M401 Spring 2010, Assignment 8 Solutions

1. [10 pts] Constanda Exercise 12.3, Parts (i) and (ii).

Solution to Part (i). In order to be consistent with our calculations from class I'll write the equation as

$$u_x + \frac{1}{2}u_y = \frac{1}{2}u.$$

Now, set $U(x) = u(x, y(x))$, so that $\frac{dU}{dx} = u_x + u_y\frac{dy}{dx}$, and we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2}; \quad y(x_0) = x_0 - 1\\ \frac{dU}{dx} &= \frac{1}{2}U; \quad U(x_0) = 2x_0 - y_0 = x_0 + 1.\end{aligned}$$

The first equation gives the characteristic lines $y = \frac{1}{2}x + C$, and $y(x_0) = x_0 - 1$ implies that $C = \frac{1}{2}x_0 - 1$. I.e., $y = \frac{1}{2}x + \frac{1}{2}x_0 - 1$, so that $x_0 = 2y - x + 2$. From the second equation, we have

$$U(x) = Ce^{\frac{1}{2}x}; \quad U(x_0) = x_0 + 1 \Rightarrow U(x) = (x_0 + 1)e^{-\frac{1}{2}x_0}e^{\frac{1}{2}x}.$$

We conclude

$$u(x, y) = (2y - x + 3)e^{x-y-1}.$$

Solution to Part (ii). Proceeding as in Part (i) (though without re-scaling), we set

$$\begin{aligned}\frac{dy}{dx} &= -1; \quad y(x_0) = \frac{1}{2}(1 - x_0)\\ \frac{dU}{dx} &= -2y(x); \quad U(x_0) = x_0y_0 = \frac{1}{2}x_0(1 - x_0).\end{aligned}$$

From the first equation, we have $y(x) = -x + C$, and $y(x_0) = \frac{1}{2}(1 - x_0)$ implies $y(x) = -x + \frac{1}{2}(1 + x_0)$. We have, then, $x_0 = 2y + 2x - 1$. The second equation becomes

$$\frac{dU}{dx} = -2(-x + \frac{1}{2}(1 + x_0)) \Rightarrow U(x) = x^2 - (1 + x_0)x + C.$$

The condition $U(x_0) = \frac{1}{2}x_0(1 - x_0)$ implies $C = \frac{3}{2}x_0 - \frac{1}{2}x_0^2$, and so

$$U(x) = x^2 - (1 + x_0)x + \frac{3}{2}x_0 - \frac{1}{2}x_0^2.$$

We conclude

$$u(x, y) = x^2 - (2y + 2x)x + (2y + 2x - 1)(2 - x - y),$$

which (a little algebra shows) is the same as Constanda's solution

$$u(x, y) = (1 - x - y)(3x + 3y - 2) + y^2.$$

2. [10 pts] Constanda Exercise 12.4, Parts (i) and (ii).

Solution to Part (i). We return to the notation $U(t) = u(x(t), t)$, with $\frac{dU}{dt} = u_x + u_t \frac{dx}{dt}$. We have

$$\begin{aligned} \frac{dx}{dt} &= U(t) + t; \quad x(0) = x_0 \\ \frac{dU}{dt} &= 1; \quad U(0) = x_0. \end{aligned}$$

In this case, we solve the second equation first, $U(t) = t + x_0$. The first equation becomes

$$\frac{dx}{dt} = 2t + x_0 \Rightarrow x(t) = t^2 + x_0 t + x_0,$$

so that

$$x_0 = \frac{x - t^2}{1 + t}.$$

We conclude

$$u(x,t) = t + \frac{x - t^2}{1+t} = \frac{t + t^2 + x - t^2}{1+t} = \frac{t + x}{1 + t}.$$

Solution to Part (ii). In the notation of Part (i), we set

$$\begin{aligned} \frac{dx}{dt} &= 1; \quad x(0) = x_0 \\ \frac{dU}{dt} &= -2tU^2; \quad U(0) = e^{-x_0}. \end{aligned}$$

The first equation gives $x = t + x_0$, so that $x_0 = x - t$, while the second equation must be solved by separation of variables. We set

$$\frac{dU}{U^2} = -2tdt \Rightarrow -\frac{1}{U} = -t^2 + C,$$

and $U(0) = e^{-x_0}$ implies $C = -e^{x_0}$. We have

$$U(t) = \frac{1}{t^2 + e^{x_0}},$$

and so we conclude

$$u(x,t) = \frac{1}{t^2 + e^{x-t}}.$$

3. [10 pts] Constanda Exercise 12.5, Parts (i) and (ii).

