# M401 Spring 2010, Assignment 5 Solutions

1. [10 pts] We saw in Problem 5 from Assignment 3 that for

$$\begin{aligned} y'' + k^2 y &= \epsilon y^2 \\ y(0) &= 1 \\ y'(0) &= 0, \end{aligned}$$

the function $y_1(t)$ in the perturbation expansion $y(t) = y_0(t) + \epsilon y_1(t) + \dots$ does not have a secular term to remove. It turns out that the function $y_2(t)$ does have a secular term. Find a Poincare expansion for the solution to this equation, with terms $y(t) \approx y_0 + \epsilon y_1$ and $\lambda(\epsilon) \approx k + \lambda_1 \epsilon + \lambda_2 \epsilon^2$. (You will have to write down the $y_2$ equation in order to find $\lambda_2$, but you won't have to solve it.) Be clear about the size of the error for your approximation, and the interval of time for which it is valid. If you combine this calculation with that of Problem 5 from Assignment 3 then you've solved Exercise 4.3 from p. 55 of Simmonds and Mann Jr.

**Solution.** We change variables with $\tau = \lambda(\epsilon)t$ and $Y(\tau) = y(t)$, and in the new variables, our equation becomes

$$\begin{aligned} \lambda(\epsilon)^2 Y'' + k^2 Y &= \epsilon Y^2 \\ Y(0) &= 1 \\ Y'(0) &= 0. \end{aligned}$$

We now expand $\lambda(\epsilon)$ and $Y(\tau)$ as

$$\begin{aligned} \lambda(\epsilon) &= k + \lambda_1 \epsilon + \lambda_2 \epsilon^2 + \dots \\ Y &= Y_0 + \epsilon Y_1 + \epsilon^2 Y_2 \dots. \end{aligned}$$

Our equation becomes

$$\begin{aligned} (k + \lambda_1 \epsilon + \lambda_2 \epsilon^2 + \dots)^2 (Y_0 + \epsilon Y_1 + \epsilon^2 Y_2 + \dots)'' & \\ + k^2 (Y_0 + \epsilon Y_1 + \epsilon^2 Y_2 + \dots) &= \epsilon (Y_0 + \epsilon Y_1 + \epsilon^2 Y_2 + \dots)^2, \\ Y_0(0) + \epsilon Y_1(0) + \epsilon^2 Y_2(0) + \cdots &= 1 \\ Y_0'(0) + \epsilon Y_1'(0) + \epsilon^2 Y_2'(0) + \cdots &= 0. \end{aligned}$$

The $\epsilon^0$ equation is

$$\begin{aligned} k^2 Y_0'' + k^2 Y_0 &= 0 \\ Y_0(0) &= 1 \\ Y_0'(0) &= 0, \end{aligned}$$

for which we have

$$Y_0(\tau) = \cos \tau.$$

The $\epsilon^1$ equation is

$$\begin{aligned} k^2 Y_1'' + 2k\lambda_1 Y_0'' + k^2 Y_1 &= Y_0^2 \\ Y_1(0) &= 0 \\ Y_1'(0) &= 0. \end{aligned}$$

The equation is

$$k^2 Y_1'' + k^2 Y_1 = -2k\lambda_1 Y_0'' + Y_0^2 = 2k\lambda_1 \cos\tau + \cos^2\tau = 2k\lambda_1 \cos\tau + \frac{1}{2} + \frac{1}{2}\cos 2\tau.$$

The only term that will lead to a secular term is $2k\lambda_1 \cos\tau$, so we simply take $\lambda_1 = 0$. We have, then,

$$Y_1'' + Y_1 = \frac{1}{2k^2} + \frac{1}{2k^2}\cos 2\tau,$$

for which particular solutions have the form

$$Y_{1_p}(\tau) = A + B\cos 2\tau + C\sin 2\tau.$$

Substituting this into our equation we obtain

$$-4B\cos 2\tau - 4C\sin 2\tau + A + B\cos 2\tau + C\sin 2\tau = \frac{1}{2k^2} + \frac{1}{2k^2}\cos 2\tau,$$

from which we conclude $A = \frac{1}{2k^2}$, $C = 0$, and $B = -\frac{1}{6k^2}$. We conclude that

$$Y_1(\tau) = C_1 \cos\tau + C_2 \sin\tau + \frac{1}{2k^2} - \frac{1}{6k^2}\cos 2\tau.$$

