M401 Spring 2010, Assignment 3 Solutions

1. [10 pts] Exercise 1.5 on p. 15 of Simmonds and Mann Jr.

Solution. We first observe that for $\epsilon = 0$, we have $-x(0)^2 + 1 = 0 \Rightarrow x(0) = \pm 1$, and since these roots are not repeated the Implicit Function Theorem will hold for them. For these roots, we take the standard expansion,

$$x(\epsilon) = a_0 + a_1\epsilon + a_2\epsilon^2 + \dots.$$

Substituting this expansion into our equation, we find

$$\epsilon(a_0 + a_1\epsilon + a_2\epsilon^2 + \dots)^3 - (a_0 + a_1\epsilon + a_2\epsilon^2 + \dots)^2 + 1 = 0.$$

Equating the coefficients of powers of ϵ, we have

$$\begin{aligned}
1 :& -a_0^2 + 1 = 0 \Rightarrow a_0 = \pm 1 \\
\epsilon :& a_0^3 - 2a_0a_1 = 0 \Rightarrow a_1 = \frac{a_0^2}{2} = \frac{1}{2} \\
\epsilon^2 :& 3a_0^2a_1 - a_1^2 - 2a_0a_2 = 0 \Rightarrow a_2 = \frac{3a_0^2a_1 - a_1^2}{2a_0} = \pm\frac{5}{8}.
\end{aligned}$$

We conclude that the non-singular roots satisfy

$$x(\epsilon) = \pm 1 + \frac{1}{2}\epsilon \pm \frac{5}{8}\epsilon^2 + \mathbf{O}(\epsilon^2).$$

For the singular root, we observe that for x large, $\epsilon x^3 - x^2 \cong 0 \Rightarrow x \cong \frac{1}{\epsilon}$. (Note particularly that since x will be large the terms x^2 and 1 cannot possibly cancel.) We make the change of variables, $z = \epsilon x$, for which

$$\epsilon\frac{z^3}{\epsilon^3} - \frac{z^2}{\epsilon^2} + 1 = 0 \Rightarrow z^3 - z^2 + \epsilon^2 = 0.$$

Finally, we make the substitution $\beta = \epsilon^2$ to obtain

$$z^3 - z^2 + \beta = 0.$$

Applying now the standard expansion,

$$z(\beta) = a_0 + a_1\beta + a_2\beta^2 + \mathbf{O}(\beta^3),$$

we have,

$$(a_0 + a_1\beta + a_2\beta^2 + \mathbf{O}(\beta^3))^3 - (a_0 + a_1\beta + a_2\beta^2 + \mathbf{O}(\beta^3))^2 + \beta = 0.$$

Equating coefficients of powers of β, we find (for the non-singular solution only)

$$\begin{aligned}
1 :& a_0^3 - a_0^2 = 0 \Rightarrow a_0 = 1 \\
\beta :& 3a_0^2a_1 - 2a_0a_1 + 1 = 0 \Rightarrow a_1 = \frac{1}{2a_0 - 3a_0^2} = -1 \\
\beta^2 :& 3a_0a_1^2 + 3a_0^2a_2 - a_1^2 - 2a_0a_2 = 0 \Rightarrow a_2 = \frac{a_1^2 - 3a_0a_1^2}{3a_0^2 - 2a_0} = -2.
\end{aligned}$$

(Keep in mind that the repeated roots $a_0 = 0$ correspond with the roots we have already found, so they aren't considered here.) We conclude that

$$z(\beta) = 1 - \beta - 2\beta^2 + \mathbf{O}(\beta^3),$$

and consequently,

$$x(\epsilon) = \frac{1}{\epsilon} - \epsilon - 2\epsilon^3 + \mathbf{O}(\epsilon^5).$$

In summary

$$x(\epsilon) = \begin{cases} -1 + \frac{1}{2}\epsilon - \frac{5}{8}\epsilon^2 + \mathbf{O}(\epsilon^3) \\ +1 + \frac{1}{2}\epsilon + \frac{5}{8}\epsilon^2 + \mathbf{O}(\epsilon^3) \\ \frac{1}{\epsilon} - \epsilon - 2\epsilon^3 + \mathbf{O}(\epsilon^5). \end{cases}$$

2. [10 pts] Find the first two terms in the expansions of each of the four roots of

$$\epsilon x^4 + \epsilon x^3 - x^2 + 2x - 1 = 0,$$

for $\epsilon \geq 0$.

