M401 Spring 2010, Assignment 2 Solutions

1. [10 pts] In Assignment 1 we used Taylor's Theorem to solve Exercise 1.4 on p. 12 of Simmonds and Mann Jr. Solve the same problem using the expansion method.

Solution. We have already checked that the Implicit Function Theorem applies for this equation, so we can immediately look for solutions of the form

$$z(\epsilon) = a_0 + a_1\epsilon + \dots.$$

Subsituting this into

$$z^3 - z + \epsilon = 0,$$

we have

$$(a_0 + a_1\epsilon + \dots)^3 - (a_0 + a_1\epsilon + \dots) + \epsilon = 0.$$

Equating coefficients of powers of ϵ, we find

$$\begin{aligned} &1 : a_0^3 - a_0 = 0 \Rightarrow a_0 = -1, 0, 1 \\ &\epsilon : 3a_0^2 a_1 - a_1 + 1 = 0 \Rightarrow a_1 = -\frac{1}{3a_0^2 - 1} = -\frac{1}{2}, +1, -\frac{1}{2}. \end{aligned}$$

We conclude

$$z(\epsilon) = \begin{cases} -1 - \frac{1}{2}\epsilon + \mathbf{O}(\epsilon^2) \\ \epsilon + \mathbf{O}(\epsilon^2) \\ 1 - \frac{1}{2}\epsilon + \mathbf{O}(\epsilon^2). \end{cases}$$

2. [10 pts] Use the expansion method to find $\mathbf{O}(\epsilon^2)$ approximations for each of the three roots of

$$x^3 + \epsilon x^2 + 1 = 0.$$

Note. For this problem it's useful to make use of the identity $e^{i\pi} = -1$.

Solution. First, for $\epsilon = 0$ we have

$$x(0)^3 + 1 = 0 \Rightarrow x(0)^3 = -1 = -e^{2\pi n i} = e^{(\pi + 2\pi n)i},$$

$n = 0, 1, 2, \dots$. We have, then,

$$x(0) = e^{\frac{(1+2n)\pi i}{3}},$$

so that

$$\begin{aligned} &n = 0 : x(0) = e^{\frac{\pi}{3}i} = \cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3}) = \frac{1}{2} + i\frac{\sqrt{3}}{2} \\ &n = 1 : x(0) = e^{\pi i} = \cos(\pi) + i\sin(\pi) = -1 \\ &n = 2 : x(0) = e^{\frac{5\pi}{3}i} = \cos(\frac{5\pi}{3}) + i\sin(\frac{5\pi}{3}) = \frac{1}{2} - i\frac{\sqrt{3}}{2}. \end{aligned}$$

(Alternatively, you can simply rearrange our figure from class, starting with the value -1 instead of +1.) We set

$$f(x, \epsilon) = x^3 + \epsilon x^2 + 1$$

and compute

$$f_x(x,\epsilon) = 3x^2 + 2\epsilon x.$$

Clearly,

$$f_x(x(0),0) \neq 0$$

for each of these values. (**Note.** It is acceptable at this point to simply state that since f is continuously differentiable the Implicit Function Theorem holds for any non-repeated root; at the same time, it will also be acceptable to state that the Implicit Function Theorem does not hold for repeated roots.) This justifies looking for solutions of the form

$$x(\epsilon) = a_0 + a_1\epsilon + \dots,$$

for which we have

$$(a_0 + \epsilon a_1 + \dots)^3 + \epsilon(a_0 + a_1\epsilon + \dots)^2 + 1 = 0.$$

Equating coefficients of powers of ϵ, we have

$$\begin{aligned} &1 : a_0^3 + 1 = 0 \Rightarrow a_0 = (\frac{1}{2} + i\frac{\sqrt{3}}{2}), -1, (\frac{1}{2} - i\frac{\sqrt{3}}{2}) \\ &\epsilon : 3a_0^2 a_1 + a_0^2 = 0 \Rightarrow a_1 = -\frac{1}{3}. \end{aligned}$$

We conclude

$$x(\epsilon) = \begin{cases} (\frac{1}{2} + i\frac{\sqrt{3}}{2}) - \frac{1}{3}\epsilon + \mathbf{O}(\epsilon^2) \\ -1 - \frac{1}{3}\epsilon + \mathbf{O}(\epsilon^2) \\ (\frac{1}{2} - i\frac{\sqrt{3}}{2}) - \frac{1}{3}\epsilon + \mathbf{O}(\epsilon^2). \end{cases}$$

3. [10 pts] Exercise 1.7 on p. 17 of Simmonds and Mann Jr.

