M401 Spring 2010, Assignment 6 Solutions

Note on Problems 1–3. In approximating solutions of equations of the form

$$y'' + k^2 y = \epsilon f(y, y')$$

with the two-scale method it's often convenient to use *polar form*, especially when $f(y, y')$ is not odd in y'. In Problem 1 of this assignment, we will investigate the difficulties encountered when our usual approach to the two-scale method is taken with a problem of this form. In Problem 2 we will discuss what we mean by polar form, and then in Problem 3 we will use polar form along with the two-scale method to (more successfully) approximate the solution to the problem we considered in Problem 1.

1. [10 pts] Carry out the two-scale method on

$$\begin{aligned} y'' + k^2 y &= \epsilon y^3 \\ y(0) &= 1 \\ y'(0) &= 0, \end{aligned}$$

as far as writing down ODE for C_1 and C_2. You don't have to solve these equations.

Solution. Setting $\tau = \epsilon t$ and $Y(t, \tau) = y(t)$, we obtain the equation

$$\begin{aligned} Y_{tt} + 2\epsilon Y_{t\tau} + \epsilon^2 Y_{\tau\tau} + k^2 Y &= \epsilon Y^3 \\ Y(0,0) &= 1 \\ Y_t(0,0) + \epsilon Y_\tau(0,0) &= 0. \end{aligned}$$

We now look for solutions

$$Y = Y_0 + \epsilon Y_1 + \dots,$$

for which we have

$$\begin{aligned} (Y_0 + \epsilon Y_1 + \dots)_{tt} + 2\epsilon(Y_0 + \epsilon Y_1 + \dots)_{t\tau} + \epsilon^2(Y_0 + \epsilon Y_1 + \dots)_{\tau\tau} & \\ + k^2(Y_0 + \epsilon Y_1 + \dots) &= \epsilon(Y_0 + \epsilon Y_1 + \dots)^3 \\ Y_0(0,0) + \epsilon Y_1(0,0) + \dots &= 1 \\ Y_{0_t}(0,0) + \epsilon Y_{1_t}(0,0) + \dots + \epsilon(Y_{0_\tau}(0,0) + \epsilon Y_{1_\tau}(0,0) + \dots) &= 0. \end{aligned}$$

The ϵ^0 equation is

$$\begin{aligned} Y_{0_{tt}} + k^2 Y_0 &= 0 \\ Y_0(0,0) &= 1 \\ Y_{0_t}(0,0) &= 0, \end{aligned}$$

from which we conclude

$$Y_0(t, \tau) = C_1(\tau) \cos kt + C_2(\tau) \sin kt.$$

From the initial conditions we find that $C_1(0) = 1$ and $C_2(0) = 0$. It's important to observe, however, that since $f(y, y') = y^3$ is not odd in y', we cannot take $C_2(\tau) = 0$ for all τ. The ϵ^1 equation is

$$Y_{1_{tt}} + 2Y_{0_{t\tau}} + k^2 Y_1 = Y_0^3,$$

where

$$\begin{aligned}
Y_0^3 &= (C_1(\tau)\cos kt + C_2(\tau)\sin kt)^3 \\
&= C_1^3\cos^3 kt + 3C_1^2C_2\cos^2 kt\sin kt + 3C_1C_2^2\cos kt\sin^2 kt + C_2^3\sin^3 kt \\
&= C_1^3(\frac{3}{4}\cos kt + \frac{1}{4}\cos 3kt) + 3C_1^2C_2(\frac{1}{4}\sin kt + \frac{1}{4}\sin 3kt) \\
&+ 3C_1C_2^2(\frac{1}{4}\cos kt - \frac{1}{4}\cos 3kt) + C_2^3(\frac{3}{4}\sin kt - \frac{1}{4}\sin 3kt),
\end{aligned}$$

and likewise

$$2Y_{0_{t\tau}} = -2C_1'(\tau)k\sin kt + 2C_2'(\tau)k\cos kt.$$

In order to eliminate secular terms we require

$$\begin{aligned}
2C_1'k + \frac{3}{4}C_1^2C_2 + \frac{3}{4}C_2^3 &= 0 \\
-2C_2'k + \frac{3}{4}C_1^3 + \frac{3}{4}C_1C_2^2 &= 0.
\end{aligned}$$

