$ if and only if (10.6) holds. Thus, (i) clearly implies (ii), since then $\ell = \dim \Sigma'_{0,1}(L^*) = \dim \Sigma'_{0,1}(L)$. By the same reasoning, (ii) implies $\ell = \dim \Sigma'_{0,1}(L^*) = \dim \Sigma'_{0,1}(L)$. But, also, (ii) implies $\Sigma'_{0,2}(L^*) \setminus \Sigma'_{0,1}(L^*) = \emptyset$, since there can be no constant solution of the (inhomogeneous) generalized eigenvalue equation

$$(B^* z')' = -A^* z' + \tilde{\psi}.$$

It follows that $\dim \Sigma'_0(L^*) = \ell$, from which we obtain (9.6) and thus (i). $\square$

***Remark 10.4.*** In the case that the linear equations under study in fact arise from linearization about an underlying shock, the bounded solutions constructed in the proof have an alternative description in terms of the traveling wave ODE. The nondegeneracy condition of Definition 10.1 implies, by the Implicit Function Theorem, that the family of traveling wave solutions extends as a smooth manifold of solutions $\bar{u}(\delta, u_-, s)$, $\delta \in R^\ell$ parametrized by $(u_-, s)$, connecting $u_-$ to $u_+(u_-, s)$. The bounded solutions are then spanned by the partial derivatives of $\bar{u}^\delta(u_-, s)$ with respect to *both* $\delta$ and the parameters $(u_-, s)$. Likewise, (10.4) can be recognized as simply a linearized Rankine–Hugoniot relation. The relation of asymptotic dynamics to the underlying traveling wave ODE is an observation that goes back to [J].

Both conditions (i) and (ii) are closely related to conservation of mass. The same argument as in the proof of Proposition 10.3 (ii) shows that a *stable* standard shock for which $\{r_j^\pm : a_j^\pm \lessgtr 0\}$ are independent is of Lax or overcompressive type *if and only if* $\Sigma_0(L^*)$ consists entirely of constant functions, or equivalently the projection $\mathcal{P}_0 v_0$ of the initial perturbation onto $\mathrm{Ker}(L)$ is determined *entirely by the perturbation mass*, $\int v_0(y)dy$. For stable Lax and overcompressive shocks, this means that the $L^\infty$ time-asymptotic state (corresponding to shift and deformation of the shock profile) is determined by the mass of the initial perturbation. The ($L^\infty$) time-asymptotic state of undercompressive shocks on the other hand is generically *not* determined by the initial perturbation mass, perhaps the main distinction of their behavior from that of Lax and overcompressive shocks (see [LZ.2], Proposition 4.3.1 and just below, for a related, heuristic discussion). This clarifies the meaning of condition (ii).

Condition (i) has a similar interpretation. Recall, for stable shocks, that the $L^1$ time-asymptotic state consists of stationary excited modes $c_j\partial\bar{u}^\delta/\partial\delta_j$, plus outgoing Gaussian signals $\theta_j^\pm r_j^\pm$, $a_j^\pm \lessgtr 0$, the remainder of the solution decaying in $L^1$. Equating initial and time-asymptotic mass, we obtain

$$\int v_0\,dy = \sum_{1\le j\le \ell} c_j \int \frac{\partial \bar{u}^\delta}{\partial \delta_j}\,dy + \sum_{a_j^\pm \lessgtr 0}\left(\int \theta_j^\pm\,dy\right) r_j^\pm \tag{10.7}$$

by conservation of mass. Condition (i) is thus necessary and sufficient that the time-asymptotic state of a stable shock be determined by initial perturbation mass. More, it is clearly *necessary* for stability, since otherwise (10.7) is insoluble in general.

This observation, and a heuristic version of the above argument, was first given by Liu [L.1]. Indeed, for Lax shocks, condition (i) reduces to the familiar Liu–Majda condition,

$$\{r_j^\pm : a_j^\pm \lessgtr 0\} \cup (u_+ - u_-) \text{ is a basis for } R^n.$$

This has a second interpretation as the condition for *hyperbolic* stability of the associated inviscid (ideal) shock [M].

Proposition 10.3 has the following, extremely useful consequences:

***Proposition 10.5.*** *For Lax and overcompressive shocks, linearized orbital stability is equivalent to either of:*

(i) *linearized asymptotic stability with respect to zero-mass perturbations* $v_0 \in C_0^\infty$, $\int v_0 = 0$, *plus the condition that* $\{r_j^\pm : a_j^\pm \lessgtr 0\}$ *are independent.*

(ii) *linearized orbital stability with respect to zero-mass perturbations, plus the condition that*

$$\{r_j^\pm : a_j^\pm \lessgtr 0\} \cup \left\{ \int_{-\infty}^{+\infty} \frac{\partial \bar{u}^\delta}{\partial \delta_j} \, dx : \ 1 \le j \le \ell \right\}$$

*is a basis for* $R^n$.

*Proof.*

(i) Zero-mass stability is equivalent to the statement that $\Sigma'_{\lambda^*}(L^*)$ consists entirely of constant solutions whenever $\mathrm{Re}\lambda \ge 0$. It thus follows trivially from orbital stability, by Proposition 10.3(ii) and the observation that $\Sigma'_{\lambda^*}(L^*) = \emptyset$ for $\mathrm{Re}\lambda \ge 0$ and $\lambda \ne 0$. The converse follows by Proposition 10.3(ii) and the trivial observation that the adjoint eigenvalue equation

$$(B^* z')' = -A^* z' + \lambda^* z$$

can have no nontrivial constant solutions except when $\lambda = 0$, since all terms other than $\lambda z$ then vanish.

(ii) We need only show that (ii) implies stability, since the converse follows by (i). By the argument of the previous case, (ii) gives $\Sigma'_{\lambda^*}(L^*) = \emptyset$ and thus $\Sigma_{\lambda^*}(L) = \emptyset$ for $\mathrm{Re}\lambda \ge 0$ and $\lambda \ne 0$. Stability then follows from Lemma 9.3 together with Proposition 10.3. □

***10.3. The integrated equations.*** Zero-mass stability is often proved by studying the integrated equations

$$V_t = \tilde{L}V := -AV_x + BV_{xx}, \tag{10.8}$$

where

$$V(x,t) := \int_{-\infty}^{x} v(y,t)\, dy.$$

This point of view is profitable also within the Evans function framework. Let $\tilde{D}_L(\lambda)$ denote the Evans function associated with the "integrated" operator $\tilde{L}$.

***Proposition 10.6.*** *For Lax and overcompressive shocks, linearized orbital stability is equivalent to each of:*

(i) *linearized asymptotic stability of* (10.8), *plus independence of the vectors* $\{r_j^\pm : a_j^\pm \lessgtr 0\}$.

