M401 Spring 2010, Assignment 11 Solutions

1a. [5 pts] Compute the Fourier cosine series for $f(x) = \sin x$ on $[0, \pi]$.

Solution. Since $L = \pi$, the series will have the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx,$$

where

$$a_0 = \frac{1}{\pi}\int_0^\pi \sin x dx = \frac{1}{\pi}(-\cos x)\Big|_0^\pi = \frac{1}{\pi}(-\cos\pi + 1) = \frac{2}{\pi}$$
$$a_n = \frac{2}{\pi}\int_0^\pi \sin x \cos nx dx.$$

For the second integral, we recall (see Assignment 4) the trigonometric identity

$$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)].$$

This gives (for $n = 2, 3, \dots$)

$$\begin{aligned}\frac{2}{\pi}\int_0^\pi \sin x \cos nx dx &= \frac{2}{\pi}\int_0^\pi \frac{1}{2}[\sin((n+1)x) + \sin((1-n)x)]dx \\ &= \frac{1}{\pi}[-\frac{1}{n+1}\cos((n+1)x) - \frac{1}{1-n}\cos((1-n)x)]_0^\pi \\ &= \frac{1}{\pi}[-\frac{1}{n+1}\cos((n+1)\pi) + \frac{1}{n+1} - \frac{1}{1-n}\cos((1-n)\pi) + \frac{1}{1-n}].\end{aligned}$$

We observe

$$\cos((n+1)\pi) = (-1)^{n+1}$$
$$\cos((1-n)\pi) = (-1)^{n+1},$$

and also

$$\frac{1}{n+1} + \frac{1}{1-n} = \frac{-2}{n^2-1}.$$

We conclude

$$a_n = \frac{1}{\pi}[\frac{2}{n^2-1}(-1)^{n+1} - \frac{2}{n^2-1}] = -\frac{2}{\pi(n^2-1)}[1 + (-1)^n], \quad n = 2, 3, \dots$$

For $n = 1$ (omitted above, since it would require dividing by 0),

$$a_1 = \frac{2}{\pi}\int_0^\pi \sin x \cos x dx = \frac{2}{\pi}\int_0^\pi \frac{1}{2}\sin 2x dx = -\frac{1}{2\pi}\cos 2x\Big|_0^\pi = 0.$$

That is,

$$\sin x = \frac{2}{\pi} - \sum_{n=2}^{\infty} \frac{2}{\pi(n^2-1)}[1 + (-1)^n]\cos nx.$$

1b. [3 pts] Find an upper bound on the error obtained if the first N terms of the Fourier cosine series from Part (a) are used as an approximation for $f(x) = \sin x$, where N denotes a positive *even* integer.

Solution. If N denotes a positive integer, then

$$\sin x = \frac{2}{\pi} - \sum_{n=2}^{N} \frac{2}{\pi(n^2-1)}[1+(-1)^n]\cos nx + R_N(x),$$

where

$$R_N(x) = -\sum_{n=N+2}^{\infty} \frac{2}{\pi(n^2-1)}[1+(-1)^n]\cos nx,$$

where we are using the fact that the summand for $n = N+1$ is 0. We compute an upper bound on $R_N(x)$ as follows:

$$\begin{aligned} |R_N(x)| &\leq \frac{4}{\pi}\sum_{n=N+2}^{\infty} \frac{1}{n^2-1} \leq \frac{4}{\pi}\int_{N+1}^{\infty} \frac{1}{x^2-1}dx \\ &= \frac{4}{\pi}\frac{1}{2}\ln|\frac{x-1}{x+1}|_{N+1}^{\infty} = -\frac{2}{\pi}\ln\frac{N}{N+2}. \end{aligned}$$

Here, we have used our Theorem from class on bounding series with integrals. The antiderivative of $\frac{1}{x^2-1}$ is easy to compute with partial fractions.

1c. [2 pts] Explain how your result from Part (b) ensures that there cannot be a Gibbs Phenomenon for this series.

