M401 Spring 2010, Assignment 1, due Thursday January 28

1. [10 pts] In many introductory textbooks on ODE, air resistance is modeled with a linear term (because this gives rise to equations that are easy to solve). In this case the equation for a falling object is

$$y'' = -g - by',$$

where $y(t)$ denotes the height of the object at time t, g denotes gravitational acceleration at the earth's surface, and b is the coefficient of air resistance. Determine the dimensions of b and nondimensionalize this equation. Specify your initial conditions, assuming the original initial conditions are $y(0) = h$ and $y'(0) = v$.

Solution. First, $[by'] = LT^{-2}$, so $[b] = T^{-1}$. Now set

$$\tau = \frac{t}{A}, \quad Y(\tau) = \frac{y(t)}{B},$$

so that

$$y'(t) = B\frac{d}{dt}Y(\tau) = BY'(\tau)\frac{d\tau}{dt} = \frac{B}{A}Y'(\tau),$$

and

$$y''(t) = \frac{B}{A^2}Y''(\tau).$$

We have, then,

$$\frac{B}{A^2}Y'' = -g - b\frac{B}{A}Y' \Rightarrow Y'' = -g\frac{A^2}{B} - bAY'.$$

If we choose $A = b^{-1}$ and $B = \frac{g}{b^2}$ (which have dimensions $[A] = T$ and $[B] = L$) we find

$$Y'' = -1 - Y',$$

with $Y(0) = \frac{hb^2}{g}$ and $Y'(0) = \frac{bv}{g}$.

2. [10 pts] Consider an object of mass m moving along a frictionless surface and attached to a wall by a spring. If we let $y(t)$ denote the spring's displacement from equilibrium, and we assume $y(t)$ is small, then the motion can often be modeled by an equation of the form

$$my'' = -k_1 y + k_3 y^3,$$

where k_1 is Hooke's constant and the term $k_3 y^3$ is a nonlinear correction corresponding with the fact that the spring will weaken if either strongly stretched or strongly compressed. Find the dimensions of k_1 and k_3 and nondimensionalize this equation.

Solutions. First, $[k_1 y] = MLT^{-2}$, so $[k_1] = MT^{-2}$. Likewise, $[k_3 y^3] = MLT^{-2}$, so $[k_3] = ML^{-2}T^{-2}$. Now (precisely as above) set

$$\tau = \frac{t}{A}, \quad Y(\tau) = \frac{y(t)}{B},$$

so that

$$y'(t) = B\frac{d}{dt}Y(\tau) = BY'(\tau)\frac{d\tau}{dt} = \frac{B}{A}Y'(\tau),$$

and

$$y''(t) = \frac{B}{A^2} Y''(\tau).$$

We have, then,

$$m\frac{B}{A^2}Y'' = -k_1 BY + k_2 B^3 Y^3 \Rightarrow Y'' = -k_1 \frac{A^2}{m} Y + k_2 \frac{A^2 B^2}{m} Y^3.$$

If we choose $A = \sqrt{\frac{m}{k_1}}$ and $B = \sqrt{\frac{k_1}{k_2}}$ (with dimensions $[A] = T$ and $[B] = L$) we have

$$Y'' = -Y + Y^3.$$

3. [10 pts] Exercise 1.14 on p. 24 of Simmonds and Mann Jr.

Notes and suggestions. Begin by specifying the dimensions of EI (combined) and k. (You don't need to know anything about the physics to do this, though see my solutions for a brief discussion.) I suggest replacing the letter T with F (for force) since tension is a force and we will typically use T to denote the dimension time. Either at the beginning or (as I recommend) at a convenient later stage you'll need to divide by a constant with dimensions of p to make each term dimensionless. Keep in mind that p is force per unit length, so it's not pressure. Be clear in the end about your definitions of ϵ, β, and f.

Solution. First, $[kw] = [p] = MT^{-2}$, so $[k] = ML^{-1}T^{-2}$, and likewise $[EI\frac{d^4w}{ds^4}] = MT^{-2}$, so $[EI] = ML^3T^{-2}$. (The constant E is Young's modulus, a measure of an object's tendency to deform under stress. It's a pressure, so $[E] = ML^{-1}T^{-2}$. The constant I is the area moment of inertia, or second moment of inertia of the beam. It's dimensions are $[I] = L^4$.) As suggested, I will write the equation as

$$EI\frac{d^4w}{ds^4} - F\frac{d^2w}{ds^2} + kw = p.$$

Now set

$$x = \frac{s}{A} \quad y(x) = \frac{w(s)}{B},$$

so that (as above)

$$w^{(k)}(s) = \frac{B}{A^k} y^{(k)}(x), \quad k = 1, 2, 3, 4.$$

the equation becomes

$$EI\frac{B}{A^4}\frac{d^4y}{dx^4} - F\frac{B}{A^2}\frac{d^2y}{dx^2} + kBy = p(xA).$$

We know from the problem statement that we want the coefficients of $y^{(4)}$ and y to be the same, so we equate

$$EI\frac{B}{A^4} = kB \Rightarrow A = (\frac{EI}{k})^{1/4}.$$

(At this point we could have divided out, as usual, by the coefficient of the highest derivative, but since we want a multiplier on this derivative of ϵ^2 it's clear that we would have to multiply back later.) If we now divide the equation by $F\frac{B}{A^2}$ we obtain (observing that the coefficients of $y^{(4)}$ and y are now both kB, by our choice of A)

$$\frac{kA^2}{F}(\frac{d^4y}{dx^4}+y)-\frac{d^2y}{dx^2}=\frac{A^2}{BF}p(xA).$$

Clearly, we make the choice

$$\epsilon=\sqrt{\frac{kA^2}{F}}.$$

In defining β and $f(x)$ we notice that there's some freedom, in that we only require

$$\frac{A^2}{BF}p(xA)=\beta f(x).$$

In order to keep both β and $f(x)$ dimensionless, we can set

$$\beta=\frac{A}{B},$$

and

$$f(x)=\frac{A}{F}p(xA).$$

On the other hand, we could also set

$$\beta=\epsilon^2$$

and

$$f(x)=\frac{1}{kB}p(xA).$$

Finally, we haven't yet chosen B, so it's clear that we can take any constant with dimension $[B]=L$. For example, we could take $B=A$.