(ii) *linearized orbital stability of* (10.8), *plus independence of the vectors*

$$\left\{ \int_{-\infty}^{+\infty} (\partial \bar{u}^\delta / \partial \delta_j)\, dx : \ 1 \le j \le \ell \right\}.$$

(iii) $\tilde{D}_L(\lambda) \neq 0$ *on* $\{\mathrm{Re}\lambda \ge 0\}$ *plus independence of the vectors* $\{r_j^\pm : a_j^\pm \lessgtr 0\}$.

*Proof.* (i) and (iii): First, observe that (10.8) is of the form treated by our stability theory, hence asymptotic stability is equivalent to $\tilde{D}_L(\lambda) \neq 0$ on $\{\mathrm{Re}\lambda \ge 0\}$, or alternatively to $\Sigma'_\lambda(\tilde{L}) = \emptyset$ for $\mathrm{Re}\lambda \ge 0$.

Clearly, $(\partial/\partial x)\Sigma'_\lambda(\tilde{L}) \subset \Sigma'_\lambda(L)$, with equality if and only if $\int \Sigma'_\lambda(L) = 0$, meaning $\int \phi\, dx = 0$ for all $\phi \in \Sigma_\lambda(L)$. It follows that $\dim \Sigma'_\lambda(\tilde{L}) = \dim \Sigma'_\lambda(L)$ whenever $\int \Sigma'_\lambda(L) = 0$ and $\Sigma'_\lambda(\tilde{L})$ contains no constant functions. For $\lambda \neq 0$, the functions in $\Sigma'_\lambda(L)$ decay at $\pm\infty$. Thus, integrating the eigenvalue equation $0 = (Bw')' - (Aw)' - \lambda w$ over $(-\infty, +\infty)$, we find that $\lambda \int w = 0$, verifying $\int \Sigma'_\lambda(L) = 0$. Similarly, for constant solutions $W$ of the integrated eigenvalue equation, the only nonzero term is $\lambda W$, or $W = 0$, hence $\Sigma'_\lambda(\tilde{L})$ contains no constant functions. Thus, $\Sigma'_\lambda(\tilde{L}) = \emptyset$ is equivalent to $\Sigma'_\lambda(L) = \emptyset$ for $\lambda \neq 0$.

To complete the proof, it thus remains only to study $\lambda = 0$. It is easily seen, similarly as in the proof of Proposition 10.3(ii), that the constant functions in $\Sigma'_0(\tilde{L})$ are exactly (using the notation of that Proposition) $\mathcal{R}^+ \cap \mathcal{R}^- = \emptyset$. Thus, $(\partial/\partial x)\Sigma'_0(\tilde{L}) = \mathrm{Ker}(\tilde{L})$ consists of functions exponentially decaying at $\pm\infty$, hence must be empty if zero-mass stability holds (for unintegrated equations). For, otherwise $(\partial/\partial x)\mathrm{Ker}(\tilde{L}) \subset \mathrm{Ker}(L)$ would consist of $L^1$ stationary states with zero mass, contradicting zero-mass stability. By Proposition 10.5, this shows that (unintegrated) linearized orbital stability implies (i) and (iii).

On the other hand, the adjoint eigenvalue equation associated with $\tilde{L}$,

$$(B^*Z)'' = -(A^*Z)' + \lambda^* Z,$$

is the "differentiated equation" for the adjoint eigenvalue equation for $L$, i.e. can be obtained by setting $Z = z'$. Thus, $(\partial/\partial x)\Sigma'_0(L^*) \subset \Sigma'_0(\tilde{L}^*)$, so that $\Sigma'_0(\tilde{L}^*) = \emptyset$ only if $\Sigma'_0(L^*)$ consists of constant functions. By Proposition 10.3(ii), therefore, (i) or (iii) implies linearized orbital stability.

(ii) This follows by the argument for (i), together with the observation that independence of

$$\left\{ \int_{-\infty}^{+\infty} \frac{\partial \bar{u}^\delta}{\partial \delta_j}\, dx : 1 \le j \le \ell \right\}$$

alone is equivalent to (9.6). $\square$

***10.4. Lax and undercompressive shocks.*** For pure Lax and undercompressive waves, the condition of Lemma 9.4 simplifies to $z = c_1 \bar{u}_x$ for all bounded solutions of

$$(Bz')' = (Az)' + c_2 \bar{u}_x \tag{10.9}$$

such that $z(\pm\infty) \in \mathcal{R}^{\pm}$, since $\ell = 1$.

This leads to another simple interpretation of (9.6). Recall the full traveling wave ODE

$$(B(\bar{u})\bar{u}')' = (f(\bar{u}) - s\bar{u})'; \quad \bar{u}(-\infty) = u_-,$$

parametrized by $\alpha = (u_-, s)$. Differentiating with respect to $\alpha$, we obain the variational equation

$$(Bz')' = (Az)' - \alpha_2 \bar{u}_x; \quad z(-\infty) = \alpha_1,$$

for $z = \bar{u}_\alpha$, exactly the form of (10.9). Thus, we can interpret the bounded solutions of (10.9) as the tangent manifold about $\bar{u}$ of the *extended stationary manifold* of all viscous shock connections $(\tilde{u}_-, \tilde{u}_+, \tilde{s})$ nearby $(u_-, u_+, s)$.

By a counting argument similar to that in Proposition 10.3, it can be seen that the dimension of the manifold of parameters $(\tilde{u}_-, \tilde{s})$ for which such connections exist is of codimension equal to the degree of undercompressivity $\ell - 1$; that is, the requirement that a connection exist imposes an *extra* $\ell - 1$ constraints on $(\tilde{u}_-, \tilde{u}_+, \tilde{s})$ beyond the $n$ constraints of the Rankine–Hugoniot jump conditions.

As pointed out in [ZMP, Fre.2], this gives the right number of constraints that the hyperbolic equation $u_t + f(u)_x = 0$ be linearly well-posed for $C_0^\infty$ perturbations of the associated ideal (inviscid) shock, or, alternatively, that the Riemann problem for this hyperbolic equation be linearly well-posed in the vicinity of the shock. This can be seen by simply counting incoming and outgoing characteristics: there are $n + \ell - 2$ outgoing modes whose values must be determined at the free boundary, along with the shock speed $s$. Indeed, it is not difficult to see that the precise transversality condition determining linearized well-posedness, as determinded by the Implicit Function Theorem, is none other than (10.9). For details, we refer the reader to [ZMP, Fre.2].

We thus see that in the Lax and undercompressive case, (9.6) is equivalent to linearized stability of the associated hyperbolic shock/well-posedness of the associated Riemann solution. Condition (10.9) can be expressed compactly as the nonvanishing of an appropriate Melnikov integral. However, it must in general be checked numerically.

***10.5. Mixed type waves.*** In the case of waves of mixed type, (9.6) is again equivalent to nonvanishing of a certain Melnikov integral associated with (9.7). However, we have no straightforward interpretation of this quantity.