Solution. Setting $\epsilon = 0$, we have $-x(0)^2 + 2x(0) - 1 = -(x(0) - 1)^2 = 0$, which means we have two solutions that approach $x(0) = 1$ as $\epsilon \to 0$. Since $x(0) = 1$ is a repeated root, these solutions will be raised to a fractional power (the Implicit Function Theorem will not apply). Re-writing our equation as

$$\epsilon x^4 + \epsilon x^3 - (x - 1)^2 = 0,$$

we make the substitution $z = x - 1$, so that

$$\epsilon(z + 1)^4 + \epsilon(z + 1)^3 - z^2 = 0.$$

Now, we search for our scaling by making the substitution $z = \epsilon^p w$, which gives

$$\epsilon(\epsilon^p w + 1)^4 + \epsilon(\epsilon^p w + 1)^3 - \epsilon^{2p} w^2 = 0.$$

Matching powers, we find that the order ϵ terms from the first two summands must cancel with the order ϵ^{2p} power (they are $\epsilon + \epsilon$, so that they cannot cancel one another). We conclude that $p = \frac{1}{2}$, and consequently

$$(\epsilon^{1/2} w + 1)^4 + (\epsilon^{1/2} w + 1)^3 - w^2 = 0.$$

Finally, making the substitution $\beta = \epsilon^{1/2}$, we have

$$(\beta w + 1)^4 + (\beta w + 1)^3 - w^2 = 0.$$

Observing that we already have the first term of x ($x(0) = 1$), we need only consider the expansion $z(\beta) = a_0 + \mathbf{O}(\beta)$. But this is equivalent to simply setting $\beta = 0$. We conclude that $2 - a_0^2 = 0 \Rightarrow a_0 = \pm\sqrt{2}$. Working back through our substitutions, we have

$$x(\epsilon) = 1 \pm \sqrt{2}\epsilon^{1/2} + \mathbf{O}(\epsilon).$$

Now, we proceed with the singular roots. Observing that for x large the two dominant terms are ϵx^4 and $-x^2$, we have $\epsilon x^4 - x^2 \cong 0 \Rightarrow x \cong \epsilon^{-1/2}$. This suggests the substitution $z = \epsilon^{1/2} x$, which leads to

$$\frac{z^4}{\epsilon} + \frac{z^3}{\epsilon^{1/2}} - \frac{z^2}{\epsilon} + 2\frac{z}{\epsilon^{1/2}} - 1 = 0.$$

Multiplying by ϵ, we have

$$z^4 + \epsilon^{1/2} z^3 - z^2 + 2\epsilon^{1/2} z - \epsilon = 0.$$

Finally, we make the substitution $\beta = \epsilon^{1/2}$, to get

$$z^4 + \beta z^3 - z^2 + 2\beta z - \beta^2 = 0,$$

for which we take the regular expansion

$$z(\beta) = a_0 + a_1\beta + \dots,$$

which leads to

$$(a_0 + a_1\beta + \dots)^4 + \beta(a_0 + a_1\beta + \dots)^3 - (a_0 + a_1\beta + \dots)^2 + 2\beta(a_0 + a_1\beta + \dots) - \beta^2 = 0.$$

Equating coefficients of powers of β, we find (considering only the singular solutions)

$$\begin{aligned} 1 :& a_0^4 - a_0^2 = 0 \Rightarrow a_0 = 1, -1 \\ \beta :& 4a_0^3 a_1 + a_0^3 - 2a_0 a_1 + 2a_0 = 0 \Rightarrow a_1 = \frac{-a_0^3 - 2a_0}{4a_0^3 - 2a_0} = -\frac{3}{2}, -\frac{3}{2}. \end{aligned}$$