Our initial conditions give

$$\begin{aligned} C_1 + \frac{1}{2k^2} - \frac{1}{6k^2} &= 0 \Rightarrow C_1 = -\frac{1}{3k^2} \\ C_2 &= 0. \end{aligned}$$

We have

$$Y_1(\tau) = -\frac{1}{3k^2}\cos\tau + \frac{1}{2k^2} - \frac{1}{6k^2}\cos 2\tau.$$

Finally, the $\epsilon^2$ equation is

$$k^2 Y_2'' + 2k\lambda_2 Y_0'' + k^2 Y_2 = 2Y_0 Y_1,$$

so that

$$k^2 Y_2'' + k^2 Y_2 = 2Y_0 Y_1 - 2k\lambda_2 Y_0'' = 2\cos\tau(-\frac{1}{3k^2}\cos\tau + \frac{1}{2k^2} - \frac{1}{6k^2}\cos 2\tau) + 2k\lambda_2 \cos\tau.$$

The right hand side can be re-written as

$$\begin{aligned} &-\frac{2}{3k^2}\cos^2\tau + \frac{1}{k^2}\cos\tau - \frac{1}{3k^2}\cos\tau\cos 2\tau + 2k\lambda_2\cos\tau \\ &\qquad = -\frac{2}{3k^2}(\frac{1}{2} + \frac{1}{2}\cos 2\tau) + \frac{1}{k^2}\cos\tau - \frac{1}{3k^2}\frac{1}{2}(\cos 3\tau + \cos\tau) + 2k\lambda_2\cos\tau. \end{aligned}$$

We eliminate secular terms by choosing $\lambda_2$ so that

$$\frac{1}{k^2} - \frac{1}{6k^2} + 2k\lambda_2 = 0 \Rightarrow \lambda_2 = -\frac{5}{12k^3}.$$

We have then

$$Y(\tau) = \cos\tau + \epsilon\Big( - \frac{1}{3k^2}\cos\tau + \frac{1}{2k^2} - \frac{1}{6k^2}\cos 2\tau\Big) + \mathbf{O}(\epsilon^2),$$

with

$$\tau = (k - \frac{5}{12k^3}\epsilon^2 + \dots)t.$$

Our approximate solution is

$$y(t) = \cos[(k-\frac{5}{12k^2}\epsilon^2)t] + \epsilon\Big(-\frac{1}{3k^2}\cos[(k-\frac{5}{12k^2}\epsilon^2)t] + \frac{1}{2k^2} - \frac{1}{6k^2}\cos 2[(k-\frac{5}{12k^2}\epsilon^2)t]\Big) + \mathbf{O}(\epsilon^2).$$

This error is uniform for $|t| \leq t_1/\epsilon$.

2. [10 pts] Try applying Poincare's method to the van der Pol equation

$$\begin{aligned} y'' + \epsilon y'(y^2 - 1) + y &= 0 \\ y(0) &= 1 \\ y'(0) &= 0, \end{aligned}$$

and explain where it fails and why.

**Solution.** We set $\tau = \lambda(\epsilon)t$ and $Y(\tau) = y(t)$, and in these new variables our equation becomes

$$\begin{aligned} \lambda(\epsilon)^2 Y'' + \epsilon\lambda(\epsilon)Y'(Y^2 - 1) + Y &= 0 \\ Y(0) &= 1 \\ Y'(0) &= 0. \end{aligned}$$

We look for a solution with

$$\begin{aligned} Y &= Y_0 + \epsilon Y_1 + \dots \\ \lambda(\epsilon) &= 1 + \lambda_1\epsilon + \dots, \end{aligned}$$

for which we have

$$\begin{aligned} &(1 + \lambda_1\epsilon + \dots)^2(Y_0 + \epsilon Y_1 + \dots)'' \\ &+\epsilon(1 + \lambda_1\epsilon + \dots)(Y_0 + \epsilon Y_1 + \dots)'((Y_0 + \epsilon Y_1 + \dots)^2 - 1) + Y_0 + \epsilon Y_1 + \dots = 0 \\ &Y_0(0) + \epsilon Y_1(0) + \dots = 1 \\ &Y_0'(0) + \epsilon Y_1'(0) + \dots = 0. \end{aligned}$$

