M401 Spring 2010, Assignment 4 Solutions

1. [10 pts] In Problem 5 in Assignment 3, we used Taylor's Theorem to find the first two terms in a perturbation expansion of the solution of the ODE

$$\begin{aligned} y'' + k^2 y &= \epsilon y^2, \\ y(0) &= 1 \\ y'(0) &= 0. \end{aligned}$$

1a. Write this equation as a first order system of equations. Write it in vector notation, and identify the vector $\vec{f}(t, \vec{y}; \epsilon)$.

Solution. We define the variables $y_1 = y$ and $y_2 = y'$, so that

$$\begin{aligned} y_1' &= y_2; \quad y_1(0) = 1 \\ y_2' &= \epsilon y_1^2 - k^2 y_1; \quad y_2(0) = 0. \end{aligned}$$

This has the form

$$\frac{d\vec{y}}{dt} = \vec{f}(t, \vec{y}; \epsilon); \quad \vec{y}(0) = \vec{y}_0,$$

with

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}; \quad \vec{f}(t, \vec{y}; \epsilon) = \begin{pmatrix} y_2 \\ \epsilon y_1^2 - k^2 y_1 \end{pmatrix}; \quad \vec{y}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

1b. Explain what Taylor's Theorem and Picard's Theorem guarantee about the accuracy or your perturbation expansion.

Solution. The expansion from Problem 5 in Assignment 3 has the form

$$y(t) = \cos(kt) + \epsilon\Big((1 - \frac{1}{3k^2})\cos(kt) + \frac{1}{2k^2} - \frac{1}{6k^2}\cos(2kt)\Big) + \frac{y_{\epsilon\epsilon}(t; c)}{2}\epsilon^2,$$

where c is between 0 and ϵ and depends on t. Picard's Theorem guarantees that $y(t; \epsilon)$ and all its derivatives are continuous in both t and ϵ, so $y_{\epsilon\epsilon}(t; \epsilon)$ is continuous for $|t| < t_0$ and $|\epsilon| < \epsilon_0$ for some values $t_0 > 0$ and $\epsilon_0 > 0$. This means that for any $t_1 < t_0$ and $\epsilon_1 < \epsilon_0$ $y_{\epsilon\epsilon}(t; \epsilon)$ is continuous on the closed and bounded rectangle $|t| \le t_1$ and $|\epsilon| \le \epsilon_1$. We can conclude from the Extreme Value Theorem that $y_{\epsilon\epsilon}(t; \epsilon)$ is bounded on this rectangle. We have, then,

$$y(t) = \cos(kt) + \epsilon\Big((1 - \frac{1}{3k^2})\cos(kt) + \frac{1}{2k^2} - \frac{1}{6k^2}\cos(2kt)\Big) + \mathbf{O}(\epsilon^2),$$

uniformly for $|t| \le t_1$.

2. [10 pts] In this problem we will see how trigonometric identities follow from Euler's formula

$$e^{iA} = \cos A + i \sin A,$$

and the usual rules of exponentiation.

2a. Use the identity $e^{iA}e^{-iA} = 1$ to show that

$$\cos^2 A + \sin^2 A = 1.$$

Solution. We have

$$1 = e^{iA}e^{-iA} = (\cos A + i \sin A)(\cos A - i \sin A) = \cos^2 A + \sin^2 A.$$

2b. Use the identity $e^{i(A+B)} = e^{iA}e^{iB}$ to show both of the following:

$$\begin{aligned} \cos(A + B) &= \cos A \cos B - \sin A \sin B \\ \sin(A + B) &= \sin A \cos B + \cos A \sin B. \end{aligned}$$

Solution. We have

$$e^{i(A+B)} = \cos(A + B) + i \sin(A + B),$$

and

$$e^{iA}e^{iB} = (\cos A+i \sin A)(\cos B+i \sin B) = \cos A \cos B-\sin A \sin B+i(\cos A \sin B+\sin A \cos B).$$

Equating real and imaginary parts gives the identities.

2c. Use (2b) with B replaced by $-B$ to show

$$\begin{aligned} \cos(A - B) &= \cos A \cos B + \sin A \sin B \\ \sin(A - B) &= \sin A \cos B - \cos A \sin B. \end{aligned}$$

Solution. We have

$$\begin{aligned} \cos(A - B) &= \cos A \cos(-B) - \sin A \sin(-B) = \cos A \cos B + \sin A \sin B \\ \sin(A - B) &= \sin A \cos(-B) + \cos A \sin(-B) = \sin A \cos B - \cos A \sin B. \end{aligned}$$

2d. Combine (2b) and (2c) to show

$$\begin{aligned} \cos A \cos B &= \frac{1}{2}(\cos(A + B) + \cos(A - B)) \\ \sin A \cos B &= \frac{1}{2}(\sin(A + B) + \sin(A - B)) \\ \sin A \sin B &= \frac{1}{2}(\cos(A - B) - \cos(A + B)). \end{aligned}$$

Solution. For the first, add the first identity in (2b) to the first identity in (2c). For the second, add the second identity in (2b) to the second identity in (2c). For the third, subtract the first identity in (2a) from the first identity in (2b).

