

M401 Spring 2010, Assignment 10 Solutions

1a. [5 pts] In class we solved

$$\begin{aligned}u_t &= \frac{1}{2}u_{xx} \\u_x(0, t) &= 0; \quad u_x(3, t) = 0; \quad t \geq 0 \\u(x, 0) &= x, \quad x \in [0, 3].\end{aligned}$$

and we found the solution to be

$$u(x, t) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{6}{n^2\pi^2} \left((-1)^n - 1 \right) e^{-\frac{1}{2} \frac{n^2\pi^2}{9} t} \cos \frac{n\pi x}{3}.$$

Write down the two-term approximation for $u(x, t)$ (including $\frac{3}{2}$ as one of the terms) and find an upper bound on its error. Also, write down the three-term approximation for $u(x, t)$ (including $\frac{3}{2}$ as one of the terms) and find an upper bound on its error.

Solution. The two-term approximation is

$$u(x, t) = \frac{3}{2} - \frac{12}{\pi^2} e^{-\frac{1}{2} \frac{\pi^2}{9} t} \cos \frac{\pi x}{3} + R_1(x, t),$$

where

$$R_1(x, t) = \sum_{n=3}^{\infty} \frac{6}{n^2\pi^2} \left((-1)^n - 1 \right) e^{-\frac{1}{2} \frac{n^2\pi^2}{9} t} \cos \frac{n\pi x}{3}.$$

(The subscript on R corresponds with the last approximating term, in this case u_1 . Note that u_2 is 0.) We have

$$\begin{aligned}|R_1(x, t)| &\leq \sum_{n=3}^{\infty} \left| \frac{6}{n^2\pi^2} \left((-1)^n - 1 \right) e^{-\frac{1}{2} \frac{n^2\pi^2}{9} t} \cos \frac{n\pi x}{3} \right| \\&\leq \frac{12}{\pi^2} e^{-\frac{1}{2} \pi^2 t} \sum_{n=3}^{\infty} \frac{1}{n^2}.\end{aligned}$$

According to our theorem from class,

$$\sum_{n=3}^{\infty} \frac{1}{n^2} \leq \int_2^{\infty} \frac{1}{x^2} dx = -\frac{1}{x} \Big|_2^{\infty} = \frac{1}{2}.$$

We conclude

$$|R_1(x, t)| \leq \frac{6}{\pi^2} e^{-\frac{1}{2} \pi^2 t}.$$

Likewise, the three-term approximation is

$$u(x, t) = \frac{3}{2} - \frac{12}{\pi^2} e^{-\frac{1}{2} \frac{\pi^2}{9} t} \cos \frac{\pi x}{3} - \frac{4}{3\pi^2} e^{-\frac{1}{2} \pi^2 t} \cos \pi x + R_3(x, t).$$

where

$$R_3(x, t) = \sum_{n=5}^{\infty} \frac{6}{n^2 \pi^2} \left((-1)^n - 1 \right) e^{-\frac{1}{2} \frac{n^2 \pi^2}{9} t} \cos \frac{n \pi x}{3}.$$

We have

$$\begin{aligned} |R_3(x, t)| &\leq \sum_{n=5}^{\infty} \left| \frac{6}{n^2 \pi^2} \left((-1)^n - 1 \right) e^{-\frac{1}{2} \frac{n^2 \pi^2}{9} t} \cos \frac{n \pi x}{3} \right| \\ &\leq \frac{12}{\pi^2} e^{-\frac{25}{18} \pi^2 t} \sum_{n=5}^{\infty} \frac{1}{n^2}. \end{aligned}$$

According to our theorem from class,

$$\sum_{n=5}^{\infty} \frac{1}{n^2} \leq \int_4^{\infty} \frac{1}{x^2} dx = -\frac{1}{x} \Big|_4^{\infty} = \frac{1}{4}.$$