The $\epsilon^0$ equation is

$$\begin{aligned} Y_0'' + Y_0 &= 0 \\ Y_0(0) &= 1 \\ Y_0'(0) &= 0, \end{aligned}$$

with solution $Y_1(\tau) = \cos\tau$. The $\epsilon^1$ equation is

$$Y_1'' + 2\lambda_1 Y_0'' + Y_0'(Y_0^2 - 1) + Y_1 = 0,$$

or

$$\begin{aligned} Y_1'' + Y_1 = & -2\lambda_1 Y_0'' + Y_0'(1 - Y_0^2) = 2\lambda_1 \cos\tau - \sin\tau(1 - \cos^2\tau) \\ & = 2\lambda_1 \cos\tau - \sin^3\tau = 2\lambda_1 \cos\tau - \frac{3}{4}\sin\tau + \frac{1}{4}\sin 3\tau. \end{aligned}$$

Secular terms will arise from both $\cos\tau$ and $\sin\tau$, and we see that there is no way to remove them.

3. [10 pts]

3a. Find all values $a$ for which the difficulty you had with Poincare's method in problem 2 does not arise for the van der Pol equation

$$\begin{aligned} y'' + \epsilon y'(y^2 - 1) + y &= 0 \\ y(0) &= a \\ y'(0) &= 0. \end{aligned}$$

(Notice that the initial condition has changed.)

3b. Choose any positive value of $a$ from your list in (3a) and find the Poincare solution of the van der Pol equation with terms $y(t) \approx y_0 + \epsilon y_1$ and $\lambda(\epsilon) \approx 1 + \lambda_1 \epsilon + \lambda_2 \epsilon^2$.

**Solution.** First, following the calculations for Problem 2 we arrive at the $\epsilon^0$ equation

$$\begin{aligned} Y_0'' + Y_0 &= 0 \\ Y_0(0) &= a \\ Y_0'(0) &= 0, \end{aligned}$$

with solution $Y_0(\tau) = a\cos\tau$. In this way, the $\epsilon^1$ equation is now

$$\begin{aligned} Y_1'' + Y_1 &= 2\lambda_1 a \cos\tau - a\sin\tau(1 - a^2\cos^2\tau) \\ &= 2\lambda_1 \cos\tau - a\sin\tau + a^3\left(\frac{1}{4}\sin\tau + \frac{1}{4}\sin 3\tau\right). \end{aligned}$$

In the event that $-a + \frac{a^3}{4} = 0$, there is no $\sin\tau$ term to remove, and we can simply choose $\lambda_1 = 0$. The possible values for $a$ are clearly $-2, 0, 2$. If we take $a = 2$ (the only positive value), the $Y_1$ equation becomes

$$\begin{aligned} Y_1'' + Y_1 &= 2\sin 3\tau, \\ Y_1(0) &= 0 \\ Y_1'(0) &= 0. \end{aligned}$$

Particular solutions have the form

$$Y_{1_p}(\tau) = A\sin 3\tau + B\cos 3\tau,$$

with

$$Y''_{1_p}(\tau) = -9A\sin 3\tau - 9B\cos 3\tau.$$

This gives

$$-9A\sin 3\tau - 9B\cos 3\tau + A\sin 3\tau + B\cos 3\tau = 2\sin 3\tau,$$

from which we find $A = -\frac{1}{4}$ and $B = 0$. We have, then,

$$Y_1(\tau) = C_1\cos\tau + C_2\sin\tau - \frac{1}{4}\sin 3\tau,$$

with

$$Y_1'(\tau) = -C_1\sin\tau + C_2\cos\tau - \frac{3}{4}\cos 3\tau,$$

so that the initial conditions give

$$\begin{aligned} C_1 &= 0 \\ C_2 - \frac{3}{4} &= 0 \Rightarrow C_2 = \frac{3}{4}. \end{aligned}$$

Finally,

$$Y_1(\tau) = \frac{3}{4}\sin\tau - \frac{1}{4}\sin 3\tau.$$

In order to find a value for $\lambda_2$ we need the $\epsilon^2$ equation. We have (keeping in mind that $\lambda_1 = 0$)

$$\begin{aligned} &(1+\lambda_2\epsilon^2+\dots)^2(Y_0+\epsilon Y_1+\epsilon^2 Y_2\dots)'' \\ &+\epsilon(1+\lambda_2\epsilon^2+\dots)(Y_0+\epsilon Y_1+\epsilon^2 Y^2\dots)'((Y_0+\epsilon Y_1+\epsilon^2 Y_2\dots)^2-1)+Y_0+\epsilon Y_1+\epsilon^2 Y_2\dots = 0. \end{aligned}$$