Solution. Since $\ln 1 = 0$, we observe that $|R_N(x)| \to 0$ as $N \to \infty$, and $R_N(x)$ is the error for *all* values of x. In the Gibbs Phenomenon there is some value v so that for all values of N there exists some x so that $|R_N(x)| \geq v$. That cannot happen if $|R_N(x)| \to 0$.

2a. [5 pts] Compute the Fourier sine series for $f(x) = \cos x$ on $[0, \pi]$.

Solution. In this case, the series has the form

$$\cos x = \sum_{n=1}^{\infty} b_n \sin nx,$$

where (for $n = 2, 3, \dots$)

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} \cos x \sin nx dx = \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}[\sin((n+1)x) + \sin((n-1)x)]dx \\ &= \frac{1}{\pi}[-\frac{1}{n+1}\cos((n+1)x) - \frac{1}{n-1}\cos((n-1)x)]_0^{\pi} \\ &= \frac{1}{\pi}[-\frac{1}{n+1}\cos((n+1)\pi) + \frac{1}{n+1} - \frac{1}{n-1}\cos((n-1)\pi) + \frac{1}{n-1}]. \end{aligned}$$

We observe

$$\begin{aligned} \cos((n+1)\pi) &= (-1)^{n+1} \\ \cos((n-1)\pi) &= (-1)^{n+1}, \end{aligned}$$

and also

$$\frac{1}{n+1} + \frac{1}{n-1} = \frac{2n}{n^2-1}.$$

We conclude

$$b_n = \frac{1}{\pi}[-\frac{2n}{n^2-1}(-1)^{n+1} + \frac{2n}{n^2-1}] = \frac{2n}{\pi(n^2-1)}[1+(-1)^n], \quad n = 2, 3, \dots$$

Likewise, for $n = 1$

$$b_1 = \frac{2}{\pi}\int_0^\pi \cos x \sin x dx = \frac{2}{\pi}\int_0^\pi \frac{1}{2}\sin 2x dx = -\frac{1}{2\pi}\cos 2x\Big|_0^\pi = 0.$$

This gives

$$\cos x = \sum_{n=1}^\infty \frac{2n}{\pi(n^2-1)}[1+(-1)^n]\sin nx.$$

2b. [3 pts] Explain why our method of error estimation fails for this Fourier sine series.

Solution. Our method fails because

$$\frac{2x}{x^2-1}$$

is not integrable on an infinite domain.

2c. [2 pts] Explain why there *is* a Gibbs Phenomenon for this series, and specify the values $x \in [0, \pi]$ where it occurs.

Solution. The difficulty here is that $\cos 0 = 1$ and $\cos \pi = -1$, whereas the Fourier sine series is clearly 0 at $x = 0$ and $x = \pi$. This means that even though $f(x) = \cos x$ is a continuous function, it's periodic extension is discontinuous at $x = 0$ and $x = \pi$. These, then, are the values at which there is a Gibbs Phenomenon.

3. [10 pts] Exercise 5.6 in Constanda, Parts (i) and (iv).

Solution to Part (i). For this problem $c = 1$ and $L = 1$, so we have

$$u(x,t) = \sum_{n=1}^\infty (c_{1n}\cos n\pi t + c_{2n}\sin n\pi t)\sin n\pi x,$$

where

$$f(x) = \sum_{n=1}^\infty c_{1n}\sin n\pi x$$
$$g(x) = \sum_{n=1}^\infty c_{2n} n\pi \sin n\pi x.$$

For Part (i) we simply match terms with

$$f(x) = -3\sin(2\pi x) + 4\sin(7\pi x)$$
$$g(x) = \sin(3\pi x).$$

Clearly, $c_{12} = -3$, $c_{17} = 4$, and $c_{23} = \frac{1}{3\pi}$, and the other coefficients are all 0. This gives

$$u(x,t) = -3\cos(2\pi t)\sin(2\pi x) + 4\cos(7\pi t)\sin(7\pi x) + \frac{1}{3\pi}\sin(3\pi t)\sin(3\pi x).$$