***Remark.*** The conclusions of Sections 10.2–10.5 extend and explain observations made in [GZ] by explicit evaluation of (9.6). In the case of an extreme overcompressive shock, for example, one obtains (by calculations described in [GZ])

$$(d/d\lambda)^{\ell} D_L(0) = \det\left(r_1^-, \cdots, r_{n-\ell}^-, \int \partial \bar{u}^{\delta}/\partial\delta_1, \cdots, \int \partial \bar{u}^{\delta}/\partial\delta_{\ell}\right) \\ \times \det\left(r_1^-, \cdots, r_{n-\ell}^-, s_1^-, \cdots, s_{\ell}^-\right),$$

where as usual $s_j^{\pm}$ denote eigenvectors of $B_{\pm}^{-1}A_{\pm}$. In the general case, one obtains a similar formula, the first term always corresponding to condition (i). The second term can be rather complicated; however, Proposition 10.3 guarantees that this term never vanishes, hence can be ignored.

The latter observation can be quite nontrivial in some cases. For example, in the case above, we find from Proposition 10.3 that

$$\det\left(r_1^-, \cdots, r_{n-\ell}^-, s_1^-, \cdots, s_{\ell}^-\right) \neq 0, \tag{10.10}$$

whenever $A_{\pm}, B_{\pm}$ are Majda–Pego stable pairs, with an appropriate shock connection between $u_{\pm}$. But, provided $A_-$, $B_-$ are Majda–Pego stable, we can always provide such a connection to a nearby Majda–Pego stable state $u_+$, by constructing an appropriate decoupled system and considering arbitrarily weak shocks. This establishes (very indirectly!) the linear algebraic result that (10.10) holds whenever $A_-$, $B_-$ are Majda–Pego stable. This result is important in the study of initial–boundary value problems, as described in [Se.1, Yo]. Indeed, the conclusion for Majda–Pego stable pairs resolves a question left open in [Se.1]; for $n > 2$, (10.10) was previously established only for $A$ and $B$ *symmetrizable.*

## 11. Nonlinear stability.

Using the pointwise bounds developed in the previous sections, one can prove the very general nonlinear stability result (Result 6) stated in the introduction. That will be the topic of future work [Z.2]. For now, we only point out that, in the Lax and overcompressive cases, we can immediately obtain nonlinear stability in the case $B \equiv$ *constant* by combining our pointwise bounds with a modified version of the argument used by Liu to treat the weak Lax case with artificial viscosity $B \equiv I$ [L.3]. We then show how to extend this result to the full, variable viscosity case, using the pointwise smoothing properties of strictly parabolic systems.

***11.1. The argument of Liu.*** There are three requirements that must be satisfied in order to apply the stability argument of [L.3]:

(i) First, the (nonlinear) $L^1$-asymptotic state of the perturbed shock must be formally determined by conservation of mass, in the sense that

$$\int_{-\infty}^{+\infty} v_0(y)\,dy = \int (\overline{u}^\delta(y) - \overline{u}(y))\,dy + \sum_{a_j^\pm \gtrless 0} m_j^\pm r_j^\pm, \tag{11.1}$$

must be soluble for $m_j^\pm$, $\delta$, at least for small $v_0$. Here, $v_0(y)$ is the initial perturbation of $\overline{u}(x)$, $\bar{u}^\delta$ is the asymptotic shift/deformation of $\bar{u}$, $a_j^\pm$ and $r_j^\pm$ are the eigenvalues and eigenvectors of $A_\pm = f'(u_\pm)$, and $m_j^\pm$ denotes the asymptotic mass in the $j$th outgoing characteristic field at $\pm\infty$. By the Implicit Function Theorem, this requirement is equivalent to the transversality condition (9.6), which follows from *linearized stability*, $(\mathcal{D})$.

(ii) Second, the Green's functions both for the linearized equation $v_t = Lv$ with *zero-mass perturbation* $v = V_x$ and the integrated equation $V_t = \tilde{L}V$ must satisfy the approximate bounds derived by Liu, with an error of order $\varepsilon e^{-\eta|x|}$, $\varepsilon$ sufficiently small. Since we deal in this paper with the exact Green's function, we have no such error, i.e. $\varepsilon = 0$. The appropriate bounds on the Green's function for the integrated equation follow from the observation in Proposition 10.6 that linearized stability implies $\Sigma_0'(\tilde{L}) = \emptyset$. Thus, the $\mathbf{O}(1)$ excited term $E$ does not appear in the bounds of Theorem 8.3, and the remaining terms satisfy the required estimates: namely, they are bounded by Gaussian signals scattering from the shock (cf. estimates in Sections 3–4 of [L.3]).

For the unintegrated equation with data $v = V_x$, Liu writes the solution as

$$\int G_y(x,t;y)V(y,0)\,dy, \tag{11.2}$$

after integrating by parts. More precisely, therefore, the improved bounds we require are on $|G_y|$. These follow, again, by the previously observed properties of Lax and overcompressive shocks. By Proposition 10.3, $\Sigma_0'(L^*)$ consists entirely of constant solutions. Thus, the dominant terms $\phi(x)\pi(y)$, $\pi \in \Sigma_0'(L^*)$ in the excited mode $E$ disappear under differentiation by $y$, and we again obtain the improved estimate needed for the argument. Alternatively, (11.2) can be written as

$$\int \tilde{G}_x(x,t;y)V(y,0)\,dy,$$

where $\tilde{G}$ is the Green's function for the integrated equations, and the same result obtained more transparently from the improved bounds for the integrated equations. [5] We omit the details.

[5] These considerations are well-illustrated in the case of the scalar Burgers equation,

Note, again, that the weak shock assumption in [L.3] was necessary only to control the error in the construction of an approximate Green's function. This error is mainly due to approximate diagonalization of the system. We do not have any such errors, since we work with the exact Green's function; in particular, we do not perform any approximate diagonalization. Our estimates are nonetheless *asymptotically diagonal* as a consequence of (**H2**), just as are those of [L.3]. Beyond this requirement, the only smallness assumption needed is on the perturbation. Thus, in our argument *there is no limitation on the strength of the shock.*

(iii) The third, purely technical requirement of the argument in [L.3] is that the viscosity matrix be constant. This has the consequence that the quadratic source term $Q$ appearing in the perturbation equation depends on $v$ alone, and not $v_x$, eliminating the need to "gain a derivative" (see (11.5), below). We will show in a moment how to remove this hypothesis.

***11.2. Basic stability result.*** By the observations in the previous subsection, we can immediately apply the pointwise Green's function argument of [L.3] to obtain a stability result in the case $B \equiv$ constant.

