

M401 Spring 2010, Assignment 9, due Thursday April 15

1a. [6 pts] Solve the quarter-plane problem

$$\begin{aligned}u_{tt} &= c^2 u_{xx}; & (x, t) &\in (0, \infty) \times (0, \infty) \\u_x(0, t) &= 0; & t &\geq 0 \\u(x, 0) &= f(x); & x &\geq 0 \\u_t(x, 0) &= g(x); & x &\geq 0.\end{aligned}$$

Notice that the difference between this problem and the quarter-plane problem we solved in class is the condition $u_x(0, t) = 0$ (replacing $u(0, t) = 0$).

1b. [2 pts] Solve the equation from Part (a) with $c = 2$, $g(x) = 0$, and

$$f(x) = \begin{cases} x - 2 & 2 \leq x \leq 3 \\ 4 - x & 3 < x \leq 4 \\ 0 & \text{otherwise} \end{cases}.$$

Sketch graphs of $u(x, 0)$, $u(x, 1)$, and $u(x, 2)$.

1c. [2 pts] Solve the equation from Part (a) with $c = 2$, $f(x) = 0$ and

$$g(x) = \frac{1}{x^2 + 1}.$$

Sketch a graph of $u(x, 1)$.

2a. [5 pts] Suppose a chemical is to be combined with a homogeneous fluid such as water in a thin cylindrical tube (i.e., a test tube). For example, you might think of mixing food coloring with water. Let $u(x, t)$ denote the concentration of chemical at time t and distance x along the tube. According to *Fick's law of diffusion*, the flux associated with u is

$$f = -ku_x,$$

where k is referred to as the chemical diffusivity. Explain what Fick's law of diffusion means physically, and use it to derive a PDE for the concentration $u(x, t)$.

2b. [5 pts] Suppose $u(x, t)$ denotes traffic density (number of cars per unit length of road) along a certain stretch of road. In class, we discussed models in which the traffic flux depends only on traffic density u . One drawback of such models is that they do not capture a driver's reaction to what he sees ahead. For example, a driver who sees a higher density of traffic ahead will often slow down, while a driver who sees a lower density of traffic ahead will often speed up. Incorporate this idea to revise our model from class.

3. For the PDE

$$\begin{aligned}u_t - V(x)u &= ku_{xx} \\u(0, t) &= 0; & u(L, t) &= 0 \\u(x, 0) &= f(x),\end{aligned}$$

suppose $V(x) \geq 0$ for all $x \in [0, L]$.

3a. [3 pts] Write down equations for $X(x)$ and $T(t)$ under the separation assumption $u(x, t) = X(x)T(t)$.

3b. [3 pts] Show that the eigenvalue problem for $X(x)$ has no negative eigenvalues and that 0 is not an eigenvalue.

3c. [4 pts] Show that if λ_1 and λ_2 are two different eigenvalues for this problem, and $X_1(x)$ and $X_2(x)$ are the associated eigenfunctions, then $X_1(x)$ and $X_2(x)$ are *orthogonal* in the following sense:

$$\int_0^L X_1(x)X_2(x)dx = 0.$$

Hint. Write out the eigenvalue equation for X_1 and multiply it by X_2 , then write out the eigenvalue problem for X_2 and multiply it by X_1 . Now subtract and integrate.

4. Consider the fourth order eigenvalue problem

$$\begin{aligned} X'''' - \lambda X &= 0 \\ X(0) &= 0; \quad X(L) = 0 \\ X''(0) &= 0; \quad X''(L) = 0. \end{aligned}$$

4a. [5 pts] Show that there are no negative eigenvalues for this problem, and that $\lambda = 0$ is not an eigenvalue.

4b. [5 pts] Find the eigenvalues and eigenfunctions for this problem.

5. Exercise 5.1 in Constanda, Parts (i) and (iii).