2e. Use (2a) and (2b) to show

$$\cos^2 A = \frac{1 + \cos 2A}{2}$$
$$\sin^2 A = \frac{1 - \cos 2A}{2}.$$

Solution. First, taking $B = A$ in the first identity in (2b)

$$\cos 2A = \cos^2 A - \sin^2 B.$$

If we now use (2a) to replace $\sin^2 A$ with $1 - \cos^2 A$ we obtain

$$\cos 2A = \cos^2 A - (1 - \cos^2 A) = -1 + 2\cos^2 A \Rightarrow \cos^2 A = \frac{1 + \cos 2A}{2}.$$

Likewise, if we use (2a) to replace $\cos^2 A$ with $1 - \sin^2 A$ we obtain

$$\cos 2A = (1 - \sin^2 A) - \sin^2 A = 1 - 2\sin^2 A \Rightarrow \sin^2 A = \frac{1 - \cos 2A}{2}.$$

2f. Use (2a) and (2d) to show

$$\cos^3 A = \frac{3}{4}\cos A + \frac{1}{4}\cos 3A$$
$$\sin^3 A = \frac{3}{4}\sin A - \frac{1}{4}\sin 3A$$
$$\cos A \sin^2 A = \frac{1}{4}\cos A - \frac{1}{4}\cos 3A$$
$$\cos^2 A \sin A = \frac{1}{4}\sin A + \frac{1}{4}\sin 3A.$$

Solution. For the first, we compute

$$\cos^3 A = \cos A \cos^2 A = \cos A(\frac{1}{2} + \frac{1}{2}\cos 2A) = \frac{1}{2}\cos A + \frac{1}{2}\cos A \cos 2A$$
$$= \frac{1}{2}\cos A + \frac{1}{2} \cdot \frac{1}{2}(\cos 3A + \cos A) = \frac{3}{4}\cos A + \frac{1}{4}\cos 3A.$$

Likewise,

$$\sin^3 A = \sin A \sin^2 A = \sin A(\frac{1}{2} - \frac{1}{2}\cos 2A) = \frac{1}{2}\sin A - \frac{1}{2}\sin A \cos 2A$$
$$= \frac{1}{2}\sin A - \frac{1}{2} \cdot \frac{1}{2}(\sin 3A - \sin A) = \frac{3}{4}\sin A - \frac{1}{4}\sin 3A.$$

For the third,

$$\cos A \sin^2 A = \cos A(1 - \cos^2 A) = \cos A - \frac{3}{4}\cos A - \frac{1}{4}\cos 3A$$
$$= \frac{1}{4}\cos A - \frac{1}{4}\cos 3A,$$

where I've used the identity for $\cos^3 A$. Finally,

$$\begin{aligned}\cos^2 A \sin A &= (1-\sin^2 A)\sin A = \sin A - \frac{3}{4}\sin A + \frac{1}{4}\sin 3A\\ &= \frac{1}{4}\sin A + \frac{1}{4}\sin 3A.\end{aligned}$$

3. [10 pts] Use the expansion method to find the first two terms in a perturbation expansion of the solution of the ODE

$$\begin{aligned}y'' + \epsilon y'(y^2-1) + y &= 0\\ y(0) &= 1\\ y'(0) &= 0.\end{aligned}$$

This equation is known as the van der Pol equation, and was proposed by Balthasar van der Pol (1889-1959) in 1920.

Solution. We look for solutions of the form

$$y(t;\epsilon) = y_0(t) + \epsilon y_1(t) + \dots,$$

for which we have

$$\begin{aligned}(y_0+\epsilon y_1+\dots)'' + \epsilon(y_0+\epsilon y_1+\dots)'((y_0+\epsilon y_1+\dots)^2-1) + y_0 + \epsilon y_1 + \dots &= 0\\ y_0(0) + \epsilon y_1(0) + \dots &= 1\\ y_0'(0) + \epsilon y_1'(0) + \dots &= 0.\end{aligned}$$