Solution. We observe that for $\epsilon = 0$, we have $x(0)^3 = 0$, which has a triple root at 0. If we set

$$f(x,\epsilon) = x^3 - \epsilon x^2 - \epsilon^2,$$

we find $f_x(x,\epsilon) = 3x^2 - 2\epsilon x$, so that $f_x(0,0) = 0$ and the Implicit Function Theorem does not apply. (Notice that I've re-written the equation in terms of x instead of z, consistent with our notation in class.) Hence, we expect our expansion to be in terms of a non-integer power of ϵ. Accordingly, we make the substitution $x = \epsilon^p z$, for which z is considered to be $\mathbf{O}(1)$ as $\epsilon \to 0$. We have

$$\epsilon^{3p} z^3 - \epsilon^{1+2p} z^2 - \epsilon^2 = 0.$$

For these terms to cancel, we must have at least two exponents of agreeing. The possibilities are $3p = 1 + 2p \Rightarrow p = 1$, $1 + 2p = 2 \Rightarrow p = \frac{1}{2}$, and $3p = 2 \Rightarrow p = \frac{2}{3}$. For $p = 1$, we would have $\epsilon^3 z^3 - \epsilon^3 z^2 - \epsilon^2 = 0$. Dividing by ϵ^3, however, and taking a limit as $\epsilon \to 0$, we find that this implies $-\infty = 0$, which is a contradition. Similarly, $p = \frac{1}{2}$ can be eliminated, and we try $p = \frac{2}{3}$, for which we have

$$\epsilon^2 z^3 - \epsilon^{7/3} z^2 - \epsilon^2 = 0 \Rightarrow z^3 - \epsilon^{1/3} z^2 - 1 = 0.$$

Finally, we make the substitution $\beta = \epsilon^{1/3}$ to arrive at the regular equation

$$z^3 - \beta z^2 - 1 = 0.$$

Posing now the standard expansion,

$$z(\beta) = a_0 + a_1\beta + \mathbf{O}(\beta^2),$$

we have

$$(a_0 + a_1\beta + \mathbf{O}(\beta^2))^3 - \beta(a_0 + a_1\beta + \mathbf{O}(\beta^2))^2 - 1 = 0.$$

Equating coefficients of powers of β, we find

$$\begin{aligned} 1 :& a_0^3 - 1 = 0 \Rightarrow a_0 = e^{\frac{2\pi n i}{3}}, n = 0, 1, 2 \Rightarrow a_0 = 1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2} \\ \beta :& 3a_0^2 a_1 - a_0^2 = 0 \Rightarrow a_0 = \frac{1}{3}. \end{aligned}$$

We conclude that

$$x(\epsilon) = \begin{cases} \epsilon^{2/3} + \frac{1}{3}\epsilon + \mathbf{O}(\epsilon^{4/3}) \\ (-\frac{1}{2} + i\frac{\sqrt{3}}{2})\epsilon^{2/3} + \frac{1}{3}\epsilon + \mathbf{O}(\epsilon^{4/3}) \\ (-\frac{1}{2} - i\frac{\sqrt{3}}{2})\epsilon^{2/3} + \frac{1}{3}\epsilon + \mathbf{O}(\epsilon^{4/3}). \end{cases}$$

4. [10 pts] Approximate the real-valued root of

$$(x-1)^3 + \epsilon x = 0$$

with an error $\mathbf{o}(\epsilon)$.

