

M401 Spring 2010, Assignment 7 Solutions

1a. [5 pts] Non-dimensionalize the wave equation

$$\begin{aligned}u_{tt} &= c^2 u_{xx}; & (x, t) &\in (0, L) \times (0, \infty) \\u(0, t) &= U_1; & u(L, t) &= U_2; & t > 0 \\u(x, 0) &= f(x); & x &\in [0, L] \\u_t(x, 0) &= g(x), & x &\in [0, L].\end{aligned}$$

Solution. We set

$$\tau = \frac{t}{A}, \quad \xi = \frac{x}{B}, \quad v(\xi, \tau) = \frac{u(x, t)}{D},$$

where

$$[A] = T; \quad [B] = L; \quad [D] = L.$$

The equation becomes

$$\begin{aligned}\frac{D}{A^2} v_{\tau\tau} &= c^2 \frac{D}{B^2} v_{\xi\xi}; & (\xi, \tau) &\in (0, \frac{L}{B}) \times (0, \infty) \\Dv(0, \tau) &= U_1; & Dv(\frac{L}{B}, \tau) &= U_2; & \tau > 0 \\Dv(\xi, 0) &= f(B\xi); & \xi &\in [0, \frac{L}{B}] \\ \frac{D}{A} v_\tau(\xi, 0) &= g(B\xi); & \xi &\in [0, \frac{L}{B}].\end{aligned}$$

We choose A , B , and D so that

$$c^2 \frac{A^2}{B^2} = 1; \quad D = U_1; \quad \frac{L}{B} = 1.$$

This gives

$$A = \frac{L}{c}; \quad B = L; \quad D = U_1,$$

and the equation becomes

$$\begin{aligned}v_{\tau\tau} &= v_{\xi\xi}; & (\xi, \tau) &\in (0, 1) \times (0, \infty) \\v(0, \tau) &= 1; & v(1, \tau) &= \frac{U_2}{U_1}; & \tau > 0 \\v(\xi, 0) &= \frac{f(\xi L)}{U_1}; & \xi &\in [0, 1] \\v_\tau(\xi, 0) &= \frac{L}{cU_1} g(\xi L); & \xi &\in [0, 1].\end{aligned}$$

1b. [5 pts] Non-dimensionalize the Navier-Stokes momentum equation

$$\begin{aligned}\rho(u_t + uu_x) &= \mu u_{xx} - p_x + f(x, t); & (x, t) &\in (0, L) \times (0, \infty) \\u(0, t) &= U_1; & u(L, t) &= U_2; & t > 0 \\u(x, 0) &= g(x); & x &\in [0, L].\end{aligned}$$

Take both f and p to be given functions. (Though p is typically an unknown, and this equation is typically coupled with the continuity equation to give a system of two equations for the unknowns u and p .) Begin by specifying the dimensions of the viscosity coefficient μ . Your non-dimensionalized equation should have the form

$$v_\tau + vv_\xi = \frac{1}{R}v_{\xi\xi} - \phi_\xi + g(\xi, \tau),$$

where R denotes the *Reynold's number*

$$R = \frac{LU_1\rho}{\mu}.$$

Note. The Reynold's number is an important dimensionless constant in the theory of fluids. In the case of three space dimensions, the values L and U_1 are typically chosen to correspond with the geometry of the particular problem.

Solution. First, $[\rho]u_t = ML^{-2}T^{-2}$ and $[\mu u_{xx}] = [\mu]L^{-1}T^{-1}$, so

$$[\mu] = ML^{-1}T^{-1}.$$

Now, we set

$$\tau = \frac{t}{A}, \quad \xi = \frac{x}{B}, \quad v(\xi, \tau) = \frac{u(x, t)}{D}, \quad \phi(\xi, \tau) = \frac{p(x, t)}{E}$$

where

$$[A] = T; \quad [B] = L; \quad [D] = LT^{-1}; \quad [E] = ML^{-1}T^{-2}.$$

The equation becomes

$$\begin{aligned} \rho\left(\frac{D}{A}v_\tau + \frac{D^2}{B}vv_\xi\right) &= \mu\frac{D}{B^2}v_{\xi\xi} - \frac{E}{B}\phi_\xi + f(B\xi, \tau A); \quad (\xi, \tau) \in \left(0, \frac{L}{B}\right) \times (0, \infty) \\ Dv(0, \tau) &= U_1; \quad Dv\left(\frac{L}{B}, t\right) = U_2; \quad t > 0 \\ Dv(\xi, 0) &= g(B\xi); \quad \xi \in \left[0, \frac{L}{B}\right]. \end{aligned}$$

In order to make each term dimensionless, we divide by $\rho\frac{D}{A}$, which puts the equation in the form

$$v_\tau + \frac{DA}{B}vv_\xi = \mu\frac{1}{B^2}\frac{A}{\rho}v_{\xi\xi} - \frac{E}{B}\frac{A}{D\rho}\phi_\xi + \frac{A}{D\rho}f(B\xi, \tau A).$$