The $\epsilon^2$ equation is

$$Y_2'' + 2\lambda_2 Y_0'' + Y_1'(Y_0^2 - 1) + Y_0' 2Y_0 Y_1 + Y_2 = 0,$$

which we can re-write as

$$Y_2'' + Y_2 = -2\lambda_2 Y_0'' + Y_1'(1 - Y_0^2) - 2Y_0' Y_0 Y_1.$$

This left hand side is

$$
\begin{aligned}
&4\lambda_2\cos\tau+(\frac{3}{4}\cos\tau-\frac{3}{4}\cos 3\tau)(1-4\cos^2\tau)+8\sin\tau\cos\tau(\frac{3}{4}\sin\tau-\frac{1}{4}\sin 3\tau)\\
&=4\lambda_2\cos\tau+\frac{3}{4}\cos\tau-3\cos^3\tau-\frac{3}{4}\cos 3\tau\\
&\quad+3\cos 3\tau\cos^2\tau+6\cos\tau\sin^2\tau-2\cos\tau\sin\tau\sin 3\tau\\
&=4\lambda_2\cos\tau+\frac{3}{4}\cos\tau-3[\frac{3}{4}\cos\tau+\frac{1}{4}\cos 3\tau]-\frac{3}{4}\cos 3\tau\ +3\cos 3\tau(\frac{1}{2}+\frac{1}{2}\cos 2\tau)\\
&\quad+6[\frac{1}{4}\cos\tau-\frac{1}{4}\cos 3\tau]-\cos\tau[\cos 2\tau-\cos 4\tau]\\
&=4\lambda_2\cos\tau+\frac{3}{2}\cos 3\tau\cos 2\tau-\frac{3}{2}\cos 3\tau-\cos\tau\cos 2\tau+\cos\tau\cos 4\tau\\
&=4\lambda_2\cos\tau+\frac{3}{4}[\cos 5\tau+\cos\tau]-\frac{3}{2}\cos 3\tau\\
&\quad-\frac{1}{2}[\cos 3\tau+\cos\tau]+\frac{1}{2}[\cos 5\tau+\cos 3\tau]\\
&=4\lambda_2\cos\tau+\frac{1}{4}\cos\tau-\frac{3}{2}\cos 3\tau+\frac{5}{4}\cos 5\tau
\end{aligned}
$$

We must choose $\lambda_2$ so that

$$4\lambda_2+\frac{1}{4}=0\Rightarrow\lambda_2=-\frac{1}{16}.$$

We conclude that

$$\lambda(\epsilon)=1-\frac{1}{16}\epsilon^2+\dots,$$

and so

$$
\begin{aligned}
y(t)=&\,2\cos[(1-\frac{1}{16}\epsilon^2)t]\\
&+\epsilon\Big(\frac{3}{4}\sin[(1-\frac{1}{16}\epsilon^2)t]-\frac{1}{4}\sin 3[(1-\frac{1}{16}\epsilon^2)t]\Big)+\mathbf{O}(\epsilon^2),
\end{aligned}
$$

where now the error is uniform for $|t|\le t_1/\epsilon$.

4. [10 pts] Find the first order two-scale approximation for

$$
\begin{aligned}
y''+\epsilon y^4y'+k^2y&=0\\
y(0)&=1\\
y'(0)&=0.
\end{aligned}
$$

Be clear about the size of the error for your approximation, and the interval of time for which it is valid. **Hint.** Use $\cos^4\theta\sin\theta=\frac{1}{8}\sin\theta+\frac{3}{16}\sin 3\theta+\frac{1}{16}\sin 5\theta$.