4. [10 pts] Exercise 1.4 on p. 12 of Simmonds and Mann Jr.

Notes and suggestions. Use Taylor's Theorem (instead of the expansion method). Before solving this problem, show that the Implicit Function Theorem will apply in this case and explain what Taylor's Theorem and the Implicit Function Theorem guarantee about the existence and accuracy of your approximation.

Solution. We are trying to solve

$$f(z,\epsilon)=z^3-z+\epsilon=0,$$

and we will proceed by Taylor's Theorem with

$$z(\epsilon)=z(0)+z'(0)\epsilon+\frac{1}{2}z''(c)\epsilon^2,$$

where c is some value between 0 and ϵ. First, when $\epsilon=0$, we have

$$z(0)^3-z(0)=0\Rightarrow z(0)(z(0)^2-1)=0\Rightarrow z(0)=-1,0,1.$$

Since $f_z(z, \epsilon) = 3z^2 - 1$,we have in all cases

$$\begin{aligned} f(z(0), 0) &= 0 \\ f_z(z(0), 0) &\neq 0. \end{aligned}$$

According to the Implicit Function Theorem, for each $z(0)$ there exist a unique infinitely differentiable function $z(\epsilon)$ so that

$$f(z(\epsilon), \epsilon) = 0,$$

for all ϵ sufficiently small. This means, by Taylor's Theorem, that the root $z(\epsilon)$ certainly exists and certainly has the form we are looking for. Now, we differentiate our equation with respect to ϵ to find

$$3z^2 z' - z' + 1 = 0 \Rightarrow z'(\epsilon) = -\frac{1}{3z^2 - 1}.$$

We have, respectively,

$$z'(0) = \begin{cases} -\frac{1}{2} \\ +1 \\ -\frac{1}{2}. \end{cases}$$

Our approximations are

$$z(\epsilon) = \begin{cases} -1 - \frac{1}{2}\epsilon + \mathbf{O}(\epsilon^2) \\ \epsilon + \mathbf{O}(\epsilon^2) \\ +1 - \frac{1}{2}\epsilon + \mathbf{O}(\epsilon^2). \end{cases}$$

5. [10 pts] Find power series approximations for the roots of

$$x^3 + \epsilon x^2 - x + \epsilon = 0,$$

with an error of order ϵ^3. Follow the same notes and suggestions as in Problem 4.

Solution. In this case, our starting values $x(0)$ satisfy

$$x(0)^3 - x(0) = 0 \Rightarrow x(0) = -1, 0, 1.$$

In this case $f_x(x, \epsilon) = 3x^2 + 2\epsilon x - 1$, and so again in all cases

$$\begin{aligned} f(x(0), 0) &= 0 \\ f_x(x(0), 0) &\neq 0. \end{aligned}$$

Precisely as discussed in associate with Problem 4 Taylor's Theorem and the Implicit Function Theorem guarantee the existence of unique solutions (near each starting value) of the form

$$x(\epsilon) = x(0) + x'(0) + \frac{1}{2}x''(0) + \frac{1}{6}x'''(c)\epsilon^3,$$

for some value c between 0 and ϵ. Differentiating our equation with respect to ϵ, we obtain

$$3x^2 x' + x^2 + \epsilon 2xx' - x' + 1 = 0,$$

so that

$$x'(\epsilon) = -\frac{1+x^2}{3x^2+\epsilon 2x-1}.$$

For $\epsilon = 0$ we have

$$x'(0) = -\frac{1+x(0)^2}{3x(0)^2-1} = \begin{cases} -1 \\ +1 \\ -1. \end{cases}$$

For $x''(0)$, we have two choices: (1) differentiate our expression for $x'(\epsilon)$ or take a second derivative of the original equation. In this case, it's probably a little faster proceeding the second way, but I'll work through both. Computing $x''(\epsilon)$ directly, we find

$$x''(\epsilon) = -\frac{(3x^2+\epsilon 2x-1)2xx' - (1+x^2)(6xx'+2x+\epsilon 2x')}{(3x^2+\epsilon 2x-1)^2},$$

so that

$$x''(0) = -\frac{(3x(0)^2-1)2x(0)x'(0) - (1+x(0)^2)(6x(0)x'(0)+2x(0))}{(3x(0)^2-1)^2}.$$

Substituting values, we find

$$x''(0) = \begin{cases} +1 \\ 0 \\ -1 \end{cases}$$

If we proceed by the second method, we compute

$$6xx'^2 + 3x^2x'' + 2xx' + 2xx' + \epsilon 2x'^2 + \epsilon 2xx'' - x'' = 0,$$

so that for $\epsilon = 0$ we have

$$x''(0) = -\frac{6x(0)x'(0)^2 + 4x(0)x'(0)}{3x(0)^2-1} = \begin{cases} +1 \\ 0 \\ -1. \end{cases}$$

We conclude

$$x(\epsilon) = \begin{cases} -1 - \epsilon + \frac{1}{2}\epsilon^2 + \mathbf{O}(\epsilon^3) \\ \epsilon + \mathbf{O}(\epsilon^3) \\ 1 - \epsilon - \frac{1}{2}\epsilon^2 + \mathbf{O}(\epsilon^3). \end{cases}$$