Solution to Part (i). With $c = 1$, the d'Alembert solution is

$$u(x,t) = \frac{1}{2}(f(x-t) + f(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} g(y)dy,$$

which for this data becomes

$$\begin{aligned} u(x,t) =& \frac{1}{2}(2x-4)+\frac{1}{2}\int_{x-t}^{x+t} -2dy \\ =& x-2+\frac{1}{2}(-2(x+t)+2(x-t)) \\ =& x-2-2t. \end{aligned}$$

Solution to Part (ii). With $c = 3$, the d'Alembert solution is

$$u(x,t) = \frac{1}{2}(f(x-3t)+f(x+3t)) + \frac{1}{6}\int_{x-3t}^{x+3t} g(y)dy,$$

which for this data becomes

$$\begin{aligned} u(x,t) =& \frac{1}{2}((x-3t)^2+(x+3t)^2)+\frac{1}{6}\int_{x-3t}^{x+3t} y+1dy \\ =& \frac{1}{2}(2x^2+2\cdot 9t^2)+\frac{1}{6}(\frac{1}{2}(x+3t)^2+(x+3t)-\frac{1}{2}(x-3t)^2-(x-3t)) \\ =& x^2+9t^2+\frac{1}{6}(6xt+6t) = x^2+9t^2+xt+t. \end{aligned}$$

4. [10 pts] Constanda Exercise 12.6, Parts (i) and (ii). Explain why the solution to Part (ii) is continuous, while the solution to Part (i) is not. For Part (ii) show that your solution is equivalent to Constanda's.

Solution to Part (i). First, the solution is not continuous at $x = t$, because $f(0) = -1$, which is inconsistent with $u(0,t) = 0$ for all $t > 0$.

With $c = 1$ the quarter-plane solution is

$$u(x,t) = \begin{cases} \frac{1}{2}(f(x-t)+f(x+t))+\frac{1}{2}\int_{x-t}^{x+t} g(y)dy & x > t \\ \frac{1}{2}(-f(t-x)+f(x+t))+\frac{1}{2}\int_{t-x}^{x+t} g(y)dy & 0 \le x < t, \end{cases}$$

which for this data becomes

$$\begin{aligned} u(x,t) =& \begin{cases} \frac{1}{2}(4x-2)+\frac{1}{2}\int_{x-t}^{x+t} 3dy & x > t \\ \frac{1}{2}(4x)+\frac{1}{2}\int_{t-x}^{x+t} 3dy & 0 \le x < t \end{cases} \\ =& \begin{cases} \frac{1}{2}(4x-2)+\frac{1}{2}(6t) & x > t \\ \frac{1}{2}(4x)+\frac{1}{2}(6x) & 0 \le x < t \end{cases} \\ =& \begin{cases} 2x-1+3t & x > t \\ 5x & 0 \le x < t. \end{cases} \end{aligned}$$

Solution to Part (ii). In this case $f(0) = 0$, and so the equation is consistent, and the solution is continuous.