Following Liu, let the vectors $r_j^\pm$ be normalized so that either $\nabla_u a_j^\pm(u_\pm) \cdot r_j^\pm = 1$, or $\nabla_u a_j^\pm(u_\pm) \cdot r_j^\pm = 0$, where $a_j(u)$ here refer to the eigenvalues of the matrix $A(u) := f'(u)$, which are well-defined near $u_\pm$ by assumption (**H2**). In the first case, we refer to the corresponding field $j^\pm$ as *genuinely nonlinear*, in the second case *linearly degenerate.* Associated with the outgoing modes $a_j^\pm \gtrless 0$, define *diffusion waves*, governed accordingly by the Burgers or heat equation:

$$(\theta_j^\pm)_t + a_j^\pm(\theta_j^\pm)_x + \left(\frac{1}{2}(\theta_j^\pm)^2\right)_x = \beta_j^\pm(\theta_j^\pm)_{xx}, \quad j^\pm \text{ genuinely nonlinear,} \tag{11.3}$$

$$(\theta_j^\pm)_t + a_j^\pm(\theta_j^\pm)_x = \beta_j^\pm(\theta_j^\pm)_{xx}, \quad j^\pm \text{ linearly degenerate,}$$

$$\int_{-\infty}^{+\infty} \theta_j^\pm(x,t)\,dx = m_j^\pm.$$

Here, $m_j^\pm$ are as determined in (11.1) and the effective diffusion coefficients $\beta_j^\pm$ are as in (**K3**), Section 2, i.e. $\beta_j^\pm := l_j^\pm B_\pm r_j^\pm$, where $l_j^\pm$ are the left eigenvectors of $A_\pm$ corresponding to the right eigenvectors $r_j^\pm$.

From [LZe], it is known that $\theta_j^\pm r_j^\pm$ well-approximates the $L^1$-asymptotic state of perturbations in the far field, in each outgoing mode $j^\pm$. Define the

Example 8.6, where we have already seen that our bounds are sharp. Recall that Liu's bounds derive from scalar equations, and *approximate* the exact Burgers bounds.

remainder $v(x,t)$ by

$$u(x,t) \equiv \bar{u}^{\delta}(x) + \sum_{a_j^{\pm} \gtrless 0} \theta_j^{\pm} r_j^{\pm} + v(x,t), \tag{11.4}$$

where $\delta$ and $\theta_j^{\pm}$ are as determined in (11.1) and (11.3). By subtracting off the $L^1$-asymptotic state in this way, we obtain the *zero mass condition*,

$$\int_{-\infty}^{+\infty} v(x,t)\, dx = 0.$$

The remainder $v$ satisfies a modified perturbation equation

$$v_t - Lv = Q(\theta, v)_x + M(\theta)_x, \tag{11.5}$$

where $Q$ as usual denotes a quadratic order source term and

$$\theta = \sum_{a_j^{\pm} \gtrless 0} r_j^{\pm} \theta_j^{\pm}.$$

The new, inhomogeneous term $M$ is of linear order,

$$\begin{aligned} M(\theta) &= -(\partial/\partial t - L)\theta \\ &= \sum_{a_j \pm \gtrless 0} \big((A(x) - a_j^{\pm} I) r_j^{\pm} \theta_j^{\pm}\big)_x - \big((B(x) - \beta_j^{\pm} I) r_j^{\pm} (\theta_j^{\pm})_x\big)_x \end{aligned} \tag{11.6}$$

***Proposition 11.1.*** *Let* (**H0**)–(**H4**) *hold, with $B \equiv$ constant, and let $\bar{u}$ be a pure Lax or overcompressive shock satisfying the stability condition $(\mathcal{D})$. Then, $\bar{u}$ is $L^p$ nonlinearly orbitally stable, $p > 1$, with respect to perturbations in*

$$\mathcal{A}_{\zeta} := \{f : |f(x)| \le \zeta (1 + |x|)^{-3/2}\},$$

*for $\zeta$ sufficiently small. Moreover, for $v(x,t)$ defined as in* (11.4)*, it holds that*

$$\begin{aligned} v(x,t) &= \mathbf{O}(\zeta(|x|+1)^{-1}(t+|x|+1)^{-1/2}) \\ &\quad + \sum_{a_j^{\pm} \gtrless 0} r_j^{\pm} \mathbf{O}(\zeta[\psi_j^{\pm}(x,t)^{3/2} + \chi_j^{\pm}(x,t)]) + \mathbf{O}(\zeta \overline{\psi}_j^{\pm}(x,t)^{3/2}), \end{aligned} \tag{11.7}$$

*where*

$$\begin{aligned}
(11.8)\qquad \psi_j^{\pm}(x,t) &:= [(x-a_j^{\pm}(t+1))^2+t+1]^{-1/2},\\
\overline{\psi}_j^{\pm}(x,t) &:= [(x-a_j^{\pm}(t+1))^3+(t+1)^2]^{-1/3},\\
\chi_j^{\pm}(x,t) &:= \min(\bar{\chi}_j^{\pm}(x,t),(|x|+1)^{-1/2}(t+1)^{-1/2}),\\
\bar{\chi}_j^{\pm}(x,t) &:= |x-a_j^{\pm}t|^{-1}(1+|x-a_j^{\pm}t|)^{-1/2}\chi_{\{x\in[0,a_j^{\pm}(t+1)(1+(t+1)^{1/2})]\}}.
\end{aligned}$$

*Proof.* Following [L.3], the variable $v$ is augmented with $v_x$, $w := \int_{-\infty}^{x} v$ and $z := w_t$, and to these variables there are associated corresponding template functions $h_v(x,t)$, $h_{v_x}$, $h_w(x,t)$, and $h_z(x,t)$, given by

$$\begin{aligned}
(11.9)\qquad h_v(x,t) &:= (|x|+1)^{-1}(t+|x|+1)^{-1/2}+\sum_{a_j^{\pm}\gtrless 0}(\psi_j^{\pm}(x,t)^{3/2}+\chi_j^{\pm}(x,t)),\\
h_{v_x}(x,t) &:= (|x|+1)^{-1}(t+|x|+1)^{-1/2}+\sum_{a_j^{\pm}\gtrless 0}\bar{\psi}_j^{\pm}(x,t)^2,\\
h_w(x,t) &:= (|x|+1)^{-1}(t+|x|+1)^{-1/4}+\sum_{a_j^{\pm}\gtrless 0}\psi_j^{\pm}(x,t)^{1/2},\\
h_z(x,t) &:= (t+|x|+1)^{-1}+(|x|+1)^{-1}(t+|x|+1)^{-1/2}\\
&\qquad +\sum_{a_j^{\pm}\gtrless 0}(\alpha_j^{\pm}(x,t)+\psi_j^{\pm}(x,t)^{3/2}+\bar{\chi}_j^{\pm}(x,t)),
\end{aligned}$$

where $\alpha_j^{\pm}(x,t) := (t+1)^{-1/2}e^{(x-a_j^{\pm}(t+1))^2/M(t+1)}$, and $\chi_j^{\pm}$, $\bar{\chi}_j^{\pm}$, $\psi_j^{\pm}$, and $\bar{\psi}_j^{\pm}$ are as in (11.8) above. Note that the template function $h_v$ is roughly $1/\zeta$ times the modulus of the right hand side of (11.7). All four template functions are related in this way to the right hand sides of equations (7.8)–(7.14) of [L.3], p. 56.