I.e., we have the system of nonlinear ODE

$$\begin{aligned}
\frac{dC_1}{dt} &= -\frac{3}{8k}C_1^2C_2 - \frac{3}{8k}C_2^3; \quad C_1(0) = 1 \\
\frac{dC_2}{dt} &= \frac{3}{8k}C_1^3 + \frac{3}{8k}C_1C_2^2; \quad C_2(0) = 0.
\end{aligned}$$

While it is possible to solve this system, it requires a clever trick, and there are no general systematic methods for solving such systems.

2. [10 pts]

2a. We have seen that the general solution of the ODE

$$y'' + k^2 y = 0$$

is

$$y(t) = C_1\cos kt + C_2\sin kt.$$

Alternatively, it's clear that the functions

$$y(t) = r\cos(\theta - kt)$$

and

$$y(t) = \tilde{r}\sin(\tilde{\theta} - kt)$$

also solve this equation. (Here, r, θ, $\tilde{r}$, and $\tilde{\theta}$ are all constants.) Use appropriate trigonometric identities to identify C_1 and C_2 in terms of r and θ. (While the version involving the sine function is given for completeness, we won't work with it in this assignement.)

Solution. We write

$$r\cos(\theta - kt) = r(\cos\theta\cos kt - \sin\theta\sin(-kt)) = r\cos\theta\cos kt + r\sin\theta\sin kt,$$

so

$$\begin{aligned} C_1 &= r\cos\theta \\ C_2 &= r\sin\theta. \end{aligned}$$

That is, (r, θ) is the polar designation of the cartesian coordinate (C_1, C_2).

2b. Use the general solution

$$y(t) = r\cos(\theta - kt)$$

to solve the ODE

$$\begin{aligned} y'' + 4y &= 0 \\ y(0) &= 1 \\ y'(0) &= 2. \end{aligned}$$

Solution. We already have that the general solution is

$$y(t) = r\cos(\theta - 2t),$$

so we only need to find values for r and θ from the initial conditions. Since

$$y'(t) = 2r\sin(\theta - 2t),$$

we have

$$\begin{aligned} r\cos\theta &= 1 \\ 2r\sin\theta &= 2. \end{aligned}$$

If we divide the second equation by 2 and sum the squares we obtain $r^2 = 2$, so that $r = \sqrt{2}$. Likewise,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = 1 \Rightarrow \theta = \frac{\pi}{4}.$$

We conclude

$$y(t) = \sqrt{2}\cos(\frac{\pi}{4} - 2t).$$

Note. While $\frac{\pi}{4}$ can be replaced with $\frac{\pi}{4} \pm n\pi$ for any $n = 1, 2, 3, \ldots$, and $k = -2$ can be used in place of $k = 2$, these choices all give the same solution as the one above.

3. [10 pts] Use polar form to find the first term in a two-scale approximation for the solution of

$$\begin{aligned} y'' + k^2 y &= \epsilon y^3 \\ y(0) &= 1 \\ y'(0) &= 0. \end{aligned}$$

Compare your result with the expansion we found in class for this problem using Poincare's method.