Solution to Part (iv). In this case, we require the formulas

$$\begin{aligned} c_{1n} &= 2\int_0^1 f(x)\sin n\pi x dx \\ c_{2n} &= \frac{2}{n\pi}\int_0^1 g(x)\sin n\pi x dx. \end{aligned}$$

With $f(x) = 1$, we have

$$c_{1n} = 2\int_0^1 \sin n\pi x dx = -\frac{2}{n\pi}\cos n\pi x\Big|_0^1 = -\frac{2}{n\pi}[\cos n\pi - 1] = \frac{2}{n\pi}[1 - (-1)^n],$$

while with $g(x) = x$ we have

$$\begin{aligned} c_{2n} &= \frac{2}{n\pi}\int_0^1 x\sin n\pi x = \frac{2}{n\pi}[-\frac{x}{n\pi}\cos n\pi x\Big|_0^1 + \int_0^1 \frac{1}{n\pi}\cos n\pi x dx] \\ &= \frac{2}{n\pi}[-\frac{1}{n\pi}\cos n\pi + \frac{1}{n^2\pi^2}\sin n\pi x\Big|_0^1] = -\frac{2}{n^2\pi^2}(-1)^n. \end{aligned}$$

Putting these together, we find

$$u(x,t) = \sum_{n=1}^\infty \Big[\frac{2}{n\pi}[1 - (-1)^n]\cos n\pi t + \frac{2}{n^2\pi^2}(-1)^{n+1}\sin n\pi t\Big]\sin n\pi x.$$

4. [10 pts] Solve the wave equation

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \\ u_x(0,t) &= 0; \quad u(L,t) = 0; \quad t \geq 0 \\ u(x,0) &= f(x); \quad x \in [0,L] \\ u_t(x,0) &= g(x); \quad x \in [0,L]. \end{aligned}$$

Solution. Separating variables with $u(x,t) = X(x)T(t)$ we find (proceeding as in class)

$$\begin{aligned} T''(t) + \lambda c^2 T(t) &= 0 \\ X''(x) + \lambda X(x) &= 0, \end{aligned}$$

where the $X(x)$ equation has boundary values $X'(0) = 0$ and $X(L) = 0$. In order to check that there are no negative eigenvalues we multiply by $X(x)$ and integrate on $[0,L]$:

$$\int_0^L X''(x)X(x)dx + \lambda\int_0^L X(x)^2 dx = 0.$$

Integrating the first integral by parts, we find

$$\int_0^L X''(x)X(x)dx = X'(x)X(x)\Big|_0^L - \int_0^L X'(x)^2 dx = -\int_0^L X'(x)^2 dx.$$

We see that

$$\lambda = \frac{\int_0^L X'(x)^2 dx}{\int_0^L X(x)^2 dx} \geq 0,$$

and we can only have $\lambda = 0$ if $X(x)$ is a constant. But the boundary condition $X(L) = 0$ ensures that the constant would be 0, and so $\lambda = 0$ is not an eigenvalue. For $\lambda > 0$,

$$X(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x,$$

with also

$$X'(x) = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}x + C_2\sqrt{\lambda}\cos\sqrt{\lambda}x.$$

The condition $X'(0) = 0$ immediately gives $C_2 = 0$, while the condition $X(L) = 0$ gives

$$0 = C_1 \cos\sqrt{\lambda}L \Rightarrow \sqrt{\lambda}L = n\pi - \frac{\pi}{2}, \quad n = 1, 2, \dots,$$

so that

$$\lambda_n = \frac{(n-\frac{1}{2})^2\pi^2}{L^2}, \quad n = 1, 2, \dots.$$

The associated eigenfunctions are

$$X_n(x) = \cos\frac{(n-\frac{1}{2})\pi x}{L}, \quad n = 1, 2, \dots.$$

Returning to the $T(t)$ equation, we find

$$T_n(t) = c_{1n}\cos\frac{(n-\frac{1}{2})\pi ct}{L} + c_{2n}\sin\frac{(n-\frac{1}{2})\pi ct}{L},$$

and so

$$u(x,t) = \sum_{n=1}^{\infty}\Big(c_{1n}\cos\frac{(n-\frac{1}{2})\pi ct}{L} + c_{2n}\sin\frac{(n-\frac{1}{2})\pi ct}{L}\Big)\cos\frac{(n-\frac{1}{2})\pi x}{L}.$$