We conclude

$$|R_3(x, t)| \leq \frac{3}{\pi^2} e^{-\frac{25}{18} \pi^2 t}.$$

1b. [5 pts] In class we solved

$$\begin{aligned} u_t &= 2u_{xx} \\ u(0, t) &= 0; \quad u(1, t) = 0; \quad t \geq 0 \\ u(x, 0) &= x(1 - x), \quad x \in [0, 1], \end{aligned}$$

and we found the solution to be

$$u(x, t) = \sum_{n=1}^{\infty} \frac{4}{n^3 \pi^3} \left(1 - (-1)^n \right) e^{-2n^2 \pi^2 t} \sin n \pi x.$$

We also found that if we use the first approximation

$$u(x, t) = \frac{8}{\pi^3} e^{-2\pi^2 t} \sin \pi x + R_1(x, t),$$

then we can get an upper bound on the error

$$|R_1(x, t)| \leq \frac{1}{\pi^3} e^{-18\pi^2 t}.$$

On the other hand, since we only have a non-zero summand when n is odd, we could write $u(x, t)$ as

$$u(x, t) = \sum_{n=1}^{\infty} \frac{8}{(2n-1)^3 \pi^3} e^{-2(2n-1)^2 \pi^2 t} \sin(2n-1)\pi x.$$

Show that if we use this form we do not get quite as good an error estimate as before on the one-term approximation.

Solution. In this case, we have

$$u(x, t) = \frac{8}{\pi^3} e^{-2\pi^2 t} \sin \pi x + R_1(x, t),$$

where

$$R_1(x, t) = \sum_{n=2}^{\infty} \frac{8}{(2n-1)^3 \pi^3} e^{-2(2n-1)^2 \pi^2 t} \sin(2n-1)\pi x.$$

We have

$$|R_1(x, t)| \leq \frac{8}{\pi^2} e^{-18\pi^2 t} \sum_{n=2}^{\infty} \frac{1}{(2n-1)^3}.$$

Using our theorem from class, we find

$$\sum_{n=2}^{\infty} \frac{1}{(2n-1)^3} \leq \int_1^{\infty} \frac{1}{(2x-1)^3} dx = -\frac{1}{4} \frac{1}{(2x-1)^2} \Big|_1^{\infty} = \frac{1}{4}.$$

We conclude

$$|R_1(x, t)| \leq \frac{2}{\pi^3} e^{-18\pi^2 t},$$

which is twice as large as the estimate obtained in class.

2. [10 pts] Constanda Exercise 5.5, Parts (i) and (ii).

Solution to (i). We have $L = 1$ and $k = 1$, so

$$u(x, t) = a_0 + \sum_{n=1}^{\infty} e^{-n^2 \pi^2 t} (a_n \cos n\pi x + b_n \sin n\pi x),$$

with

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\pi x + b_n \sin n\pi x).$$

For Part (i), we have $f(x) = 2 \sin(2\pi x) - \cos(5\pi x)$, and matching this with our expansion we conclude

$$a_5 = -1; \quad b_2 = 2,$$

and the remaining coefficients are all 0. That is,

$$u(x, t) = 2e^{-4\pi^2 t} \sin(2\pi x) - e^{-25\pi^2 t} \cos(5\pi x).$$

Solution to (ii). In this case, we have $f(x) = 3x - 2$, and we need to use the expressions

$$\begin{aligned} a_0 &= \frac{1}{2} \int_{-1}^1 f(x) dx \\ a_n &= \int_{-1}^1 f(x) \cos n\pi x dx \\ b_n &= \int_{-1}^1 f(x) \sin n\pi x dx. \end{aligned}$$

We compute:

$$a_0 = \frac{1}{2} \int_{-1}^1 3x - 2dx = \frac{1}{2} \left(\frac{3}{2}x^2 - 2x \Big|_{-1}^1 \right) = -2,$$

and

$$\begin{aligned} a_n &= 3 \int_{-1}^1 x \cos n\pi x dx - 2 \int_{-1}^1 \cos n\pi x dx \\ &= 3 \left[\frac{x}{n\pi} \sin n\pi x \Big|_{-1}^1 - \int_{-1}^1 \frac{1}{n\pi} \sin n\pi x dx \right] - \frac{2}{n\pi} \sin n\pi x \Big|_{-1}^1 \\ &= 3 \left[+ \frac{1}{n^2\pi^2} \cos n\pi x \Big|_{-1}^1 \right] = \frac{1}{n^2\pi^2} (\cos n\pi - \cos(-n\pi)) = 0. \end{aligned}$$