Solution. We can use the calculations in Problem 1, except now we write

$$Y_0(t,\tau) = r(\tau)\cos(\theta(\tau) - kt),$$

with

$$Y_{0_t}(t,\tau) = kr(\tau)\sin(\theta(\tau) - kt),$$

and because we'll need it below,

$$Y_{0_{t\tau}}(t,\tau) = kr'(\tau)\sin(\theta(\tau) - kt) + kr(\tau)\theta'(\tau)\cos(\theta(\tau) - kt).$$

The initial conditions give

$$\begin{aligned} r(0)\cos\theta(0) &= 1 \\ r(0)k\sin\theta(0) &= 0. \end{aligned}$$

The second condition is satisfied by $\theta(0) = 0$ (and integer multiples of π), and this choice gives $r(0) = 1$ from the first equation. Notice that even integer multiples of π give precisely the same thing, by periodicity, and odd multiples of π would give $r(0) = -1$, which simply corresponds with a phase shift by π and again gives the same solution. (Though typically we avoid negative values of r, since we are interpreting r as a polar length.) In this case the ϵ^1 equation is

$$\begin{aligned} Y_{1_{tt}} + k^2 Y_1 &= r(\tau)^3\cos^3(\theta(\tau) - kt) - 2kr'(\tau)\sin(\theta(\tau) - kt) - 2kr(\tau)\theta'(\tau)\cos(\theta(\tau) - kt) \\ &= r(\tau)^3\left(\frac{3}{4}\cos(\theta(\tau) - kt) + \frac{1}{4}\cos(3(\theta(\tau) - kt))\right) \\ &\quad - 2kr'(\tau)\sin(\theta(\tau) - kt) - 2kr(\tau)\theta'(\tau)\cos(\theta(\tau) - kt). \end{aligned}$$

In order to remove the secular terms involving $\sin(\theta(\tau) - kt)$ we require $r'(t) = 0$ for all t, so $r(t) = 1$ for all t. In order to remove the secular terms involving $\cos(\theta(\tau) - kt)$ we must choose $\theta(\tau)$ so that (keeping in mind that we have already taken $r(\tau) = 1$)

$$\frac{3}{4} - 2k\theta'(\tau) = 0,$$

with also $\theta(0) = 0$. This means

$$\theta(\tau) = \frac{3}{8k}\tau.$$

We have, then,

$$Y_0(t,\tau) = \cos(\frac{3}{8k}\tau - kt),$$

and so

$$y(t) = \cos(\frac{3}{8k}\epsilon t - kt) + \mathbf{O}(\epsilon),$$

where the error is uniform on $|t| \leq t_1/\epsilon$. This is precisely the y_0 approximation we obtained with Poincare's method.

4. [10 pts] In class we non-dimensionalized

$$\begin{aligned} my'' =& -k_1 y - by' \\ y(0) =& \, 0 \\ y'(0) =& \frac{p_0}{m}, \end{aligned} \tag{1}$$

as

$$\begin{aligned} \epsilon Y'' =& -Y - Y' \\ Y(0) =& \, 0 \\ Y'(0) =& \frac{1}{\epsilon}. \end{aligned}$$

We then changed variables to get an equation of the form

$$\begin{aligned} x'' =& -\epsilon x - x' \\ x(0) =& \, 0 \\ x'(0) =& \, 1. \end{aligned} \tag{2}$$

Show how (2) can be obtained directly from an alternative non-dimensionalization of (1). Your choice of ϵ should be the same as our choice from class. Check that your non-dimensional τ is a fast time.

Solution. We re-scale as usual with

$$\tau = \frac{t}{A}; \quad Y(\tau) = \frac{y(t)}{B},$$

where τ and Y are of course re-defined from the choices we made in class. The equation becomes

$$\begin{aligned} m\frac{B}{A^2}Y'' =& -k_1 BY - b\frac{B}{A}Y' \\ Y(0) =& \, 0 \\ \frac{B}{A}Y'(0) =& \frac{p_0}{m}. \end{aligned}$$