Setting $u(x,0) = f(x)$ we have

$$f(x) = \sum_{n=1}^{\infty} c_{1n}\cos\frac{(n-\frac{1}{2})\pi x}{L}.$$

We multiply by $\cos\frac{(m-\frac{1}{2})\pi x}{L}$, integrate on $[0, L]$, and use the orthogonality relation

$$\int_0^L \cos\frac{(n-\frac{1}{2})\pi x}{L}\cos\frac{(m-\frac{1}{2})\pi x}{L}dx = \begin{cases} \frac{L}{2} & m = n \\ 0 & m \neq n \end{cases},$$

to find

$$c_{1n} = \frac{2}{L}\int_0^L f(x)\cos\frac{(n-\frac{1}{2})\pi x}{L}dx.$$

Likewise, setting $u_t(x,0) = g(x)$ we have

$$g(x) = \sum_{n=1}^{\infty} c_{2n}\frac{(n-\frac{1}{2})\pi c}{L}\cos\frac{(n-\frac{1}{2})\pi x}{L},$$

from which we obtain

$$c_{2n} = \frac{2}{(n-\frac{1}{2})\pi c}\int_0^L g(x)\cos\frac{(n-\frac{1}{2})\pi x}{L}dx.$$

In this way, the solution is entirely characterized by

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty}\Big(c_{1n}\cos\frac{(n-\frac{1}{2})\pi ct}{L} + c_{2n}\sin\frac{(n-\frac{1}{2})\pi ct}{L}\Big)\cos\frac{(n-\frac{1}{2})\pi x}{L}\\
c_{1n} &= \frac{2}{L}\int_0^L f(x)\cos\frac{(n-\frac{1}{2})\pi x}{L}dx\\
c_{2n} &= \frac{2}{(n-\frac{1}{2})\pi c}\int_0^L g(x)\cos\frac{(n-\frac{1}{2})\pi x}{L}dx.
\end{aligned}$$

5. [10 pts] Solve the inhomogeneous heat equation

$$\begin{aligned}
u_t &= ku_{xx} + e^x + 1\\
u(0,t) &= 1; \quad u(L,t) = 5; \quad t \geq 0\\
u(x,0) &= f(x); \quad x \in [0,L].
\end{aligned}$$

Solution. We begin by looking for the equilibrium solution $\bar{u}(x)$, which solves

$$\begin{aligned}
k\bar{u}_{xx} &= -e^x - 1\\
\bar{u}(0) &= 1; \quad \bar{u}(L) = 5.
\end{aligned}$$

Integrating twice and using the boundary conditions, we find

$$\bar{u}(x) = \frac{1}{k}\Big[-e^x - \frac{x^2}{2} + (\frac{4k-1}{L} + \frac{1}{L}e^L + \frac{L}{2})x + (k+1)\Big].$$

We now set

$$v(x,t) = u(x,t) - \bar{u}(x),$$

so that v solves

$$\begin{aligned}
v_t &= kv_{xx}\\
v(0,t) &= 0; \quad v(L,t) = 0; \quad t \geq 0\\
v(x,0) &= f(x) - \bar{u}(x).
\end{aligned}$$

We have solved this equation in class by separation of variables, and we found

$$v(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},$$

where

$$b_n = \frac{2}{L} \int_0^L (f(x) - \bar{u}(x)) \sin \frac{n\pi x}{L} dx.$$

Finally,

$$u(x,t) = \bar{u}(x) + v(x,t).$$