(This is also clear without a detailed calculation since $x \cos n\pi x$ is odd and $\int_{-1}^1 \cos n\pi x dx = 0$ is clear from the graph of $\cos n\pi x$.) Finally,

$$\begin{aligned} b_n &= 3 \int_{-1}^1 x \sin n\pi x dx - 2 \int_{-1}^1 \sin n\pi x dx \\ &= 3 \left[-\frac{x}{n\pi} \cos n\pi x \Big|_{-1}^1 + \int_{-1}^1 \frac{1}{n\pi} \cos n\pi x dx \right] + \frac{2}{n\pi} \cos n\pi x \Big|_{-1}^1 \\ &= 3 \left[-\frac{1}{n\pi} \cos n\pi - \frac{1}{n\pi} \cos(-n\pi) + \frac{1}{n^2\pi^2} \sin n\pi x \Big|_{-1}^1 \right] + \frac{2}{n\pi} (\cos n\pi - \cos(-n\pi)) \\ &= 3 \left[-\frac{2}{n\pi} (-1)^n \right]. \end{aligned}$$

We conclude

$$u(x, t) = -2 - \frac{6}{\pi} \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \frac{(-1)^n}{n} \sin n\pi x.$$

Note on Problems 3–5. Constanda solves the heat equation with all its standard boundary conditions, so I can't reasonably assign any of those cases as homework problems. Problems 3–5 involve fourth order equations for which solutions are quite similar to those of the heat equation.

3. [10 pts] Solve the PDE

$$\begin{aligned} u_t &= -k u_{xxxx} \\ u(0, t) &= 0; \quad u(L, t) = 0; \quad t \geq 0 \\ u_{xx}(0, t) &= 0; \quad u_{xx}(L, t) = 0; \quad t \geq 0 \\ u(x, 0) &= f(x); \quad x \in [0, L]. \end{aligned}$$

Note. Use Problem 4 from Assignment 9.

Solution. We separate variables with $u(x, t) = X(x)T(t)$, so that

$$X(x)T'(t) = -kX''''(x)T(t).$$

Dividing by $kX(x)T(t)$ we find,

$$\frac{T'(t)}{kT(t)} = -\frac{X''''(x)}{X(x)} = -\lambda,$$

giving two ODE

$$T' = -\lambda kT,$$

and

$$X'''' - \lambda X = 0.$$

The boundary conditions on $u(x, t)$ give the following boundary conditions on X :

$$\begin{aligned} X(0) &= 0; & X(L) &= 0 \\ X''(0) &= 0; & X''(L) &= 0. \end{aligned}$$

In Problem 4 of Assignment 9 we showed that the eigenvalues and eigenfunctions for this problem are

$$\lambda_n = \frac{n^4\pi^4}{L^4}; \quad X_n(x) = \sin\left(\frac{n\pi x}{L}\right).$$

It follows that

$$T_n(t) = e^{-\frac{n^4\pi^4}{L^4}t},$$

and so

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-\frac{n^4\pi^4}{L^4}t} \sin\left(\frac{n\pi x}{L}\right).$$

From our initial condition, we have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

which is just a Fourier sine series, so

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx,$$

and this specifies the solution.

4. [10 pts] Solve the PDE

$$\begin{aligned} u_t &= -ku_{xxxx} \\ u(0, t) &= 0; \quad u_x(L, t) = 0; \quad t \geq 0 \\ u_{xx}(0, t) &= 0; \quad u_{xxx}(L, t) = 0; \quad t \geq 0 \\ u(x, 0) &= f(x); \quad x \in [0, L]. \end{aligned}$$

Note. Notice that the difference between Problem 4 and Problem 3 is the boundary condition at $x = L$.