In order to obtain the specified form, we proceed as usual, dividing by the coefficient of Y''. We obtain

$$Y'' = -\frac{k_1 A^2}{m}Y - \frac{bA}{m}Y'.$$

We set the coefficient of Y' to 1 by setting

$$A = \frac{m}{b},$$

which gives

$$\epsilon = \frac{k_1}{m} \cdot \frac{m^2}{b^2} = \frac{k_1 m}{b^2},$$

as in class. Finally, we have

$$Y'(0) = \frac{p_0}{m} \cdot \frac{m}{b}\frac{1}{B},$$

so that our choice of B is

$$B = \frac{p_0}{m}.$$

This gives the specified form,

$$\begin{aligned} Y'' &= -\epsilon Y - Y' \\ Y(0) &= 0 \\ Y'(0) &= 1. \end{aligned}$$

Notice that $A = \frac{m}{b} = \frac{b}{k_1}\epsilon$, so this τ is a fast time scale.

5. [10 pts] Find the two-term composite expansion for

$$\begin{aligned} \epsilon y'' + (1+t)y' &= 0 \\ y(0) &= 0 \\ y'(0) &= \frac{1}{\epsilon}. \end{aligned}$$

Discuss the size of your error.

Solution. We begin by looking for an outer solution of the form

$$y(t) = y_0(t) + \epsilon y_1(t) + \dots,$$

for which our equation becomes

$$\epsilon(y_0(t) + \epsilon y_1(t) + \dots)'' + (1+t)(y_0(t) + \epsilon y_1(t) + \dots)' = 0,$$

with no initial conditions. The ϵ^0 equation is

$$(1+t)y_0' = 0,$$

and we conclude

$$y_0(t) = C_1,$$

for some constant C_1. Likewise, the ϵ^1 equation is

$$y_0'' + (1+t)y_1' = 0,$$

and we conclude

$$y_1(t) = C_2$$

for some constant C_2. In this way the outer solution is

$$y(t) = C_1 + C_2\epsilon + \dots.$$

We now turn to the inner solution, for which we define the fast time $\tau = \frac{t}{\epsilon}$. Our equation becomes

$$\begin{aligned} Y'' + (1 + \epsilon\tau)Y' &= 0 \\ Y(0) &= 0 \\ Y'(0) &= 1. \end{aligned}$$

We look for solutions of the form

$$Y = Y_0 + \epsilon Y_1 + \dots,$$

for which our equation becomes

$$\begin{aligned} (Y_0 + \epsilon Y_1 + \dots)'' + (1 + \epsilon\tau)(Y_0 + \epsilon Y_1 + \dots)' &= 0 \\ Y_0(0) + \epsilon Y_1(0) + \dots &= 0 \\ Y_0'(0) + \epsilon Y_1'(0) + \dots &= 1. \end{aligned}$$

The ϵ^0 equation is

$$\begin{aligned} Y_0'' + Y_0' &= 0 \\ Y_0(0) &= 0 \\ Y_0'(0) &= 1. \end{aligned}$$

The auxiliary equation for this problem is $r^2 + r = 0$ so that $r = 0, -1$, and the general solution is

$$Y_0(\tau) = C_3 + C_4 e^{-\tau},$$

with

$$Y_0'(\tau) = -C_4 e^{-\tau}.$$

The initial conditions give

$$\begin{aligned} C_3 + C_4 &= 0 \\ -C_4 &= 1 \Rightarrow C_4 = -1, \quad C_3 = 1. \end{aligned}$$

This gives

$$Y_0(\tau) = 1 - e^{-\tau}.$$

The ϵ^1 equation is

$$\begin{aligned} Y_1'' + \tau Y_0' + Y_1' &= 0 \\ Y_1(0) &= 0 \\ Y_1'(0) &= 0. \end{aligned}$$

The equation is

$$Y_1'' + Y_1' = -\tau Y_0' = -\tau e^{-\tau}.$$

We look for particular solutions of the form

$$Y_{1_p}(\tau) = A\tau^2 e^{-\tau} + B\tau e^{-\tau}.$$

(Alternatively, we could solve a first order ODE for the variable Y_1', in which case we could simply use an integrating factor.) We have

$$Y_{1_p}'(\tau) = 2A\tau e^{-\tau} - A\tau^2 e^{-\tau} + Be^{-\tau} - B\tau e^{-\tau},$$

and

$$Y_{1_p}''(\tau) = 2Ae^{-\tau} - 4A\tau e^{-\tau} + A\tau^2 e^{-\tau} - 2Be^{-\tau} + B\tau e^{-\tau}.$$