Solution. First, for $\epsilon = 0$ we have a triple root $x(0) = 1$. Setting $f(x, \epsilon) = (x-1)^3 + \epsilon x$, we have

$$f_x(x, \epsilon) = 3(x-1)^2 + \epsilon,$$

so that

$$f_x(1, 0) = 0,$$

which means the Implicit Function Theorem does not apply. We proceed first by changing variables, setting

$$y = x - 1,$$

so that our equation becomes

$$y^3 + \epsilon(y + 1) = 0.$$

In order to find an appropriate scaling, we set

$$y = \epsilon^p z,$$

which gives

$$\epsilon^{3p} z^3 + \epsilon^{1+p} z + \epsilon = 0.$$

We have three possible scaling choices: (1) $3p = 1 + p$; (2) $3p = 1$; and (3) $1 + p = 1$. For the first we find $p = \frac{1}{2}$, and our equation becomes

$$\epsilon^{\frac{3}{2}} z^3 + \epsilon^{\frac{3}{2}} z + \epsilon = 0,$$

which is unacceptable since ϵ is lower order than the canceling terms. Case (3) is similarly unacceptable, and we take Case (2) with $p = \frac{1}{3}$. With this scaling, our equation becomes

$$z^3 + \epsilon^{\frac{1}{3}} z + 1 = 0.$$

We define the new perturbation variable $\beta = \epsilon^{\frac{1}{3}}$, so that we have

$$z^3 + \beta z + 1 = 0.$$

For $\beta = 0$ we have (see the solution to Problem 2)

$$z(0)^3 + 1 = 0 \Rightarrow z(0) = \frac{1}{2} + i\frac{\sqrt{3}}{2}, -1, \frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

If we set $f(z, \beta) = z^3 + \beta z + 1$ then

$$f_z(z, \beta) = 3z^2 + \beta,$$

so it's clear that $f_x(z(0), 0) \neq 0$ for any of these roots and the Implicit Function Theorem applies. In order to the required $\mathbf{o}(\epsilon)$ estimate we require three terms,

$$z(\beta) = b_0 + b_1\beta + b_2\beta^2 + \dots.$$

We have

$$(b_0 + b_1\beta + b_2\beta^2 + \dots)^3 + \beta(b_0 + b_1\beta + b_2\beta^2 + \dots) + 1 = 0.$$

Equating coefficients of powers of β, we find (working now for the real-valued root only)

$$\begin{aligned}
1 &: b_0^3 + 1 = 0 \Rightarrow b_0 = -1, \\
\beta &: 3b_0^2 b_1 + b_0 = 0 \Rightarrow b_1 = -\frac{1}{3b_0} = \frac{1}{3} \\
\beta^2 &: 3b_0 b_1^2 + 3b_0^2 b_2 + b_1 = 0 \Rightarrow b_2 = -\frac{b_1 + 3b_0 b_1^2}{3b_0^2} = -\frac{\frac{1}{3} - 3(\frac{1}{3})^2}{3} = 0.
\end{aligned}$$

We conclude that the real root has the form

$$x(\epsilon) = 1 - \epsilon^{\frac{1}{3}} + \frac{1}{3}\epsilon^{\frac{2}{3}} + \mathbf{O}(\epsilon^{\frac{4}{3}}).$$

5. [10 pts] Approximate the four roots of

$$(x - 1)^2(x^2 - 4) - \epsilon = 0,$$

with an error $\mathbf{o}(\epsilon)$.

Solution. First, we observe that for $\epsilon = 0$, we have four solutions, $x(0) = 1, 1, 2, -2$. If we set $f(x, \epsilon) = (x-1)^2(x^2-4) - \epsilon$, we have

$$f_x(x, \epsilon) = 2(x-1)(x^2-4) + (x-1)^2 2x.$$

Here, $f_x(1, 0) = 0$, so the Implicit Function Theorem does not apply for the repeated root, but $f_x(2, 0) = 4$ and $f_x(-2, 0) = 36$, so the Implicit Function Theorem does apply for the latter two roots. For $x(0) = \pm 2$ we take a regular expansion, for which

$$((a_0 + a_1\epsilon + \dots) - 1)^2((a_0 + a_1\epsilon + \dots)^2 - 4) - \epsilon = 0.$$