We now choose A , B , D , and E so that

$$\frac{DA}{B} = 1; \quad \frac{EA}{BD\rho} = 1; \quad D = U_1; \quad B = L.$$

We conclude

$$A = \frac{L}{U_1}; \quad B = L; \quad D = U_1; \quad E = LU_1\rho\frac{U_1}{L} = \rho U_1^2,$$

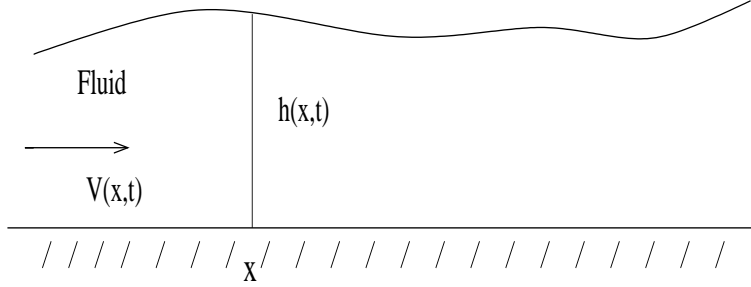


Figure 1: Flow of a thin film.

and our equation takes the form

$$\begin{aligned}
 v_\tau + vv_\xi &= \frac{1}{R}v_{\xi\xi} - \phi_\xi + g(\xi, \tau) \\
 v(0, \tau) &= 1; \quad v(1, t) = \frac{U_2}{U_1}; \quad t > 0 \\
 v(\xi, 0) &= \frac{g(L\xi)}{U_1}; \quad \xi \in [0, 1],
 \end{aligned}$$

where the Reynold's number is

$$R = \frac{LU_1\rho}{\mu},$$

and

$$g(\xi, \tau) = \frac{L}{U_1^2\rho}f\left(L\xi, \frac{L\tau}{U_1}\right).$$

2. The coating of surfaces by thin fluid films is a critical industrial process that arises in applications such as the protection of microchips, de-icing of airplane wings, and the construction of photographic film. Consider, for example, painting a wall. What you would like to do is simply brush a single thick line of paint across the top of the wall and let the paint descend in a steady sheet to the floor. Unfortunately, in most circumstances the paint drips down in *fingers*, and the wall is not smoothly covered. Through the use of mathematical modeling, we can create laboratory situations in which the paint descends as a steady wall. In this problem, we will take the first step toward such a model by deriving a partial differential equation for the height $h(x, t)$ of a thin film moving along a surface (see Figure 1). Assuming the film has constant density ρ , that it is moving with velocity $V(x, t)$ in the x -direction only, and that it is uniform in the y -direction (same height and velocity for all y over a steady width $y \in [0, L]$), derive a partial differential equation that takes a given $V(x, t)$ and describes the height $h(x, t)$. Discuss the types of initial values and boundary values your model would require, and what they mean physically.

Solution. The idea here is to proceed almost precisely as we did when deriving the continuity equation in class. First, the mass of fluid between a position x and a nearby position $x + \Delta x$ is approximately

$$m(t) = \text{Volume} \cdot \rho = (Lh(x, t)\Delta x)\rho.$$

On the other hand, the amount of fluid flowing out of this region at location $x + \Delta x$ during a time period Δt is approximately

$$m_o = (Lh(x + \Delta x, t)V(x + \Delta x, t)\Delta t)\rho,$$

while the amount flowing in at location x is

$$m_i = (Lh(x, t)V(x, t)\Delta t)\rho.$$

The change in mass over the time period $[t, t + \Delta t]$ satisfies

$$\begin{aligned} (Lh(x, t + \Delta t)\Delta x)\rho - (Lh(x, t)\Delta x)\rho &= m_i - m_o \\ &= (Lh(x, t)V(x, t)\Delta t)\rho - (Lh(x + \Delta x, t)V(x + \Delta x, t)\Delta t)\rho. \end{aligned}$$

We divide by $L\rho\Delta x\Delta t$ and take a limit as Δx and Δt go to 0 to obtain

$$h_t = -(Vh)_x.$$

I.e., the equation is

$$h_t + (Vh)_x = 0.$$

As with the continuity equation, this is typically coupled with the Navier-Stokes momentum equation for V .

3. Constanda Exercise 1, Parts (i) and (ii).

Solution to (i). We set $U(t) = u(x(t), t)$, so that $\frac{dU}{dt} = u_x \frac{dx}{dt} + u_t$, and consider the ODE system

$$\begin{aligned} \frac{dx}{dt} &= 2; & x(0) &= x_0 \\ \frac{dU}{dt} &= t; & U(0) &= 1 - x_0, \end{aligned}$$

so that $x = 2t + x_0$ and $U(t) = \frac{t^2}{2} + 1 - x_0$. We conclude that

$$u(x, t) = \frac{t^2}{2} + 1 - (x - 2t).$$

Solution to (ii). In the same notation as in Part (i) we have

$$\begin{aligned} \frac{dx}{dt} &= -3t^2; & x(0) &= x_0 \\ \frac{dU}{dt} &= 2; & U(0) &= x_0^3, \end{aligned}$$