The ϵ^0 equation is

$$\begin{aligned}y_0'' + y_0 &= 0\\ y_0(0) &= 1\\ y_0'(0) &= 0,\end{aligned}$$

with solution $y_0(t) = \cos t$. The ϵ equation is

$$\begin{aligned}y_1'' + y_1 &= y_0'(1-y_0^2) = -\sin t(1-\cos^2 t) = -\sin^3 t\\ y_1(0) &= 0\\ y_1'(0) &= 0.\end{aligned}$$

Using now the identity

$$\sin^3 t = \frac{3}{4}\sin t - \frac{1}{4}\sin 3t,$$

the inhomogeneous equation we need to solve is

$$\begin{aligned}y_1'' + y_1 &= y_0'(1-y_0^2) = \frac{1}{4}\sin 3t - \frac{3}{4}\sin t\\ y_1(0) &= 0\\ y_1'(0) &= 0.\end{aligned}$$

The solutions to the homogenous problem are $\cos t$ and $\sin t$, so we look for particular solutions of the form

$$y_{1_p}(t) = A\sin 3t + B\cos 3t + Ct\sin t + Dt\cos t.$$

We have

$$\begin{aligned} y_{1_p}'(t) =& 3A\cos 3t - 3B\sin 3t + C\sin t + Ct\cos t + D\cos t - Dt\sin t \\ y_{1_p}''(t) =& -9A\sin 3t - 9B\cos 3t + 2C\cos t - Ct\sin t - 2D\sin t - Dt\cos t. \end{aligned}$$

Substituting this into our equation gives

$$\begin{aligned} &-9A\sin 3t - 9B\cos 3t + 2C\cos t - Ct\sin t - 2D\sin t - Dt\cos t \\ &\qquad + A\sin 3t + B\cos 3t + Ct\sin t + Dt\cos t = \frac{1}{4}\sin 3t - \frac{3}{4}\sin t. \end{aligned}$$

This gives the following equations for A, B, C, and D:

$$\begin{aligned} \sin 3t :& -9A + A = \frac{1}{4} \Rightarrow A = -\frac{1}{32} \\ \cos 3t :& -9B + B = 0 \Rightarrow B = 0 \\ \cos t :& 2C = 0 \Rightarrow C = 0 \\ \sin t :& -2D = -\frac{3}{4} \Rightarrow D = \frac{3}{8}. \end{aligned}$$

We have, then,

$$y_{1_p}(t) = -\frac{1}{32}\sin 3t + \frac{3}{8}t\cos t.$$

Our general solution has the form

$$y_1(t) = C_1\cos t + C_2\sin t - \frac{1}{32}\sin 3t + \frac{3}{8}t\cos t,$$

with

$$y_1'(t) = -C_1\sin t + C_2\cos t - \frac{3}{32}\cos 3t + \frac{3}{8}\cos t - \frac{3}{8}t\sin t.$$

The initial conditions give

$$\begin{aligned} C_1 &= 0 \\ C_2 - \frac{3}{32} + \frac{3}{8} &= 0 \Rightarrow C_2 = \frac{3}{32} - \frac{12}{32} = -\frac{9}{32}. \end{aligned}$$

We conclude that

$$y_1(t) = -\frac{9}{32}\sin t - \frac{1}{32}\sin 3t + \frac{3}{8}t\cos t,$$

and finally

$$y(t) = \cos t + \epsilon\Big(-\frac{9}{32}\sin t - \frac{1}{32}\sin 3t + \frac{3}{8}t\cos t\Big) + \mathbf{O}(\epsilon^2),$$

uniformly for $|t| \leq t_1$ for some $t_1 > 0$.

4. [10 pts] For the equation

$$\begin{aligned} y'' + \epsilon y(y')^2 + k^2 y &= 0 \\ y(0) &= 1 \\ y'(0) &= 0, \end{aligned}$$

find the first two terms (in both $y(t;\epsilon)$ and $\lambda(\epsilon)$) for a Poincare expansion of the solution. Explain the expected accuracy or your approximation.

Solution. In this case, we begin by making a change of variable

$$\tau = \lambda(\epsilon)t, \quad Y(\tau) = y(t),$$

with

$$\lambda(\epsilon) = k + \lambda_1\epsilon + \dots.$$

(Keep in mind that we set $\lambda(0) = k$ because k is known to be the correct angular frequency for $\epsilon = 0$.) According to the chain rule, we have

$$y'(t) = \frac{d}{dt}Y(\tau) = \frac{d}{d\tau}Y(\tau)\frac{d\tau}{dt} = Y'(\tau)\lambda(\epsilon),$$

and likewise $y''(t) = \lambda(\epsilon)^2 Y''(\tau)$. Our equation becomes

$$\begin{aligned} \lambda(\epsilon)^2 Y'' + \epsilon Y \lambda(\epsilon)^2 (Y')^2 + k^2 Y &= 0 \\ Y(0) &= 1 \\ \lambda(\epsilon)Y'(0) &= 0. \end{aligned}$$