**Solution.** We begin by defining a slow time $\tau=\epsilon t$ and looking for solutions of the form $Y(t,\tau)=y(t)$. We have

$$
\begin{aligned}
y'(t)&=Y_t+\epsilon Y_\tau\\
y''(t)&=Y_{tt}+2\epsilon Y_{t\tau}+\epsilon^2Y_{\tau\tau},
\end{aligned}
$$

and so our equation becomes

$$\begin{aligned} Y_{tt} + 2\epsilon Y_{t\tau} + \epsilon^2 Y_{\tau\tau} + \epsilon Y^4(Y_t + \epsilon Y_\tau) + k^2 Y &= 0 \\ Y(0,0) &= 1 \\ Y_t(0,0) + \epsilon Y_\tau(0,0) &= 0. \end{aligned}$$

We now look for solutions of the form

$$Y = Y_0 + \epsilon Y_1 + \dots,$$

for which our equation becomes

$$\begin{aligned} (Y_0 + \epsilon Y_1 + \dots)_{tt} + 2\epsilon(Y_0 + \epsilon Y_1 + \dots)_{t\tau} + \epsilon^2(Y_0 + \epsilon Y_1 + \dots)_{\tau\tau} & \\ +\epsilon(Y_0 + \epsilon Y_1 + \dots)^4((Y_0 + \epsilon Y_1 + \dots)_t + \epsilon(Y_0 + \epsilon Y_1 + \dots)_\tau) + k^2(Y_0 + \epsilon Y_1 + \dots) &= 0 \\ Y_0(0,0) + \epsilon Y_1(0,0) + \dots &= 1 \\ Y_{0_t}(0,0) + \epsilon Y_{1_t}(0,0) + \dots + \epsilon(Y_{0_\tau}(0,0) + \epsilon Y_{1_\tau}(0,0) + \dots) &= 0. \end{aligned}$$

The $\epsilon^0$ equation is

$$\begin{aligned} Y_{0_{tt}} + k^2 Y_0 &= 0 \\ Y_0(0,0) &= 1 \\ Y_{0_t}(0,0) &= 0, \end{aligned}$$

for which we have

$$Y_0(t,\tau) = C_1(\tau)\cos kt + C_2(\tau)\sin kt.$$

According to the initial conditions, we have $C_1(0) = 1$ and $C_2(0) = 0$. Since $f(y, y') = -y^4 y'$ is odd in $y'$, we can proceed by taking $C_2(\tau) = 0$ for all $\tau$. We have, then,

$$Y_0(t,\tau) = C_1(\tau)\cos kt,$$

with

$$Y_{0_t}(t,\tau) = -kC_1(\tau)\sin kt,$$

and $C_1(0) = 1$. This is all the information we can glean from the $\epsilon^0$ equation, so we now consider the $\epsilon^1$ equation,

$$Y_{1_{tt}} + 2Y_{0_{t\tau}} + Y_0^4 Y_{0_t} + k^2 Y_1 = 0,$$

where the initial conditions won't play a role since we are only computing $Y_0$. We have

$$Y_{1_{tt}} + k^2 Y_1 = -2Y_{0_{\tau t}} - Y_0^4 Y_{0_t} = 2kC_1'(\tau)\sin kt + kC_1^4\cos^4 kt C_1 \sin kt.$$

At this point, we use the hint, which makes the right hand side

$$2kC_1'\sin kt + kC_1^5(\frac{1}{8}\sin kt + \frac{3}{16}\sin 3kt + \frac{1}{16}\sin 5kt).$$

In order to eliminate secular terms, we choose $C_1(\tau)$ to solve the ODE

$$\begin{aligned}\frac{dC_1}{d\tau} &= -\frac{1}{16}C_1^5\\ C_1(0) &= 1.\end{aligned}$$

We solve this by separation of variables, from which we have

$$C_1(\tau) = \frac{\sqrt{2}}{(4+\tau)^{1/4}}.$$

We conclude that

$$Y_0(t,\tau) = \frac{\sqrt{2}}{(4+\tau)^{1/4}}\cos kt,$$

so that

$$y(t) = \frac{\sqrt{2}}{(4+\epsilon t)^{1/4}}\cos kt + \mathbf{O}(\epsilon),$$

uniformly for $|t| \le t_1/\epsilon$.