These Ansatze are then verified by the continuous induction argument described in Lemma 1.6 of the introduction, with the minor modification that we must add to (1.10) the hypothesis

$$(11.10)\qquad \left|\int G_y(x,t-s;y)M_i(\theta)(y,s)\,dy\,ds\right| \le Cmh_i(x,t), \quad i=v,v_x,w,z,$$

where $m := \sum_{a_j^{\pm}\gtrless 0}|m_j| = \mathbf{O}(\zeta)$ denotes the total mass of diffusion waves. This ensures that the contribution to (1.9) from the source terms $M_i(\theta)_x$ can be absorbed in the first term on the right hand side of (1.11). For the (highly nontrivial) details of the verification of (1.10) and (11.10), we refer to [L.3].

The application of Lemma 1.6 results in modulus bounds for $v$, $v_x$, $w$, and $z$. These bounds can then be filtered back through (1.9) to give the more precise information in (11.7) of the magnitude of the perturbation in different characteristic fields.

In [L.3], the magnitude in each characteristic field is controlled throughout the iteration using a separate template function. Though this approach is convenient in the approximately decoupled context of [L.3], it is not necessary for the argument. Since the quadratic source terms couple all characteristic modes of $v$, the only benefit of separate estimates is to give a decomposition of $v$ into component waves supported mainly along characteristic directions $x = a_j^{\pm} t$, which can then be estimated separately using different techniques. This decomposition can be accomplished equally well by a partition of unity with respect to $x/(t+1)$. Note that spatial derivatives falling on a cutoff function depending on $x/(t+1)$ give an extra factor of time-decay $(t+1)^{-1}$, *more* than the difference between decay rates of $v$ and $v_x$, or $w$ and $v$. Thus, we recover all of the estimates of Liu using our simplified modulus bounds. □

***Corollary 11.2.*** *Under the hypotheses of Proposition 11.1,*

$$\|v(\cdot,t)\|_{L_p} = \begin{cases} \mathbf{O}(1)(t+1)^{(-3p-2)/(4p)}, & 1 \le p \le 2, \\ \mathbf{O}(1)(t+1)^{-1/2}, & 2 \le p \le \infty, \end{cases}$$

*and*

$$\|(u - \bar{u}^{\delta})(\cdot,t)\|_{L_p} = \mathbf{O}(1)(t+1)^{-1/2-1/2p},\ 1 \le p \le \infty.$$

As in [L.3], all decay rates given are optimal. From the stated $L^p$ bounds, we see that the dominant behavior of a perturbed shock is indeed shift/deformation of the shock plus emission of diffusion waves, as described in (11.4). This validates the heuristic picture of behavior given in [L.1].

***11.3. Short-time theory.*** Before treating the variable viscosity case $B \not\equiv$ *constant* we provide the requisite short-time existence/regularity theory for general quasilinear parabolic systems, using the *parametrix method* of Levi [Fr, GM, LU, LSU, Le]. Though rather straightforward, these results do not seem to be readily available in the literature, except for systems with special, essentially scalar viscosity matrices [LSU].

***Proposition 11.3.*** *Let $A(x,t)$, $B(x,t)$, and $C(x,t)$ be uniformly bounded in $L^\infty$ and $C^{(0,0)+(\alpha,\alpha/2)}(x,t)$, $0 < \alpha, \beta < 1$, taking values on a compact set, with* $\mathrm{Re}\,\sigma(B)$ *bounded strictly away from zero. Then, for* $0 < t < T$, *$T$ sufficiently* *small, there is a Green's function* $G(x,t;y,s) \in C^{2,1}(x,t)$ *associated with the Cauchy problem for*

$$v_t = Cv + Av_x + Bv_{xx}, \quad v \in \mathbb{R}^n, \tag{11.11}$$

*satisfying bounds*

$$|D_x^j G(x,t;y,s)| \le Ct^{-(j+1)/2} e^{-(x-y)^2/M(t-s)}, \quad j = 0,1,2, \tag{11.12}$$

*where* $C$, $M$, $T > 0$ *depend only on the bounds on the coefficients and on the lower bound on* $\operatorname{Re}\sigma(B)$.

*Likewise, for an equation in divergence form,*

$$v_t = Cv + (Av)_x + (Bv_x)_x, \quad v \in \mathbb{R}^n, \tag{11.13}$$

*there is a Green's function* $G(x,t;y,s) \in C^{1,0}(x,t)$ *in the distributional sense, satisfying* (11.12) *for* $j = 0,1$.

*Proof.* The first stated result is standard for scalar equations. The method of proof depends only on the existence of corresponding bounds for arbitrary $x$- and $t$-derivatives of the Green's functions $\Gamma_{y_0,s_0}$ of the "frozen", constant-coefficient systems

$$v_t - B(y_0,s_0)v_{xx}$$

obtained from the principal part of (11.11), for all $(y_0,s_0)$ in the domain of interest, along with Hölder bounds on these derivatives with respect to $(y_0,\ s_0)$ (see [Fr], p. 16, or [GM], p. 173). For scalar equations, these bounds are usually obtained by explicit reduction to the heat equation. They can be obtained for systems by Fourier Transform arguments like those described in Section 7 of [HoZ.1] (see especially estimate (7.26)), together with the observation that the hypotheses imply strict parabolicity, $B(y_0,s_0) \ge \eta I > 0$, after an appropriate linear change of (independent) variable, where, by compactness, $\eta$ as well as the condition number of the change of coordinates, is uniform in $(y_0,s_0)$. Alternatively, these bounds can be obtained by the methods of the present paper (sections 2-8) applied to the constant-coefficient case. The bounds on $\Gamma_{y_0,s_0}$ are then used in the explicit iteration of a Volterra integral equation for $G-\Gamma$, carried out up to two spatial derivatives, plus their Hölder quotients [Fr,GM].