Expanding Y as

$$Y(\tau) = Y_0(\tau) + \epsilon Y_1(\tau) + \dots,$$

we have

$$\begin{aligned} (k + \lambda_1\epsilon + \dots)^2(Y_0 + \epsilon Y_1 + \dots)'' + \epsilon(Y_0 + \epsilon Y_1 + \dots)(k + \lambda_1\epsilon + \dots)^2(Y_0' + \epsilon Y_1' + \dots)^2 \\ + k^2(Y_0 + \epsilon Y_1 + \dots) &= 0 \\ Y_0(0) + \epsilon Y_1(0) + \dots &= 1 \\ (k + \lambda_1\epsilon + \dots)(Y_0'(0) + \epsilon Y_1'(0) + \dots) &= 0. \end{aligned}$$

The ϵ^0 equation is

$$\begin{aligned} k^2 Y_0'' + k^2 Y_0 &= 0 \\ Y_0(0) &= 1 \\ Y_0'(0) &= 0, \end{aligned}$$

with solution $Y_0(\tau) = \cos\tau$. The ϵ equation is

$$\begin{aligned} k^2 Y_1'' + 2k\lambda_1 Y_0'' + Y_0 k^2 (Y_0')^2 + k^2 Y_1 &= 0 \\ Y_1(0) &= 0 \\ Y_1'(0) &= 0. \end{aligned}$$

Substituting $Y_0(\tau) = \cos\tau$, we get

$$Y_1'' + Y_1 = +\frac{2\lambda_1}{k}\cos\tau - \cos\tau\sin^2\tau = \frac{2\lambda_1}{k}\cos\tau - \frac{1}{4}\cos\tau + \frac{1}{4}\cos 3\tau.$$

In order to eliminate secular terms, we require

$$\frac{2\lambda_1}{k} - \frac{1}{4} = 0 \Rightarrow \lambda_1 = \frac{k}{8}.$$

We are left with the equation

$$\begin{aligned} Y_1'' + Y_1 &= \frac{1}{4}\cos 3\tau \\ Y_1(0) &= 0 \\ Y_1'(0) &= 0. \end{aligned}$$

For this equation, particular solutions will have the form

$$Y_{1_p}(\tau) = A\cos 3\tau + B\sin 3\tau,$$

for which we have

$$-9A\cos 3\tau - 9B\sin 3\tau + A\cos 3\tau + B\sin 3\tau = \frac{1}{4}\cos 3\tau.$$

This gives

$$\begin{aligned} \cos 3\tau:& -9A + A = \frac{1}{4} \Rightarrow A = -\frac{1}{32} \\ \sin 3\tau:& -9B + B = 0 \Rightarrow B = 0. \end{aligned}$$

The particular solution is

$$Y_{1_p}(\tau) = -\frac{1}{32}\cos 3\tau,$$

and so the general solution is

$$Y_1(\tau) = C_1\cos\tau + C_2\sin\tau - \frac{1}{32}\cos 3\tau,$$

with

$$Y_1'(\tau) = -C_1\sin\tau + C_2\cos\tau + \frac{3}{32}\sin 3\tau.$$

Our initial conditions for Y_1 give

$$\begin{aligned} C_1 - \frac{1}{32} &= 0 \Rightarrow C_1 = \frac{1}{32} \\ C_2 &= 0. \end{aligned}$$

We conclude that

$$Y_1(\tau) = \frac{1}{32}\cos\tau - \frac{1}{32}\cos 3\tau,$$

so that

$$Y(\tau) = \cos\tau + \epsilon\Big(\frac{1}{32}\cos\tau - \frac{1}{32}\cos 3\tau\Big) + \mathbf{O}(\epsilon^2),$$

uniformly for $|\tau| \le \tau_1$ for some $\tau_1 > 0$. In our original coordinates, we have

$$y(t) = \cos[(k + \frac{k}{8}\epsilon)t] + \epsilon\Big(\frac{1}{32}\cos[(k + \frac{k}{8}\epsilon)t] - \frac{1}{32}\cos[3(k + \frac{k}{8}\epsilon)t]\Big) + \mathbf{O}(\epsilon^2),$$

where the error is $\mathbf{O}(\epsilon^2)$ uniformly for $|t| \le t_1$ for some $t_1 > 0$ and $\mathbf{O}(\epsilon)$ uniformly on the expanding time interval $|t| \le t_1/\epsilon$.