Equating coefficients of powers of ϵ, we find (for roots ± 2 only)

$$\begin{aligned} 1 :& (a_0 - 1)^2(a_0^2 - 4) = 0 \Rightarrow a_0 = 2, -2 \\ \epsilon :& 2(a_0-1)a_1(a_0^2-4) + (a_0-1)^2 2a_0a_1 - 1 = 0 \Rightarrow a_1 = \frac{1}{2(a_0-1)^2a_0} = \frac{1}{4}, -\frac{1}{36}. \end{aligned}$$

We conclude that

$$x(\epsilon) = \begin{cases} 2 + \frac{1}{4}\epsilon + \mathbf{O}(\epsilon^2) \\ -2 - \frac{1}{36}\epsilon + \mathbf{O}(\epsilon^2). \end{cases}$$

For the roots near 1, we make the substitution $z = x - 1$, to obtain the equation

$$z^2((z+1)^2 - 4) - \epsilon = 0.$$

Next, we look for solutions of the form $z = \epsilon^p w$, which satisfy

$$\epsilon^{2p} w^2((\epsilon^p w + 1)^2 - 4) - \epsilon = 0,$$

for which we see that $p = 1/2$ is the only possible scaling. Dividing by ϵ, we find

$$w^2((\epsilon^{1/2} w + 1)^2 - 4) - 1 = 0.$$

Setting $\beta = \epsilon^{1/2}$, we can write this as

$$w^2((\beta w + 1)^2 - 4) - 1 = 0.$$

Setting $\beta = 0$ in this new equation, we obtain

$$w(0)^2(-3) - 1 = 0 \Rightarrow w(0) = \pm\frac{1}{\sqrt{-3}} = \pm i\frac{1}{\sqrt{3}}.$$

Setting $f(w, \beta) = w^2((\beta w + 1)^2 - 4) - 1 = 0$, we compute

$$f_w(w, \beta) = 2w((\beta w + 1)^2 - 4) + w^2 2(\beta w + 1)\beta,$$

so that

$$f_w(-i\frac{1}{\sqrt{3}}, 0) = 2(-i\frac{1}{\sqrt{3}})(-3) \neq 0,$$

and similarly for $+i\frac{1}{\sqrt{3}}$. In order to obtain an $\mathbf{o}(\epsilon)$ approximation we will need two terms in the expansion for w, so we write

$$w = b_0 + b_1\beta + \cdots .$$

Subsituting this into our equation, we have

$$(b_0 + b_1\beta + \cdots)^2((\beta(b_0 + b_1\beta + \cdots) + 1)^2 - 4) - 1 = 0.$$

Equating coefficients of powers of β, we find

$$\begin{aligned} 1 &: b_0^2(-3) - 1 \Rightarrow b_0 = -i\frac{1}{\sqrt{3}}, +i\frac{1}{\sqrt{3}} \\ \beta &: -6b_0b_1 + 2b_0^3 = 0 \Rightarrow b_1 = \frac{b_0^2}{3} = -\frac{1}{9}, -\frac{1}{9}. \end{aligned}$$

We conclude that

$$w(\beta) = \begin{cases} -i\frac{1}{\sqrt{3}} - \frac{1}{9}\beta + \mathbf{O}(\beta^2) \\ +i\frac{1}{\sqrt{3}} - \frac{1}{9}\beta + \mathbf{O}(\beta^2) \end{cases}$$

In Summary,

$$x(\epsilon) = \begin{cases} 2 + \frac{1}{4}\epsilon + \mathbf{O}(\epsilon^2) \\ -2 - \frac{1}{36}\epsilon + \mathbf{O}(\epsilon^2) \\ 1 - i\frac{1}{\sqrt{3}}\epsilon^{1/2} - \frac{1}{9}\epsilon + \mathbf{O}(\epsilon^{3/2}) \\ 1 + i\frac{1}{\sqrt{3}}\epsilon^{1/2} - \frac{1}{9}\epsilon + \mathbf{O}(\epsilon^{3/2}) \end{cases}$$