Working back through our substitutions, we conclude

$$x(\epsilon) = \pm\frac{1}{\epsilon^{1/2}} - \frac{3}{2} + \mathbf{O}(\epsilon^{1/2}).$$

In summary we have

$$x(\epsilon) = \begin{cases} 1 - \sqrt{2}\epsilon^{1/2} + \mathbf{O}(\epsilon) \\ 1 + \sqrt{2}\epsilon^{1/2} + \mathbf{O}(\epsilon) \\ -\frac{1}{\epsilon^{1/2}} - \frac{3}{2} + \mathbf{O}(\epsilon^{1/2}) \\ +\frac{1}{\epsilon^{1/2}} - \frac{3}{2} + \mathbf{O}(\epsilon^{1/2}). \end{cases}$$

3. [10 pts] Solve the ODE

$$\begin{aligned} y'' + 4y &= e^{5t}, \\ y(0) &= 1 \\ y'(0) &= 0. \end{aligned}$$

Solution. First, the associated homogeneous equation is $y'' + 4y = 0$, and if we look for solutions of the form $y(t) = e^{rt}$, we obtain the auxiliary equation

$$r^2 + 4 = 0 \Rightarrow r = \pm 2i.$$

This gives the linearly independent solutions $y(t) = e^{\pm 2it}$, and by using Euler's formula we can work instead with $y_1(t) = \cos(2t)$ and $y_2(t) = \sin(2t)$. For the particular solution, we look for a solution of the form

$$y_p(t) = Ae^{5t}.$$

Our equation becomes

$$25Ae^{5t} + 4Ae^{5t} = e^{5t} \Rightarrow A = \frac{1}{29}.$$

The general solution to this equation is

$$y(t) = C_1 \cos(2t) + C_2 \sin(2t) + \frac{1}{29}e^{5t}.$$

According to the initial conditions, C_1 and C_2 must satisfy

$$\begin{aligned} 1 &= C_1 + \frac{1}{29} \Rightarrow C_1 = \frac{28}{29} \\ 0 &= 2C_2 + \frac{5}{29} \Rightarrow C_2 = -\frac{5}{58}. \end{aligned}$$

The solution is

$$y(t) = \frac{28}{29}\cos(2t) - \frac{5}{58}\sin(2t) + \frac{1}{29}e^{5t}.$$

4. [10 pts] Solve the ODE

$$\begin{aligned} y'' + 4y &= 5\cos(2t) \\ y(0) &= 1 \\ y'(0) &= 0. \end{aligned}$$

Solution. As in Problem 3 the homogeneous solutions are $y_1(t) = \cos(2t)$ and $y_2(t) = \sin(2t)$. Since the right-hand side is a solution to the homogeneous problem, we look for particular solutions of the form

$$y_p(t) = At\cos(2t) + Bt\sin(2t).$$

Accordingly, we have

$$\begin{aligned} y_p'(t) &= A\cos(2t) - 2At\sin(2t) + B\sin(2t) + 2Bt\cos(2t) \\ y_p''(t) &= -4A\sin(2t) - 4At\cos(2t) + 4B\cos(2t) - 4Bt\sin(2t). \end{aligned}$$

Substituting $y_p(t)$ into our equation, we find

$$-4A\sin(2t) - 4At\cos(2t) + 4B\cos(2t) - 4Bt\sin(2t) + 4At\cos(2t) + 4Bt\sin(2t) = 5\cos(2t).$$

The coefficients of $t\cos(2t)$ and $t\sin(2t)$ cancel and we are left with the following coefficient equations:

$$\begin{aligned} \sin 2t &: -4A = 0 \Rightarrow A = 0 \\ \cos 2t &: 4B = 5 \Rightarrow B = \frac{5}{4}. \end{aligned}$$

The general solution of this equation is t

$$y(t) = C_1 \cos(2t) + C_2 \sin(2t) + \frac{5}{4} t \sin(2t).$$

According to the initial conditions C_1 and C_2 must satisfy

$$\begin{aligned} 1 &= C_1 \Rightarrow C_1 = 1 \\ 0 &= 2C_2 \Rightarrow C_2 = 0. \end{aligned}$$