With $c = \frac{1}{2}$ the quarter-plane solution is

$$u(x,t) = \begin{cases} \frac{1}{2}(f(x-\frac{1}{2}t) + f(x+\frac{1}{2}t)) + \int_{x-\frac{1}{2}t}^{x+\frac{1}{2}t} g(y)dy & x \geq \frac{1}{2}t \\ \frac{1}{2}(-f(\frac{1}{2}t - x) + f(x+\frac{1}{2}t)) + \int_{\frac{1}{2}t-x}^{x+\frac{1}{2}t} g(y)dy & 0 \leq x < \frac{1}{2}t, \end{cases}$$

which for this data becomes

$$\begin{aligned} u(x,t) &= \begin{cases} \frac{1}{2}(\sin(x-\frac{1}{2}t) + \sin(x+\frac{1}{2}t)) + \int_{x-\frac{1}{2}t}^{x+\frac{1}{2}t} y - 2dy & x \geq \frac{1}{2}t \\ \frac{1}{2}(-\sin(\frac{1}{2}t - x) + \sin(x+\frac{1}{2}t)) + \int_{\frac{1}{2}t-x}^{x+\frac{1}{2}t} y - 2dy & 0 \leq x < \frac{1}{2}t \end{cases} \\ &= \begin{cases} \frac{1}{2}(\sin(x-\frac{1}{2}t) + \sin(x+\frac{1}{2}t)) + \frac{(x+\frac{1}{2}t)^2 - (x-\frac{1}{2}t)^2}{2} - 2t & x \geq \frac{1}{2}t \\ \frac{1}{2}(\sin(x-\frac{1}{2}t) + \sin(x+\frac{1}{2}t)) + \frac{(x+\frac{1}{2}t)^2 - (\frac{1}{2}t-x)^2}{2} - 4x & 0 \leq x < \frac{1}{2}t \end{cases}, \end{aligned}$$

where in the final equality we used the fact that $\sin(x)$ is odd. We now use the trig identity (see Assignment 4)

$$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)].$$

We get

$$\begin{aligned} u(x,t) &= \begin{cases} \sin(x)\cos(\frac{1}{2}t) + \frac{2xt}{2} - 2t & x \geq \frac{1}{2}t \\ \sin(x)\cos(\frac{1}{2}t) + \frac{2xt}{2} - 4x & 0 \leq x < \frac{1}{2}t \end{cases} \\ &= \begin{cases} \sin(x)\cos(\frac{1}{2}t) + xt - 2t & x \geq \frac{1}{2}t \\ \sin(x)\cos(\frac{1}{2}t) + xt - 4x & 0 \leq x < \frac{1}{2}t \end{cases}. \end{aligned}$$

5a. [4 pts] Solve the PDE

$$\begin{aligned} u_{tt} &= c^2 u_{xx}; \quad (x,t) \in (0,\infty) \times (0,\infty) \\ u(0,t) &= h(t), \quad t \geq 0 \\ u(x,0) &= 0, \quad x \geq 0 \\ u_t(x,0) &= 0, \quad x \geq 0. \end{aligned}$$

For consistency, assume $h(0) = 0$.

Solution to Part (a). As in class we begin by looking for solutions of the form

$$u(x,t) = F(x-ct) + G(x+ct).$$

For $x - ct \geq 0$ we can use d'Alembert's solution, and since $f(x)$ and $g(x)$ are both identically 0, we have

$$u(x,t) = 0, \quad x \geq ct.$$

Since $x + ct \geq 0$ for all positive values of x, $G(x+ct) = 0$ for all (x,t) in the quarter-plane.

For $x < ct$, we obtain $F(y)$ for $y < 0$ by noting that our boundary condition gives

$$h(t) = F(-ct) + G(ct).$$

If we write $y = -ct$, this becomes

$$F(y) = h(-\frac{y}{c}) - G(-y) = h(-\frac{y}{c}),$$

again because G is 0 at postive values. We have, finally,

$$u(x,t) = F(x-ct) + G(x+ct) = h(-\frac{x-ct}{c}) = h(t - \frac{x}{c}), \quad x < ct.$$

We conclude

$$u(x,t) = \begin{cases} 0 & x \geq ct \\ h(t - \frac{x}{c}) & x < ct \end{cases}.$$

5b. [3 pts] Use your solution from (5a) to solve the specific problem

$$\begin{aligned} u_{tt} &= 4u_{xx}; \quad (x,t) \in (0,\infty) \times (0,\infty) \\ u(0,t) &= \sin(\pi t), \quad t \geq 0 \\ u(x,0) &= 0, \quad x \geq 0 \\ u_t(x,0) &= 0, \quad x \geq 0. \end{aligned}$$

Sketch plots of $u(x, \frac{1}{2})$ and $u(x,1)$.