The divergence-form result can be obtained by exactly the same iteration procedure as for the non–divergence-form equation, but iterating only spatial derivatives up to order one instead of two. The crucial step in either case is to estimate the Hölder quotients of a convolution of $\Gamma_{xx}$ (resp. $\Gamma_x$) with a $C^{(0,0)+(\alpha,\alpha/2)}$ function exhibiting appropriate Gaussian decay in its Hölder quotients: specifically, $G_{xx}$ in the non–divergence-form case, $G_x$ in the divergence-form case. These convolutions arise in the Hölder estimates of second and first spatial derivatives, respectively. In both cases, $G - \Gamma$ is simultaneously seen to be less singular than $\Gamma$, verifying the Green's function property. □

**Corollary 11.4.** *Let $f, g, h \in C^{2+\alpha}(u,x)$. Then, for initial data $u(\cdot,0)$ in $L^\infty \cap C^{0+\alpha}$, the Cauchy problem*

$$u_t + f(u,x)_x + g(u,x)u = (h(u,x)u_x)_x, \quad u \in \mathbb{R}^n, \tag{11.14}$$

*has a classical solution $u \in C^{(2,1)+(\alpha,\alpha/2)}(x,t)$ for $0 < t \le T$, $T > 0$ depending only on the $L^\infty$ and Hölder norms of $u(\cdot,0)$.*

*Proof.* Equation (11.14) can be rewritten in form (11.13), where

$$\begin{aligned} B(x,t) &:= h(u(x,t),x), \\ A(x,t) &:= \int_0^1 f_u(\gamma u, x)\, d\gamma, \\ C(x,t) &:= f_x(0,x) + g(u,x). \end{aligned} \tag{11.15}$$

Writing

$$u(x,t) = \mathcal{T}(u) := \int_{-\infty}^{+\infty} \tilde{G}(x,t;y,T_0)u(y,T_0)\, dy \tag{11.16}$$

where $\tilde{G}$ denotes the Green's function for (11.13), and applying the $\tilde{G}$ bound of (11.12), we find in the usual way that $\mathcal{T}$ is a contraction on a sufficiently large ball in $C^{(0,0)+(\alpha,\alpha/2)}(\mathbb{R}^1 \times [0,T])$, for $T > 0$ sufficiently small. Thus, we obtain a fixed point solution $u \in C^{(0,0)+(\alpha,\alpha/2)}$ of (11.16).

Differentiating,

$$u_x(x,t) = \int_{-\infty}^{+\infty} \tilde{G}_x(x,t;y,T_0)u(y,T_0)\, dy, \tag{11.17}$$

and applying now the $\tilde{G}_x$ bounds of (11.12), we find that the solution $u$ is in fact in $C^{(1,0)+(\alpha,\alpha/2)}$. By the assumed regularity of $f$, $g$, and $h$, this gives that coefficients $A$, $B$, $C$ are in $C^{(1,0)+(\alpha,\alpha/2)}$, and thus (11.13) can be rewritten as a non–divergence-form equation,

$$u_t + (C + A_x)u + (A - B_x)u_x - Bu_{xx},$$

still with $C^{(0,0)+(\alpha,\alpha/2)}$ coefficients. Applying the stronger Green's function bounds for non–divergence-form operators (and the implied bound on $\tilde{G}_t$), we find that $u$ is in fact in $C^{(2,1)+(\alpha,\alpha/2)}(x,t)$, as claimed. □

***Remark.*** For scalar equations, the result of Corollary 11.4 holds for data merely in $L^\infty$, by the Nash–Moser bounds on the fundamental solution, together with standard energy estimates and Sobolev embedding. However, it is not clear whether such a result is true for systems.

***11.4. The variable viscosity case.*** The requirement $B \equiv$ *constant* can now be removed using the short-time "pointwise" smoothing described in Proposition 11.3. A straightforward calculation (see, e.g., Lemma 5.1 [SZ]) establishes

***Lemma 11.5.*** *For $t \geq T > 0$, $|a| \leq a_0$, and any $r$,*

$$\left| \int_{t-T}^{t} \int_{-\infty}^{+\infty} (t-s)^{-1} e^{-(x-y)^2/M(t-s)} (1+|y-as|)^r \, dy \, ds \right| \\ \leq C(t-T)^{-1/2}(1+|x-at|)^r,$$

*where $C$ depends only on $M$, $r$, and $a_0$.*

More generally, we have the following result:

***Lemma 11.6.*** *Let $h_v$, $h_{v_x}$, $h_w$, and $h_z$ be the template functions described in the proof of Proposition 11.1. Then, for $t \geq 2T > 0$, $T > 0$ fixed,*

$$\int_{t-T}^{t} \int_{-\infty}^{+\infty} (t-s)^{-1} e^{-(x-y)^2/M(t-s)} h_i(y,x) \, dy \, ds \leq C h_i(x,t), \quad i = v, v_x, w, z,$$

*where $C$ depends only on $T$, $M$, and the values of $a_j^{\pm}$.*

*Proof.* The template functions $h_i(x,t)$ are the sum of algebraic waves of form

$$(1+x-at)^r(1+t)^q,$$

the characteristic cones

$$\chi_{\{x \in [0, a_j^{\pm}(t+1)(1+(t+1)^{1/2})]\}},$$

and simple combinations thereof. Bounding $(1+s)^q \leq C(1+t)^q$ for $t-T \leq s \leq t$ and $(t-T)^{-1/2} \leq T^{-1/2} \leq C$, using the facts that $t \geq 2T$ and $T$ is bounded away from zero, we can therefore obtain the result by essentially the calculation of the previous Lemma. The only new feature is diffusion at the edges of the characteristic cones $\chi$, which can be absorbed by the algebraic waves $(1+|x|)^{-1}(t+1)^{-1/2}$ and $\psi_j^{\pm}$. $\square$

**Corollary 11.7.** *The conclusions of Proposition 11.1 hold also for non-constant $B(u)$, provided the initial data $u_0$ is required additionally to lie in $C^{0+\alpha}$, $\alpha > 0$.*

*Proof.* As noted above, the new feature in this case is that the quadratic source terms $Q_i$, $i = v, v_x, w, z$ appearing in the remainder equations for $v$, $v_x$, $w$, and $z$ now depend on the derivatives of the remainders as well as the remainders themselves. The issue is therefore to obtain bounds on the derivatives equivalent to those on the undifferentiated variables. Noting that $w_x = v$, $z_x = w_{tx} = v_t \sim v_{xx}$, and $(v)_x = v_x$, we see that it is sufficient to control $v_{xx}$ by the template $h_{v_x}$ for $v_x$. For expositional reasons, however (and because it is of use in more general arguments), we will first show how to bound $v_x$ by $h_v$.

Subtracting $\bar{u}_t + f(\bar{u})_x - (B(\bar{u})\bar{u}_x)_x = 0$ from $u_t + f(u)_x - (B(u)u_x)_x = 0$, and recalling that $u = \bar{u} + \theta + v$, we can regroup terms in (11.5) to obtain the modified remainder equation

$$v_t - \tilde{L}v = \tilde{M}(\theta)_x, \qquad (11.18)$$

$$\tilde{L}v := -(\tilde{A}v)_x + (\tilde{B}v_x)_x,$$

where

$$\tilde{B}(x,t) := B(u(x,t))$$

$$\tilde{A}v := (f(u) - f(u-v)) + (B(u) - B(u-v))(\bar{u}_x + \theta_x).$$

$$\tilde{Q}(v,\theta) := f(u) - f(\bar{u}) - f'(\bar{u})(v+\theta), \qquad \text{and}$$

$$\tilde{M}(\theta) := M(\theta) + (f(\bar{u}+\theta) - f(\bar{u})) + ((B(\bar{u}+\theta) - B(\bar{u}))\theta_x)_x,$$

$M(\theta)$ as in (11.5). (Here, again, $\tilde{A}(x,t)$ is defined via the integral Mean Value Theorem, e.g. $B(u) - B(u-v) = (\int_0^1 B'(u-sv)\,ds)v$).

