M401 Spring 2010, Assignment 10, due Thursday April 22

1a. [5 pts] In class we solved

\[
\begin{align*}
    &u_t = \frac{1}{2} u_{xx} \\
    &u_x(0, t) = 0; \quad u_x(3, t) = 0; \quad t \geq 0 \\
    &u(x, 0) = x, \quad x \in [0, 3].
\end{align*}
\]

and we found the solution to be

\[
    u(x, t) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{6}{n^2 \pi^2} \left( (-1)^n - 1 \right) e^{-\frac{1}{2} n^2 \pi^2 t} \cos \frac{n \pi x}{3}.
\]

Write down the two-term approximation for \(u(x, t)\) (including \(\frac{3}{2}\) as one of the terms) and find an upper bound on its error. Also, write down the three-term approximation for \(u(x, t)\) (including \(\frac{3}{2}\) as one of the terms) and find an upper bound on its error.

1b. [5 pts] In class we solved

\[
\begin{align*}
    &u_t = 2 u_{xx} \\
    &u(0, t) = 0; \quad u(1, t) = 0; \quad t \geq 0 \\
    &u(x, 0) = x(1-x), \quad x \in [0, 1],
\end{align*}
\]

and we found the solution to be

\[
    u(x, t) = \sum_{n=1}^{\infty} \frac{4}{n^3 \pi^3} \left( 1 - (-1)^n \right) e^{-2n^2 \pi^2 t} \sin n \pi x.
\]

We also found that if we use the first approximation

\[
    u(x, t) = \frac{8}{\pi^3} e^{-\pi^2 t} \sin \pi x + R_1(x, t),
\]

then we can get an upper bound on the error

\[
    |R_1(x, t)| \leq \frac{1}{\pi^3} e^{-18 \pi^2 t}.
\]

On the other hand, since we only have a non-zero summand when \(n\) is odd, we could write \(u(x, t)\) as

\[
    u(x, t) = \sum_{n=1}^{\infty} \frac{8}{(2n-1)^3 \pi^3} e^{-2(2n-1)^2 \pi^2 t} \sin (2n - 1) \pi x.
\]

Show that if we use this form we do not get quite as good an error estimate as before on the one-term approximation.

2. [10 pts] Constanda Exercise 5.5, Parts (i) and (ii).
Note on Problems 3–5. Constanda solves the heat equation with all its standard boundary conditions, so I can’t reasonably assign any of those cases as homework problems. Problems 3–5 involve fourth order equations for which solutions are quite similar to those of the heat equation.

3. [10 pts] Solve the PDE

\[ u_t = -ku_{xxxx} \]
\[ u(0, t) = 0; \quad u(L, t) = 0; \quad t \geq 0 \]
\[ u_{xx}(0, t) = 0; \quad u_{xx}(L, t) = 0; \quad t \geq 0 \]
\[ u(x, 0) = f(x); \quad x \in [0, L]. \]

Note. Use Problem 4 from Assignment 9.

4. [10 pts] Solve the PDE

\[ u_t = -ku_{xxxx} \]
\[ u(0, t) = 0; \quad u_x(L, t) = 0; \quad t \geq 0 \]
\[ u_{xx}(0, t) = 0; \quad u_{xxx}(L, t) = 0; \quad t \geq 0 \]
\[ u(x, 0) = f(x); \quad x \in [0, L]. \]

Note. Notice that the difference between Problem 4 and Problem 3 is the boundary condition at \( x = L \).

5. [10 pts] Solve the PDE

\[ u_t = k(u_{xx} - u_{xxxx}) \]
\[ u(0, t) = 0; \quad u(L, t) = 0; \quad t \geq 0 \]
\[ u_{xx}(0, t) = 0; \quad u_{xx}(L, t) = 0; \quad t \geq 0 \]
\[ u(x, 0) = f(x); \quad x \in [0, L]. \]

Note. Brace yourself.