The equation gives

$$\begin{aligned} &2Ae^{-\tau} - 4A\tau e^{-\tau} + A\tau^2 e^{-\tau} - 2Be^{-\tau} + B\tau e^{-\tau} \\ &\quad +2A\tau e^{-\tau} - A\tau^2 e^{-\tau} + Be^{-\tau} - B\tau e^{-\tau} = -\tau e^{-\tau}. \end{aligned}$$

We have:

$$\begin{aligned} e^{-\tau} &: 2A - B = 0 \\ \tau e^{-\tau} &: -2A = -1 \Rightarrow A = \frac{1}{2} \Rightarrow B = 1. \end{aligned}$$

That is,

$$Y_{1_p}(\tau) = \frac{1}{2}\tau^2 e^{-\tau} + \tau e^{-\tau}.$$

The general solution is

$$Y_1(\tau) = C_3 + C_4 e^{-\tau} + \frac{1}{2}\tau^2 e^{-\tau} + \tau e^{-\tau},$$

with

$$Y_1'(\tau) = -C_4 e^{-\tau} - \frac{1}{2}\tau^2 e^{-\tau} + e^{-\tau}.$$

The initial conditions give

$$\begin{aligned} &C_3 + C_4 = 0 \\ &-C_4 + 1 = 0 \Rightarrow C_4 = 1 \Rightarrow C_3 = -1. \end{aligned}$$

We have

$$Y_1(\tau) = -1 + e^{-\tau} + \frac{1}{2}\tau^2 e^{-\tau} + \tau e^{-\tau},$$

which gives the inner approximation

$$Y(\tau) = 1 - e^{-\tau} + \epsilon\Big(-1 + e^{-\tau} + \frac{1}{2}\tau^2 e^{-\tau} + \tau e^{-\tau}\Big) + \ldots.$$

For the matching step, the outer solution is already trivially expanded, but we need to write the inner solution in the outer variable and expand in ϵ. We have

$$\begin{aligned} Y &= 1 - e^{-\frac{t}{\epsilon}} + \epsilon\Big(-1 + e^{-\frac{t}{\epsilon}} + \frac{1}{2}\frac{t^2}{\epsilon^2} e^{-\frac{t}{\epsilon}} + \frac{t}{\epsilon} e^{-\frac{t}{\epsilon}}\Big) \\ &= 1 - \epsilon + \ldots, \end{aligned}$$

where we have taken advantage of the observation that the terms $e^{-\frac{t}{\epsilon}}$ are transcendentally small in ϵ. Matching now, we see that $C_1 = 1$ and $C_2 = -1$, so that

$$\begin{aligned} y^c(t) &= 1 - \epsilon + 1 - e^{-\frac{t}{\epsilon}} + \epsilon\Big(-1 + e^{-\frac{t}{\epsilon}} + \frac{1}{2}\frac{t^2}{\epsilon^2} e^{-\frac{t}{\epsilon}} + \frac{t}{\epsilon} e^{-\frac{t}{\epsilon}}\Big) - (1 - \epsilon) \\ &= 1 - e^{-\frac{t}{\epsilon}} + \epsilon\Big(-1 + e^{-\frac{t}{\epsilon}} + \frac{1}{2}\frac{t^2}{\epsilon^2} e^{-\frac{t}{\epsilon}} + \frac{t}{\epsilon} e^{-\frac{t}{\epsilon}}\Big). \end{aligned}$$

We have a theorem from class that guarantees that this approximation is $\mathbf{O}(\epsilon^2)$ uniformly on any finite time interval $[0, T]$.