The solution is

$$y(t) = \cos(2t) + \frac{5}{4} t \sin(2t).$$

5. [10 pts] Use Taylor's Theorem to find the first two terms in a perturbation expansion of the solution of the ODE

$$\begin{aligned} y'' + k^2 y &= \epsilon y^2, \\ y(0) &= 1 \\ y'(0) &= 0. \end{aligned}$$

Solution. As in class, we look for solutions of the form

$$y(t;\epsilon) = y(t;0) + y_\epsilon(t;0)\epsilon + \frac{y_{\epsilon\epsilon}(t;c)}{2}\epsilon^2,$$

where c is between 0 and ϵ and depends on t. In order to compute $y(t;0)$ we simply set $\epsilon = 0$, so that if we write $y_0(t) = y(t;0)$ we have

$$\begin{aligned} y_0'' + k^2 y_0 &= 0 \\ y_0(0) &= 1 \\ y_0'(0) &= 0. \end{aligned}$$

We solved this equation in class and found

$$y_0(t) = \cos(kt).$$

For $y_\epsilon(t;0)$ we take an ϵ derivative of the ODE

$$\begin{aligned} y_\epsilon'' + k^2 y_\epsilon &= y^2 + \epsilon 2 y y_\epsilon \\ y_\epsilon(0) &= 0 \\ y_\epsilon'(0) &= 0. \end{aligned}$$

Setting $\epsilon = 0$ and writing $y_1(t) = y_\epsilon(t;0)$ for convenience, we have

$$\begin{aligned} y_1'' + k^2 y_1 &= y_0^2 \\ y_1(0) &= 0 \\ y_1'(0) &= 0. \end{aligned}$$

We recall that

$$y_0^2 = \cos^2 kt = \frac{1}{2} + \frac{1}{2}\cos(2kt),$$

so the equation we need to solve is

$$\begin{aligned} y_1'' + k^2 y_1 &= \frac{1}{2} + \frac{1}{2}\cos(2kt) \\ y_1(0) &= 0 \\ y_1'(0) &= 0. \end{aligned}$$

We already know that the linearly independent solutions of this equation are $\cos(kt)$ and $\sin(kt)$, so we proceed by looking for a particular solution of the form

$$y_p(t) = A + B\cos(2kt) + C\sin(2kt).$$

We have

$$y_p''(t) = -4k^2 B\cos(2kt) - 4k^2 C\sin(2kt),$$

so that substitution into our equation leads to

$$-4k^2 B\cos(2kt) - 4k^2 C\sin(2kt) + k^2 A + k^2 B\cos(2kt) + k^2 C\sin(2kt) = \frac{1}{2} + \frac{1}{2}\cos(2kt).$$

We equate coefficients as follows:

$$\begin{aligned} \cos(2kt) : -3k^2 B &= \frac{1}{2} \Rightarrow B = -\frac{1}{6k^2} \\ \sin(2kt) : -3k^2 C &= 0 \Rightarrow C = 0 \\ 1 : k^2 A &= \frac{1}{2} \Rightarrow A = \frac{1}{2k^2}. \end{aligned}$$

We conclude that

$$y_1(t) = C_1\cos(kt) + C_2\sin(kt) + \frac{1}{2k^2} - \frac{1}{6k^2}\cos(2kt),$$

and we now use the initial conditions to evaluate C_1 and C_2. We have

$$\begin{aligned} 0 &= C_1 + \frac{1}{3k^2} \Rightarrow C_1 = -\frac{1}{3k^2} \\ 0 &= C_2 k \Rightarrow C_2 = 0. \end{aligned}$$

We conclude

$$y_1(t) = -\frac{1}{3k^2}\cos(kt) + \frac{1}{2k^2} - \frac{1}{6k^2}\cos(2kt).$$

Finally, we have

$$y(t) = \cos(kt) + \epsilon\Big(-\frac{1}{3k^2}\cos(kt) + \frac{1}{2k^2} - \frac{1}{6k^2}\cos(2kt)\Big) + \frac{y_{\epsilon\epsilon}(t;c)}{2}\epsilon^2,$$

where c is between 0 and ϵ and depends on t.