5. [10 pts] Find the first order two-scale approximation to the van der Pol equation

$$\begin{aligned}y'' + \epsilon y'(y^2-1) + y &= 0\\ y(0) &= 1\\ y'(0) &= 0.\end{aligned}$$

Be clear about the size of the error for your approximation, and the interval of time for which it is valid. **Hint.** The ODE

$$\begin{aligned}\frac{dC_1}{d\tau} &= \frac{1}{2}C_1 - \frac{1}{8}C_1^3\\ C_1(0) &= 1,\end{aligned}$$

is fairly easy to solve with separation of variables and partial fractions. You can skip this calculation and use the result,

$$C_1(\tau) = \frac{2}{\sqrt{1+3e^{-\tau}}}.$$

**Solution.** We make the same variable definitions as we did for Problem 4, and in this case our equation becomes

$$\begin{aligned}Y_{tt} + 2\epsilon Y_{t\tau} + \epsilon^2 Y_{\tau\tau} + \epsilon(Y_t + \epsilon Y_\tau)(Y^2-1) + Y &= 0\\ Y(0,0) &= 1\\ Y_t(0,0) + \epsilon Y_\tau(0,0) &= 0.\end{aligned}$$

We now look for solutions of the form

$$Y = Y_0 + \epsilon Y_1 + \dots,$$

for which our equation becomes

$$
\begin{aligned}
(Y_0+\epsilon Y_1+\dots)_{tt}+2\epsilon(Y_0+\epsilon Y_1+\dots)_{t\tau}+\epsilon^2(Y_0+\epsilon Y_1+\dots)_{\tau\tau}\\
+\epsilon((Y_0+\epsilon Y_1+\dots)_t+\epsilon(Y_0+\epsilon Y_1+\dots)_\tau)((Y_0+\epsilon Y_1+\dots)^2-1)+Y_0+\epsilon Y_1+\dots=0\\
Y_0(0,0)+\epsilon Y_1(0,0)+\dots=1\\
Y_{0_t}(0,0)+\epsilon Y_{1_\epsilon}(0,0)+\dots+\epsilon(Y_{0_\tau}(0,0)+\epsilon Y_{1_\tau}(0,0)+\dots)=0.
\end{aligned}
$$

The $\epsilon^0$ equation is

$$
\begin{aligned}
Y_{0_{tt}}+Y_0&=0\\
Y_0(0,0)&=1\\
Y_{0_t}(0,0)&=0,
\end{aligned}
$$

for which we have

$$Y_0(t,\tau)=C_1(\tau)\cos t+C_2(\tau)\sin t.$$

According to the initial conditions, we have $C_1(0)=1$ and $C_2(0)=0$. Since $f(y,y')=y'(1-y^2)$ is odd in $y'$, we can proceed by taking $C_2(\tau)=0$ for all $\tau$. We have, then,

$$Y_0(t,\tau)=C_1(\tau)\cos t,$$

with

$$Y_{0_t}(t,\tau)=-C_1(\tau)\sin t,$$

and $C_1(0)=1$. This is all the information we can glean from the $\epsilon^0$ equation, so we now consider the $\epsilon^1$ equation,

$$Y_{1_{tt}}+2Y_{0_{t\tau}}+Y_{0_t}(Y_0^2-1)+Y_1=0,$$

for which we do not need initial conditions to get $Y_0$. We have

$$
\begin{aligned}
Y_{1_{tt}}+Y_1&=-2Y_{0_{t\tau}}+Y_{0_t}(1-Y_0^2)\\
&=2C_1'(\tau)\sin t-C_1\sin t(1-C_1^2\cos^2 t)=2C_1'\sin t-C_1\sin t+C_1^3\sin t\cos^2 t\\
&=2C_1'\sin t-C_1\sin t+C_1^3(\frac{1}{4}\sin t+\frac{1}{4}\sin 3t).
\end{aligned}
$$

In order to eliminate secular terms we choose $C_1$ to solve the ODE

$$
\begin{aligned}
\frac{dC_1}{d\tau}&=\frac{1}{2}C_1-\frac{1}{8}C_1^3\\
C_1(0)&=1.
\end{aligned}
$$

The solution to this equation is given in the hint, and we have

$$C_1(\tau)=\frac{2}{\sqrt{1+3e^{-\tau}}}.$$

We conclude that

$$Y_0(t,\tau)=\frac{2}{\sqrt{1+3e^{-\tau}}}\cos t,$$

and so

$$y(t)=\frac{2}{\sqrt{1+3e^{-\epsilon t}}}\cos t+\mathbf{O}(\epsilon),$$

uniformly for $|t|\le t_1/\epsilon$.