M401 Spring 2010, Assignment 10, due Thursday April 22

1a. [5 pts] In class we solved

$$\begin{aligned} u_t &= \frac{1}{2}u_{xx} \\ u_x(0,t) &= 0; \quad u_x(3,t) = 0; \quad t \geq 0 \\ u(x,0) &= x, \quad x \in [0,3]. \end{aligned}$$

and we found the solution to be

$$u(x,t) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{6}{n^2\pi^2}\Big((-1)^n - 1\Big)e^{-\frac{1}{2}\frac{n^2\pi^2}{9}t}\cos\frac{n\pi x}{3}.$$

Write down the two-term approximation for $u(x,t)$ (including $\frac{3}{2}$ as one of the terms) and find an upper bound on its error. Also, write down the three-term approximation for $u(x,t)$ (including $\frac{3}{2}$ as one of the terms) and find an upper bound on its error.

1b. [5 pts] In class we solved

$$\begin{aligned} u_t &= 2u_{xx} \\ u(0,t) &= 0; \quad u(1,t) = 0; \quad t \geq 0 \\ u(x,0) &= x(1-x), \quad x \in [0,1], \end{aligned}$$

and we found the solution to be

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{n^3\pi^3}\Big(1 - (-1)^n\Big)e^{-2n^2\pi^2 t}\sin n\pi x.$$

We also found that if we use the first approximation

$$u(x,t) = \frac{8}{\pi^3}e^{-2\pi^2 t}\sin \pi x + R_1(x,t),$$

then we can get an upper bound on the error

$$|R_1(x,t)| \leq \frac{1}{\pi^3}e^{-18\pi^2 t}.$$

On the other hand, since we only have a non-zero summand when n is odd, we could write $u(x,t)$ as

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8}{(2n-1)^3\pi^3}e^{-2(2n-1)^2\pi^2 t}\sin(2n-1)\pi x.$$

Show that if we use this form we do not get quite as good an error estimate as before on the one-term approximation.

2. [10 pts] Constanda Exercise 5.5, Parts (i) and (ii).

Note on Problems 3–5. Constanda solves the heat equation with all its standard boundary conditions, so I can't reasonably assign any of those cases as homework problems. Problems 3–5 involve fourth order equations for which solutions are quite similar to those of the heat equation.

3. [10 pts] Solve the PDE

$$\begin{aligned} u_t &= -ku_{xxxx} \\ u(0,t) &= 0; \quad u(L,t) = 0; \quad t \geq 0 \\ u_{xx}(0,t) &= 0; \quad u_{xx}(L,t) = 0; \quad t \geq 0 \\ u(x,0) &= f(x); \quad x \in [0,L]. \end{aligned}$$

Note. Use Problem 4 from Assignment 9.

4. [10 pts] Solve the PDE

$$\begin{aligned} u_t &= -ku_{xxxx} \\ u(0,t) &= 0; \quad u_x(L,t) = 0; \quad t \geq 0 \\ u_{xx}(0,t) &= 0; \quad u_{xxx}(L,t) = 0; \quad t \geq 0 \\ u(x,0) &= f(x); \quad x \in [0,L]. \end{aligned}$$

Note. Notice that the difference between Problem 4 and Problem 3 is the boundary condition at $x = L$.

5. [10 pts] Solve the PDE

$$\begin{aligned} u_t &= k(u_{xx} - u_{xxxx}) \\ u(0,t) &= 0; \quad u(L,t) = 0; \quad t \geq 0 \\ u_{xx}(0,t) &= 0; \quad u_{xx}(L,t) = 0; \quad t \geq 0 \\ u(x,0) &= f(x); \quad x \in [0,L]. \end{aligned}$$

Note. Brace yourself.