Solution to Part (b). From Part (a) we immediately have

$$u(x,t) = \begin{cases} 0 & x \geq 2t \\ \sin \pi(t - \frac{x}{2}) & x < 2t \end{cases}.$$

This gives

$$u(x, \frac{1}{2}) = \begin{cases} 0 & x \geq 1 \\ \sin \pi(\frac{1}{2} - \frac{x}{2}) & x < 1 \end{cases},$$

and

$$u(x,1) = \begin{cases} 0 & x \geq 2 \\ \sin \pi(1 - \frac{x}{2}) & x < 2 \end{cases}.$$

The plot of $u(x, \frac{1}{2})$ is given in Figure 1, and the plot of $u(x,1)$ is given in Figure 2.

5c. [3 pts] Use your solution from (5a), our calculations from class, and the method of superposition to write down a solution to the PDE

$$\begin{aligned} u_{tt} &= c^2 u_{xx}; \quad (x,t) \in (0,\infty) \times (0,\infty) \\ u(0,t) &= h(t), \quad t \geq 0 \\ u(x,0) &= f(x), \quad x \geq 0 \\ u_t(x,0) &= g(x), \quad x \geq 0. \end{aligned}$$

For consistency, assume $h(0) = f(0)$.

Solution to Part (c). Using superposition, we have

$$u(x,t) = \begin{cases} \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy & x > ct \\ \frac{1}{2}(-f(ct-x) + f(x+ct)) + \frac{1}{2c}\int_{ct-x}^{x+ct} g(y)dy + h(t - \frac{x}{c}) & 0 \leq x < ct, \end{cases}$$

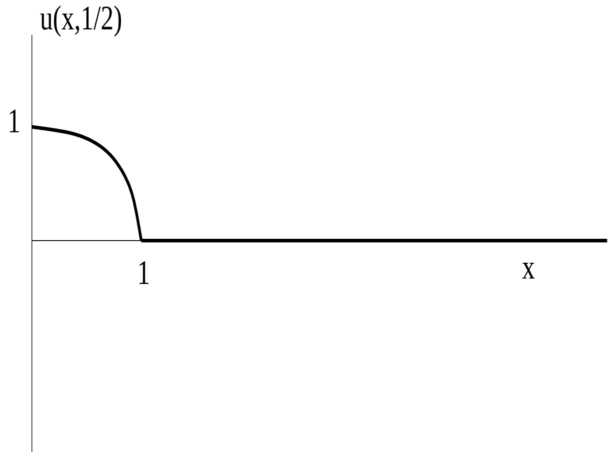

Figure 1: Plot of $u(x, \frac{1}{2})$ for Problem 5b.

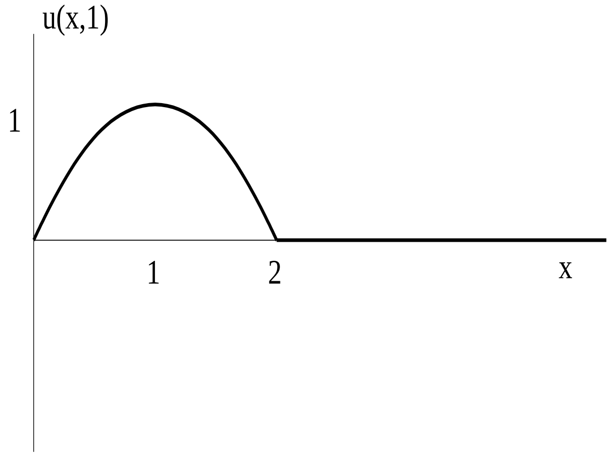

Figure 2: Plot of $u(x, 1)$ for Problem 5b.