Solution. Separating variables as in Problem 3 we obtain two ODE

$$T' = -\lambda kT,$$

and

$$X'''' - \lambda X = 0.$$

The boundary conditions on $u(x, t)$ give the following boundary conditions on X :

$$\begin{aligned} X(0) &= 0; & X'(L) &= 0 \\ X''(0) &= 0; & X'''(L) &= 0. \end{aligned}$$

First, we verify that there are no negative eigenvalues. We multiply the equation by $X''(x)$ and integrating on $[0, L]$. That is,

$$\int_0^L X''''(x)X''(x)dx - \lambda \int_0^L X''(x)X(x)dx = 0.$$

Integrating each of these by parts once, and using the boundary conditions, we find

$$-\int_0^L X'''(x)^2 dx + \lambda \int_0^L X'(x)^2 dx = 0,$$

so that

$$\lambda = \frac{\int_0^L X'''(x)^2 dx}{\int_0^L X'(x)^2 dx} \geq 0.$$

Clearly, we cannot have a solution with $X'(x) \equiv 0$, because this would be constant, and according to the boundary conditions would be 0. So we are not dividing by 0 here. We can only get $\lambda = 0$ is $X'''(x) \equiv 0$, which means

$$X(x) = C_1x^2 + C_2x + C_3,$$

for some constants C_1 , C_2 , and C_3 . But $X(0) = 0$ implies $C_3 = 0$, and $X''(0) = 0$ implies $C_1 = 0$. This leaves $0 = X'(L) = C_2$, which implies $C_2 = 0$. So $X(x) \equiv 0$, and $\lambda = 0$ is not an eigenfunction. For $\lambda > 0$ the auxiliary equation is

$$r^4 - \lambda = 0 \Rightarrow r = -\sqrt[4]{\lambda}, +\sqrt[4]{\lambda}, -i\sqrt[4]{\lambda}, +i\sqrt[4]{\lambda},$$

where $\sqrt[4]{\lambda}$ denotes the unique value of r with positive real part. (See the solution of Problem 4 from Assignment 9 for more details.)

We conclude that the general solution to this ODE is

$$X(x) = C_1e^{-\sqrt[4]{\lambda}x} + C_2e^{\sqrt[4]{\lambda}x} + C_3 \cos(\sqrt[4]{\lambda}x) + C_4 \sin(\sqrt[4]{\lambda}x),$$

with

$$X''(x) = C_1\sqrt{\lambda}e^{-\sqrt[4]{\lambda}x} + C_2\sqrt{\lambda}e^{\sqrt[4]{\lambda}x} - C_3\sqrt{\lambda} \cos(\sqrt[4]{\lambda}x) - C_4\sqrt{\lambda} \sin(\sqrt[4]{\lambda}x).$$

The conditions at $x = 0$ give

$$\begin{aligned} C_1 + C_2 + C_3 &= 0 \\ \sqrt{\lambda}(C_1 + C_2 - C_3) &= 0. \end{aligned}$$

If we multiply the first by $\sqrt{\lambda}$ and subtract the second, we find $C_3 = 0$. This leaves us with the equation

$$C_1 + C_2 = 0.$$

Also,

$$X'(L) = \sqrt[4]{\lambda} \left(-C_1 e^{-\sqrt[4]{\lambda}L} + C_2 e^{\sqrt[4]{\lambda}L} + C_4 \cos(\sqrt[4]{\lambda}L) \right),$$

with

$$X'''(L) = \sqrt[4]{\lambda^3} \left(-C_1 e^{-\sqrt[4]{\lambda}L} + C_2 \sqrt{\lambda} e^{\sqrt[4]{\lambda}L} - C_4 \cos(\sqrt[4]{\lambda}L) \right).$$

If we multiply $\sqrt[4]{\lambda}$ times the first equation and subtract the second equation, we obtain

$$2\sqrt[4]{\lambda^3} C_4 \cos(\sqrt[4]{\lambda}L) = 0.$$

If $C_4 = 0$ then (as we'll see below) C_1 and C_2 are both 0, and we do not have an eigenfunction. Setting

$$\cos(\sqrt[4]{\lambda}L) = 0,$$

we find

$$\sqrt[4]{\lambda}L = n\pi - \frac{\pi}{2}, n = 1, 2, \dots \Rightarrow \lambda_n = \frac{(n - \frac{1}{2})^4 \pi^4}{L^4}, n = 1, 2, \dots$$

