M401 Spring 2010, Assignment 6, due Thursday March 4

Note on Problems 1–3. In approximating solutions of equations of the form

$$y'' + k^2 y = \epsilon f(y, y')$$

with the two-scale method it's often convenient to use *polar form*, especially when $f(y, y')$ is not odd in y'. In Problem 1 of this assignment, we will investigate the difficulties encountered when our usual approach to the two-scale method is taken with a problem of this form. In Problem 2 we will discuss what we mean by polar form, and then in Problem 3 we will use polar form along with the two-scale method to (more successfully) approximate the solution to the problem we considered in Problem 1.

1. [10 pts] Carry out the two-scale method on

$$\begin{aligned} y'' + k^2 y &= \epsilon y^3 \\ y(0) &= 1 \\ y'(0) &= 0, \end{aligned}$$

as far as writing down ODE for C_1 and C_2. You don't have to solve these equations.

2. [10 pts]

2a. We have seen that the general solution of the ODE

$$y'' + k^2 y = 0$$

is

$$y(t) = C_1 \cos kt + C_2 \sin kt.$$

Alternatively, it's clear that the functions

$$y(t) = r \cos(\theta - kt)$$

and

$$y(t) = \tilde{r} \sin(\tilde{\theta} - kt)$$

also solve this equation. (Here, r, θ, $\tilde{r}$, and $\tilde{\theta}$ are all constants.) Use appropriate trigonometric identities to identify C_1 and C_2 in terms of r and θ. (While the version involving the sine function is given for completeness, we won't work with it in this assignement.)

2b. Use the general solution

$$y(t) = r \cos(\theta - kt)$$

to solve the ODE

$$\begin{aligned} y'' + 4y &= 0 \\ y(0) &= 1 \\ y'(0) &= 2. \end{aligned}$$

3. [10 pts] Use polar form to find the first term in a two-scale approximation for the solution of

$$\begin{aligned} y'' + k^2 y &= \epsilon y^3 \\ y(0) &= 1 \\ y'(0) &= 0. \end{aligned}$$

Compare your result with the expansion we found in class for this problem using Poincare's method.

4. [10 pts] In class we non-dimensionalized

$$\begin{aligned} my'' &= -k_1 y - by' \\ y(0) &= 0 \\ y'(0) &= \frac{p_0}{m}, \end{aligned} \tag{1}$$

as

$$\begin{aligned} \epsilon Y'' &= -Y - Y' \\ Y(0) &= 0 \\ Y'(0) &= \frac{1}{\epsilon}. \end{aligned}$$

We then changed variables to get an equation of the form

$$\begin{aligned} x'' &= -\epsilon x - x' \\ x(0) &= 0 \\ x'(0) &= 1. \end{aligned} \tag{2}$$

Show how (2) can be obtained directly from an alternative non-dimensionalization of (1). Your choice of ϵ should be the same as our choice from class. Check that your non-dimensional τ is a fast time.

5. [10 pts] Find the two-term composite expansion for

$$\begin{aligned} \epsilon y'' + (1+t)y' &= 0 \\ y(0) &= 0 \\ y'(0) &= \frac{1}{\epsilon}. \end{aligned}$$

Discuss the size of your error.