By Corollary 11.4, the solution $u(x,t)$, and thus the coefficients $\tilde{A}$ and $\tilde{B}$ are bounded in $L^\infty \cap C^{(0,0)+(\alpha,\alpha/2)}$ up to some time $2T > 0$, hence the results of Proposition 11.3 apply. Writing

$$v_x(x,t) = \int_{-\infty}^{+\infty} \tilde{G}_x(x,t;y,t-T)v(y,t-T)\,dy \qquad (11.19)$$

$$+ \int_{t-T}^{t}\int_{-\infty}^{+\infty} \tilde{G}_x(x,t;y,s)\tilde{M}(\theta)_x(y,s)\,dy\,ds.$$

by Duhamel's principle, where $\tilde{G}$ is the Green's function for $(\partial/\partial t - \tilde{L})$, and using the $\tilde{G}_x$ bounds of Lemma 11.3 for divergence-form operators, we find that

$$\|v_x(\cdot,t)\|_\infty \le C(T)^{-1/2}\|v(\cdot,t-T)\|_\infty + C(T)^{1/2}\|\tilde{M}(\theta)_x\|_\infty.$$

In particular, for $t \ge T$, we obtain a uniform Hölder (indeed, Lipschitz) bound on $v(\cdot,t)$ depending only on the $L^\infty$ norm of $v(\cdot,t-T)$. By the (standard) method of extension, we thus obtain uniform Hölder continuity of $v$ so long as $|v|$ remains bounded, in particular so long as $|v(x,t)| \le C\eta h(x,t)$ remains valid. Thus, we may assume throughout the induction that short-time Green's function bounds of (11.12) remain uniformly valid for $j = 0,1$. (Note: this step is not necessary in the argument of [L.3], since $v_x$ is also controlled. However, we include it for more general purposes).

Combining the $\tilde{G}_x$ bounds of (11.12), the ansatz $|v| \le \zeta h$, and Lemma 11.6 plus a straightforward calculation on the $\tilde{M}(\theta)$ term of the right hand side, [6] we thus obtain that

$$|v_x(x,t)| \le C\zeta h_v(x,t)$$

provided $t \ge 2T$ and $|v(x,s)| \le C_1\zeta h_v(x,s)$ for $t-T \le s \le t$.

To complete the argument for $v$, we need only observe that the induction can be started at time $t = 2T$ instead of $t = 0$. For, (11.19) together with the $\tilde{G}$ bounds of Proposition 11.3 show, by a calculation similar to but less singular than that of Lemma 11.6, that the bounds assumed on $u - \bar{u} = \theta + v$ at time $t = 0$ persist to time $t = 2T$. (Alternatively, one could perform an initial layer analyis.) Similarly, one can verify that the induction hypotheses $|v| \le C\zeta h_v$, $|v_x| \le C\zeta h_{v_x}$, $|w| \le C\zeta h_w$, and $|z| \le C\zeta h_z$, $h_i$ given by (11.9), hold at time $t = 2T$, given only the assumed bounds on $u - \bar{u}$.[7]

Similar results hold for the critical variable $v_x$ and its derivative $v_{xx}$. Differentiating (11.18), we find that $v_x$ satisfies an equation

$$(v_x)_t - \tilde{\tilde{L}}v_x = \tilde{M}_{xx} + (\tilde{A}_x v)_x$$

similar in form to (11.18), but with the new source term

$$\tilde{A}_x v = \mathbf{O}(|v|\ |\bar{u}_x| + |v|\ |\theta_x| + |v_x|).$$

The $\mathbf{O}(|v_x|)$ and $\mathbf{O}(|v|\ |\theta_x|)$ terms can be handled in the same ways as were the first and second terms of (11.19) in the previous case. The term

$$\mathbf{O}(|v|\ |\bar{u}_x|) = \mathbf{O}(t^{-1/2}e^{-\alpha|x|})$$

[6] Recall, $h(x,t)$ is designed to accomodate the effects of $M(\theta) \sim \tilde{M}(\theta)$, Section 1.1.4.

[7] This observation repairs an omission in [L.3]. In general, the hypothesis $|v_x| \le C\zeta h_{v_x}$ does *not* hold at $t = 0$, since we have made no assumptions on $v_x(\cdot,0)$.

gives rise to a contribution of similar size, $\mathbf{O}(t^{-1/2}e^{-\alpha|x|})$, provided $T$ is chosen sufficiently small, by a computation similar to that of Lemma 11.5. Noting that this can be bounded by the term $(|x|+1)^{-1}(t+|x|+1)^{-1/2}$ appearing in each of $h_v, \ldots, h_z$, we are done.

In summary, the new derivative terms appearing in the quadratic source terms $Q_i$ satisfy the same bounds as do the undifferentiated terms, hence the induction argument can be carried out as before, for all times $t \geq 2T > 0$. □

***Remark.*** For "physical", nonstrictly parabolic matrices $B(u)$, the treatment of the extra derivative term $v_x$ is by contrast highly nontrivial. In this case, stability depends in part on *nonlinear structure* as manifested by energy estimates. We refer the reader to [Ka,LZe] for further discussion.

## 12. Neutral Stability: Wave Splitting.

The ideas of this paper are perhaps best illustrated by the boundary case that $\sigma(L)\cap\{\mathrm{Re}\lambda \geq 0\} = \{0\}$ but $\Sigma_0'(L) \not\subset \mathrm{Span}\{\partial\bar u/\partial u\delta_j\}$, that is, the case that $\Sigma_0'(L)$ contains eigenfunctions not decaying at $x = \pm\infty$. This typically corresponds to the phenomenon of *wave-splitting*, an interesting variation on the standard theme of *bifurcation from instability*. We conclude with two examples of this type.

***An undercompressive family.*** Consider *Holden's model*,

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} \frac{1}{2}v^2 - \frac{1}{2}u^2 + v \\ uv \end{pmatrix}_x = \begin{pmatrix} u \\ v \end{pmatrix}_{xx}, \tag{12.1}$$

a model equation for three-phase flow.