For these values of λ_n we have that $\cos(\sqrt[4]{\lambda}L) = 0$, so

$$\begin{aligned} C_1 + C_2 &= 0 \\ -C_1 e^{-\sqrt[4]{\lambda}L} + C_2 e^{\sqrt[4]{\lambda}L} &= 0. \end{aligned}$$

If we multiply the first equation by $e^{-\sqrt[4]{\lambda}L}$ and add it to the second we obtain

$$C_2(e^{-\sqrt[4]{\lambda}L} + e^{\sqrt[4]{\lambda}L}) = 0 \Rightarrow C_2 = 0 \Rightarrow C_1 = 0.$$

We conclude that the eigenvalues and eigenvectors are

$$\lambda_n = \frac{(n - \frac{1}{2})^4 \pi^4}{L^4}; \quad X_n(x) = \sin\left(\frac{(n - \frac{1}{2})\pi x}{L}\right).$$

The solution is

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-k \frac{(n - \frac{1}{2})^4 \pi^4}{L^4} t} \sin\left(\frac{(n - \frac{1}{2})\pi x}{L}\right),$$

and so

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{(n - \frac{1}{2})\pi x}{L}\right).$$

We now multiply this equality by

$$\sin\left(\frac{(m - \frac{1}{2})\pi x}{L}\right)$$

for $m = 1, 2, \dots$ and integrate on $[0, L]$ using the orthogonality relation

$$\int_0^L \sin\left(\frac{(n - \frac{1}{2})\pi x}{L}\right) \sin\left(\frac{(m - \frac{1}{2})\pi x}{L}\right) dx = \begin{cases} \frac{L}{2} & m = n \\ 0 & m \neq n \end{cases}.$$

(Verified, as in class, with trig identities.) This gives

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{(n - \frac{1}{2})\pi x}{L}\right) dx,$$

which completely specifies the solution.

5. [10 pts] Solve the PDE

$$\begin{aligned} u_t &= k(u_{xx} - u_{xxxx}) \\ u(0, t) &= 0; \quad u(L, t) = 0; \quad t \geq 0 \\ u_{xx}(0, t) &= 0; \quad u_{xx}(L, t) = 0; \quad t \geq 0 \\ u(x, 0) &= f(x); \quad x \in [0, L]. \end{aligned}$$

Note. Brace yourself.

Solution. Separating variables with $u(x, t) = X(x)T(t)$ we find

$$X(x)T'(t) = k(X''(x) - X''''(x))T(t).$$

Dividing by $kX(x)T(t)$ we find

$$\frac{T'(t)}{kT(t)} = \frac{X''(x) - X''''(x)}{X(x)} = -\lambda,$$

which gives two ODE

$$T' = -\lambda kT,$$

and

$$X'''' - X'' - \lambda X = 0,$$

where the latter equation has boundary conditions

$$\begin{aligned} X(0) &= 0; \quad X(L) = 0 \\ X''(0) &= 0; \quad X''(L) = 0. \end{aligned}$$

First, we verify that there are no negative eigenvalues. We multiply the equation by X'' and integration on $[0, L]$ to obtain:

$$\int_0^L X'' X'''' dx - \int_0^L X''^2 dx - \lambda \int_0^L X X'' dx = 0.$$

Integrating the first and third terms by parts, and using the boundary conditions, we find

$$-\int_0^L X'''^2 dx - \int_0^L X''^2 dx = -\lambda \int_0^L X'^2 dx,$$

so that

$$\lambda = \frac{\int_0^L X'''^2 dx + \int_0^L X''^2 dx}{\int_0^L X'^2 dx} \geq 0.$$

For $\lambda = 0$ to be an eigenvalue we require $X''(x) = 0$, so that $X(x) = C_1x + C_2$. But $X(0) = 0$ implies $C_2 = 0$ and $X(L) = 0$ implies $C_1 = 0$, so $X(x) \equiv 0$ and $\lambda = 0$ is not an eigenvalue.

