

M401 Spring 2010, Assignment 4, due Thursday February 18

Note. Save this assignment page. You will be allowed to bring it to both the midterm and the final. (No, you can't bring solutions.)

1. [10 pts] In Problem 5 in Assignment 3, we used Taylor's Theorem to find the first two terms in a perturbation expansion of the solution of the ODE

$$\begin{aligned}y'' + k^2y &= \epsilon y^2, \\ y(0) &= 1 \\ y'(0) &= 0.\end{aligned}$$

1a. Write this equation as a first order system of equations. Write it in vector notation, and identify the vector $\vec{f}(t, \vec{y}; \epsilon)$.

1b. Explain what Taylor's Theorem and Picard's Theorem guarantee about the accuracy of your perturbation expansion.

2. [10 pts] In this problem we will see how trigonometric identities follow from Euler's formula

$$e^{iA} = \cos A + i \sin A,$$

and the usual rules of exponentiation.

2a. Use the identity $e^{iA}e^{-iA} = 1$ to show that

$$\cos^2 A + \sin^2 A = 1.$$

2b. Use the identity $e^{i(A+B)} = e^{iA}e^{iB}$ to show both of the following:

$$\begin{aligned}\cos(A+B) &= \cos A \cos B - \sin A \sin B \\ \sin(A+B) &= \sin A \cos B + \cos A \sin B.\end{aligned}$$

2c. Use (2b) with B replaced by $-B$ to show

$$\begin{aligned}\cos(A-B) &= \cos A \cos B + \sin A \sin B \\ \sin(A-B) &= \sin A \cos B - \cos A \sin B.\end{aligned}$$

2d. Combine (2b) and (2c) to show

$$\begin{aligned}\cos A \cos B &= \frac{1}{2}(\cos(A+B) + \cos(A-B)) \\ \sin A \cos B &= \frac{1}{2}(\sin(A+B) + \sin(A-B)) \\ \sin A \sin B &= \frac{1}{2}(\cos(A-B) - \cos(A+B)).\end{aligned}$$

2e. Use (2a) and (2b) to show

$$\cos^2 A = \frac{1 + \cos 2A}{2}$$
$$\sin^2 A = \frac{1 - \cos 2A}{2}.$$

2f. Use (2a) and (2d) to show

$$\cos^3 A = \frac{3}{4} \cos A + \frac{1}{4} \cos 3A$$
$$\sin^3 A = \frac{3}{4} \sin A - \frac{1}{4} \sin 3A$$
$$\cos A \sin^2 A = \frac{1}{4} \cos A - \frac{1}{4} \cos 3A$$
$$\cos^2 A \sin A = \frac{1}{4} \sin A + \frac{1}{4} \sin 3A.$$

3. [10 pts] Use the expansion method to find the first two terms in a perturbation expansion of the solution of the ODE

$$y'' + \epsilon y'(y^2 - 1) + y = 0$$
$$y(0) = 1$$
$$y'(0) = 0.$$

This equation is known as the van der Pol equation, and was proposed by Balthasar van der Pol (1889-1959) in 1920.

4. [10 pts] For the equation

$$y'' + \epsilon y(y')^2 + k^2 y = 0$$
$$y(0) = 1$$
$$y'(0) = 0,$$

find the first two terms (in both $y(t; \epsilon)$ and $\lambda(\epsilon)$) for a Poincare expansion of the solution. Explain the expected accuracy or your approximation.

5. [10 pts] For the equation

$$y'' + k^2 y = \epsilon y(1 - (y')^2)$$
$$y(0) = 1$$
$$y'(0) = 0,$$

find the first two terms (in both $y(t; \epsilon)$ and $\lambda(\epsilon)$) for a Poincare expansion of the solution. Explain the expected accuracy or your approximation. (**Note.** This problem is quite similar to Problem 4, and you should feel free to avoid repeating calculations where possible.)