For speed $s = 0$, it is easily verified that the traveling wave ODE (1.4) is Hamiltonian, with $H(u,v) = \frac{1}{2}(\frac{1}{3}v^3 - u^2v + v^2) - (\frac{1}{2}v_-^2 v - \frac{1}{2}u_-^2 v + v_- v - u_- v_- u)$ preserved along orbits [GZ]. In particular, for $(u_-,v_-) = (1,0)$, there is an orbit $\bar U(x) = (\bar u(x), \bar v(x))$ lying along the hyperbola

$$\frac{1}{3}v^2 + v = u^2 - 1,$$

connecting $(u_-,v_-)$ to $(u_+,v_+) = (-1,0)$.

Straightforward calculation gives

$$A_\pm = \begin{pmatrix} \pm 1 & 1 \\ 0 & \mp 1 \end{pmatrix},$$

$a_j^\pm = (-1)^j$, $j = 1,2$. Thus, the wave is undercompressive, with one incoming and one outgoing characteristic mode on each side $x = \pm\infty$, and $\ell = \dim\{\bar U^\delta\} =$

1. Indeed, $\{\bar{U}^\delta\}$ consists simply of the set of translates $\{(\bar{U}(x-\delta)\}$. Moreover, the eigenvectors corresponding to the outgoing modes $a_j^\pm \gtrless 0$ are clearly $r_1^- = r_2^+ = (1,0)^t$.

Observing that

$$\int_{-\infty}^{+\infty} (\bar{U}(x-\delta) - \bar{U}(x))\,dx = \delta(U_+ - U_-) = 2\delta(1,0)^t,$$

we find that the right hand side of (11.1) lies always parallel to $(1,0)^t$, hence (11.1) cannot be satisfied for arbitrary perturbation mass $\int_{-\infty}^{+\infty} V_0(x)\,dx$. This violates Liu's formal picture of the asymptotic behavior of a stable shock, and in fact under general perturbations the wave *does not* time-asymptotically approach one of its translates. Rather, it approaches a member of the larger family of zero-speed shocks, $\{\bar{U}^{\delta,\alpha}\} := \{\bar{U}_\alpha(x-\delta)\}$, connecting states $(\pm\alpha,0)$ along similar, hyperbolic orbits, concatenated with preceding and following Lax 1 and 2 waves lying in the outgoing directions $r_1^- = r_2^+ = (1,0)^t$. These waves do not have zero speed, but propagate toward $x = \pm\infty$ at linear rates. This kind of breakup under perturbation, of a single shock into more than one distinct wave, is known as *wave-splitting.* It is a very mild, neutral type of instability. Note that wave-splitting corresponds with the usual picture of multiple solutions bifurcating from neutral instability, but with the interesting twist that the new solutions jump to a different functional class (i.e. time-varying vs. stationary).

At the linearized level, we find, correspondingly, that the function $w = \partial\bar{U}^{\delta,\alpha}/\partial\alpha$ satisfies the zero-eigenvalue equation, with boundary values $w(\pm\infty)$ lying parallel to $r_1^- = r_2^+ = (1,0)^t$, hence is a nondecaying effective zero eigenfunction. And, according to the description of behavior in Theorem 8.3, the solution indeed moves along the tangent manifold of $\{\bar{U}^{\delta,\alpha}\}$, with the change in endstate propagating outward toward $\pm\infty$ at linear rates $a_2^+ > 0$ and $a_1^- < 0$, respectively. To verify this picture of neutral stability at the nonlinear level would require the stability of the time-varying, three wave pattern, which is more involved; in fact, the treatment of shocks and rarefactions together has not yet been carried out, though this certainly should be possible using the techniques already available.

We remark that homoclinic shock profiles occurring in the same model are strongly (exponentially) unstable, as demonstrated in [GZ].

***Fake Lax shocks.*** A second example, remarked earlier by Freistühler, [Fre.1], occurs in the cubic model,

$$U_t + (|U|^2U)_x = U_{xx}, \quad U = (u,v)^t \in \mathbb{R}^2, \tag{12.2}$$

a model equation for phenomena in MHD. For $U_- = (1,0)^t$ and any $U_+ = (-\alpha,0)$, $1/2 < \alpha < 1$, there is a pure Lax shock solution $\bar{U}_\alpha(x)$ of ODE (1.4), connecting repellor $U_-$ to saddle $U_+$, for some speed $s(\alpha)$. As in the previous example, the stationary manifold is dimension $\ell = 1$, consisting entirely of translates $\bar U(x-\delta)$. Moreover, calculation shows that, again, $(U_+ - U_-)$ and the outgoing eigenvector $r_1^-$ at $-\infty$ lie in the same direction $(1,0)^t$. Thus, $\bar U$ violates the Liu–Majda condition (1.24), indicating the existence of an additional zero effective eigenfunction not lying in the direction of the stationary manifold.

Also as before, this zero eigenfunction is given by $\partial \bar U_\alpha/\partial\alpha$, and this correspondis to a splitting of the wave into a nearby $\bar U_\alpha$ plus a following, Lax 2-wave. It behaves more like an overcompressive than a Lax shock, in the sense that it moves under perturbation within a two-parameter family of waves, and for this reason has sometimes been called a "fake Lax" shock (in this sense, the examples above are also "fake undercompressive" shocks).

However, in this case, $s(\alpha) \not\equiv$ *constant*, making the situation somewhat more interesting. This has the consequence that $\partial \bar U_\alpha/\partial\alpha$ satisfies, not the zero eigenvalue equation, but the *generalized* zero eigenvalue equation. That is, it belongs to a Jordan chain ascending from $\bar U_x$. Thus, $\bar U$ is not even neutrally stable; rather, perturbations can grow at *linear rate* in the mode $\bar U_x$, corresponding to linear, non-decaying, motion of the shock location.

This fact illustrates a second interesting point. For, the fake Lax shocks *are* orbitally stable under zero mass perturbations (though they cannot be asymptotically stable, by Proposition 10.5(i)). This shows that stability and zero-mass stability do not always agree, *even in the classical, Lax case.* Orbital zero-mass stability can be seen from Theorem 6.4(iii) together with (8.1), which together give the description of asymptotic behavior,

$$\sum_j \sum_{k\geq 0} t^k L_x^k \varphi(x)_j \pi_j(y),$$

where $\varphi_j$, $\pi_j$ are right and left zero eigenfunctions of $L$ with ascents summing to less than 1 plus the maximum ascent, 2. It follows that the single, linearly growing term is $t\bar U_x(x)\pi_j(y)$, where $L\phi_j = \bar U_x$, hence $\phi_j$ is ascent two and thus $\pi_j$ is *ascent one*, a bounded genuine eigenfunction. But, the only bounded zero-eigenfunctions of the adjoint operator $L^*$ are constants (see Section 10), hence $\pi(y) \equiv$ *constant* and so this term is not excited by zero-mass perturbations.

These examples at once indicate the kinds of subtleties that can occur in behavior of viscous shock waves, and demonstrate the utility of the effective spectrum for their explanation and study.

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