For $\lambda > 0$ the auxiliary equation is

$$r^4 - r^2 - \lambda = 0.$$

The key to solving an equation like this is to recognize that it's a quadratic equation for $s = r^2$. The solutions are

$$r^2 = \frac{1 \pm \sqrt{1 + 4\lambda}}{2}.$$

When the $+$ root is taken r^2 is positive, while when the $-$ root is taken r^2 is negative. This means we have two real roots

$$r = \pm \sqrt{\frac{1 + \sqrt{1 + 4\lambda}}{2}},$$

and two complex roots

$$r = \pm \sqrt{\frac{1 - \sqrt{1 + 4\lambda}}{2}} = \pm i \sqrt{\frac{\sqrt{1 + 4\lambda} - 1}{2}}.$$

In order to reduce notation a little, we'll set

$$r_1 = \sqrt{\frac{1 + \sqrt{1 + 4\lambda}}{2}}; \quad r_2 = \sqrt{\frac{\sqrt{1 + 4\lambda} - 1}{2}},$$

so that the roots are $\pm r_1$, and $\pm ir_2$. With this notation, the general solution is

$$X(x) = C_1 e^{r_1 x} + C_2 e^{-r_1 x} + C_3 \cos(r_2 x) + C_4 \sin(r_2 x),$$

with

$$X''(x) = C_1 r_1^2 e^{r_1 x} + C_2 r_1^2 e^{-r_1 x} - C_3 r_2^2 \cos(r_2 x) - C_4 r_2^2 \sin(r_2 x).$$

The condition $X(0) = 0$ gives

$$C_1 + C_2 + C_3 = 0,$$

while the condition $X''(0) = 0$ gives

$$C_1 r_1^2 + C_2 r_1^2 - C_3 r_2^2 = 0.$$

If we multiply the first of these equations by r_1^2 and subtract the second, we obtain

$$C_3(r_1^2 + r_2^2) = 0.$$

We notice that

$$r_1^2 + r_2^2 = \frac{1 + \sqrt{1 + 4\lambda}}{2} + \frac{\sqrt{1 + 4\lambda} - 1}{2} = \sqrt{1 + 4\lambda} \neq 0,$$

and so $C_3 = 0$. Next, $X(L) = 0$ implies

$$C_1 e^{r_1 L} + C_2 e^{-r_1 L} + C_4 \sin(r_2 L) = 0,$$

while $X''(L) = 0$ implies

$$C_1 r_1^2 e^{r_1 L} + C_2 r_1^2 e^{-r_1 L} - C_4 r_2^2 \sin(r_2 L) = 0.$$

If we multiply the first of these equations by r_1^2 and subtract the second, we obtain

$$C_4(r_1^2 + r_2^2) \sin(r_2 L) = 0.$$

Again, $r_1^2 + r_2^2 \neq 0$, and $C_4 = 0$ (as we'll see below) would leave only the trivial solution, and so we require $r_2 L = n\pi$, $n = 1, 2, \dots$. That is,

$$\sqrt{\frac{\sqrt{1 + 4\lambda} - 1}{2}} L = n\pi, \quad n = 1, 2, \dots, \Rightarrow \lambda_n = \frac{n^2 \pi^2}{L^2} + \frac{n^4 \pi^4}{L^4}, \quad n = 1, 2, \dots$$

You might notice that this is simply a sum of the eigenvalues associated with the operator D^2 and those associated with the operator $-D^4$, with these boundary conditions. The fact that the spectrum (i.e., the eigenvalues) associated with the sum of these operators is simply a sum of their eigenvalues is an example of the *Spectral Mapping Theorem*. That won't be on the final. Now, where was I? Right, with these eigenvalues the expression $C_4 \sin(r_2 L)$ is 0, so we have

$$\begin{aligned} C_1 + C_2 &= 0 \\ C_1 e^{r_1 L} + C_2 e^{-r_1 L} &= 0, \end{aligned}$$

and this implies $C_1 = C_2 = 0$. Therefore, the associated eigenfunctions are

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right).$$

The PDE solution is

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-k\left(\frac{n^2 \pi^2}{L^2} + \frac{n^4 \pi^4}{L^4}\right)t} \sin\left(\frac{n\pi x}{L}\right),$$

with also

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

This is simply a Fourier sine series for f , and we find

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx,$$

which specifies the solution.