5. [10 pts] For the equation

$$\begin{aligned} y'' + k^2 y &= \epsilon y(1 - (y')^2) \\ y(0) &= 1 \\ y'(0) &= 0, \end{aligned}$$

find the first two terms (in both $y(t;\epsilon)$ and $\lambda(\epsilon)$) for a Poincare expansion of the solution. Explain the expected accuracy or your approximation. (**Note.** This problem is quite similar to Problem 4, and you should feel free to avoid repeating calculations where possible.)

Solution. The initial set-up is precisely the same as in Problem 4, and our ODE in this case is Our equation becomes

$$\begin{aligned} \lambda(\epsilon)^2 Y'' + k^2 Y &= \epsilon Y(1 - \lambda(\epsilon)^2 (Y')^2) \\ Y(0) &= 1 \\ \lambda(\epsilon) Y'(0) &= 0. \end{aligned}$$

Expanding Y as

$$Y(\tau) = Y_0(\tau) + \epsilon Y_1(\tau) + \dots,$$

we have

$$\begin{aligned} (k + \lambda_1\epsilon + \dots)^2 (Y_0 + \epsilon Y_1 + \dots)'' + +k^2(Y_0 + \epsilon Y_1 + \dots) \\ = \epsilon(Y_0 + \epsilon Y_1 + \dots)\Big(1 - (k + \lambda_1\epsilon + \dots)^2 (Y_0' + \epsilon Y_1' + \dots)^2\Big) \\ Y_0(0) + \epsilon Y_1(0) + \dots = 1 \\ (k + \lambda_1\epsilon + \dots)(Y_0'(0) + \epsilon Y_1'(0) + \dots) = 0. \end{aligned}$$

The ϵ^0 equation is

$$\begin{aligned} k^2 Y_0'' + k^2 Y_0 &= 0 \\ Y_0(0) &= 1 \\ Y_0'(0) &= 0, \end{aligned}$$

with solution $Y_0(\tau) = \cos\tau$. The ϵ equation is

$$\begin{aligned} k^2 Y_1'' + k^2 Y_1 &= -2k\lambda_1 Y_0'' + Y_0\Big(1 - k^2 (Y_0')^2\Big) \\ Y_1(0) &= 0 \\ Y_1'(0) &= 0. \end{aligned}$$

Substituting $Y_0(\tau) = \cos\tau$, we get

$$\begin{aligned} Y_1'' + Y_1 &= +\frac{2\lambda_1}{k}\cos\tau + \frac{1}{k^2}\cos\tau(1 - k^2\sin^2\tau) \\ &= \frac{2\lambda_1}{k}\cos\tau + \frac{1}{k^2}\cos\tau - \cos\tau\sin^2\tau = \frac{2\lambda_1}{k}\cos\tau + \frac{1}{k^2}\cos\tau - \frac{1}{4}\cos\tau + \frac{1}{4}\cos 3\tau. \end{aligned}$$

In order to eliminate secular terms, we require

$$\frac{2\lambda_1}{k} + \frac{1}{k^2} - \frac{1}{4} = 0 \Rightarrow \lambda_1 = \frac{k^2 - 4}{8k}.$$

(Notice that if $k = \pm 2$ we have $\lambda_1 = 0$, which simply the means the Poincare frequency will not have a term linear in ϵ.) We are left with the equation

$$\begin{aligned} Y_1'' + Y_1 &= \frac{1}{4}\cos 3\tau \\ Y_1(0) &= 0 \\ Y_1'(0) &= 0, \end{aligned}$$

which is precisely the same as in Problem 4 and has solution

$$Y_1(\tau) = \frac{1}{32}\cos\tau - \frac{1}{32}\cos 3\tau.$$

In this case we conclude

$$Y(\tau) = \cos\tau + \epsilon\Big(\frac{1}{32}\cos\tau - \frac{1}{32}\cos 3\tau\Big) + \mathbf{O}(\epsilon^2),$$

uniformly for $|\tau| \le \tau_1$ for some $\tau_1 > 0$. In our original coordinates, we have

$$y(t) = \cos[(k + \frac{k^2-4}{8k}\epsilon)t] + \epsilon\Big(\frac{1}{32}\cos[(k + \frac{k^2-4}{8k}\epsilon)t] - \frac{1}{32}\cos[3(k + \frac{k^2-4}{8k}\epsilon)t]\Big) + \mathbf{O}(\epsilon^2),$$

where the error is $\mathbf{O}(\epsilon^2)$ uniformly for $|t| \le t_1$ for some $t_1 > 0$ and $\mathbf{O}(\epsilon)$ uniformly on the expanding time interval $|t| \le t_1/\epsilon$.