so that $x = -t^3 + x_0$ and $U(t) = 2t + x_0^3$. We conclude

$$u(x, t) = 2t + (x + t^3)^3.$$

4. Constanda Exercise 1, Parts (iii) and (iv).

Solution to (iii). In this case we have

$$\begin{aligned}\frac{dx}{dt} &= 2t + 1; & x(0) &= x_0 \\ \frac{dU}{dt} &= 2U; & U(0) &= \sin x_0,\end{aligned}$$

so that $x = t^2 + t + x_0$ and $U(t) = \sin x_0 e^{2t}$. We conclude

$$u(x, t) = e^{2t} \sin(x - t^2 - t).$$

Solution to (iv). In this case we have

$$\begin{aligned}\frac{dx}{dt} &= -3; & x(0) &= x_0 \\ \frac{dU}{dt} &= 1 - x(t); & U(0) &= x_0^2 + 1,\end{aligned}$$

so that $x = -3t + x_0$ and the second equation is

$$\frac{dU}{dt} = 1 - (-3t + x_0).$$

Integrating, we have

$$U(t) = \frac{3}{2}t^2 + (1 - x_0)t + x_0^2 + 1,$$

and we conclude

$$\begin{aligned}u(x, t) &= \frac{3}{2}t^2 + (1 - x - 3t)t + (x + 3t)^2 + 1 \\ &= \frac{15}{2}t^2 + t + 1 + 5xt + x^2.\end{aligned}$$

5. Constanda Exercise 2, Parts (i) and (ii).

Solution to (i). We begin by putting the equation in our usual form, by dividing by 2 to get

$$u_t + \frac{1}{2}u_x = u,$$

Now, we set

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{2}; & x(0) &= x_0 \\ \frac{dU}{dt} &= U; & U(0) &= e^{x_0},\end{aligned}$$

for which $x = \frac{1}{2}t + x_0$. We see that a characteristic can only be traced back to the positive x -axis if $(x_0 \geq 0) \ x - \frac{1}{2}t \geq 0$. For these values we conclude

$$U(t) = e^{x_0} e^t,$$

and so

$$u(x, t) = e^{(x-\frac{1}{2}t)}e^t = e^{x+\frac{1}{2}t}, \quad x \geq \frac{1}{2}t.$$

For $x < \frac{1}{2}t$, we have

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{2}; & x(t_0) &= 0 \\ \frac{dU}{dt} &= U; & U(t_0) &= 1, \end{aligned}$$

for which $x = \frac{1}{2}(t - t_0)$ and $U(t) = e^{-t_0}e^t$. We conclude

$$u(x, t) = e^{-(t-2x)}e^t = e^{2x}, \quad x < \frac{1}{2}t.$$

Notice that along the line $x = 2t$ the two solutions agree, so it doesn't matter which inequality we put the equality on. (Constanda makes the choice opposite to mine.) More precisely, since Constanda never specifies the boundary value $u(0, 0)$ the solution isn't specified along this line. We assume $u(0, 0)$ takes the expected value of 1. We write the solution as

$$u(x, t) = \begin{cases} e^{x+\frac{1}{2}t}, & x \geq \frac{1}{2}t \\ e^{2x}, & x < \frac{1}{2}t \end{cases}.$$

Solution to (ii). Set

$$\begin{aligned} \frac{dx}{dt} &= 2; & x(0) &= x_0 \\ \frac{dU}{dt} &= x(t)^2; & U(0) &= x_0, \end{aligned}$$

for which $x = 2t + x_0$, and we can trace characteristics back to the positive x -axis so long as $x - 2t \geq 0$. The second equation becomes

$$\frac{dU}{dt} = (2t + x_0)^2; \quad U(0) = x_0,$$

and we integrate to obtain

$$U(t) = \frac{1}{6}(2t + x_0)^3 + x_0 - \frac{1}{6}x_0^3.$$

We conclude that for $x \geq 2t$ we have

$$\begin{aligned} u(x, t) &= \frac{1}{6}(2t + x - 2t)^3 + (x - 2t) - \frac{1}{6}(x - 2t)^3 \\ &= \frac{1}{6}x^3 + (x - 2t) - \frac{1}{6}(x - 2t)^3. \end{aligned}$$

For $x < 2t$, we set

$$\begin{aligned}\frac{dx}{dt} &= 2; & x(t_0) &= 0 \\ \frac{dU}{dt} &= x(t)^2; & U(t_0) &= t_0^2,\end{aligned}$$

from which we have $x = 2(t - t_0)$, and the second equation becomes

$$\frac{dU}{dt} = 4(t - t_0)^2; \quad U(t_0) = t_0^2.$$

Integrating, we have

$$U(t) = \frac{4}{3}(t - t_0)^3 + t_0^2.$$

We conclude that for $x < 2t$

$$u(x, t) = \frac{1}{6}x^3 + \left(t - \frac{x}{2}\right)^2 = \frac{1}{6}x^3 + \frac{1}{4}(2t - x)^2.$$

We write the full solution as

$$u(x, t) = \begin{cases} \frac{1}{6}x^3 + (x - 2t) - \frac{1}{6}(x - 2t)^3, & x \geq 2t \\ \frac{1}{6}x^3 + \frac{1}{4}(2t - x)^2, & x < 2t. \end{cases}$$