1a. [6 pts] Solve the quarter-plane problem

\[ u_{tt} = c^2 u_{xx}; \quad (x, t) \in (0, \infty) \times (0, \infty) \]

\[ u_x(0, t) = 0; \quad t \geq 0 \]

\[ u(x, 0) = f(x); \quad x \geq 0 \]

\[ u_t(x, 0) = g(x); \quad x \geq 0. \]

Notice that the difference between this problem and the quarter-plane problem we solved in class is the condition \( u_x(0, t) = 0 \) (replacing \( u(0, t) = 0 \)).

1b. [2 pts] Solve the equation from Part (a) with \( c = 2 \), \( g(x) = 0 \), and

\[ f(x) = \begin{cases} 
    x - 2 & 2 \leq x \leq 3 \\
    4 - x & 3 < x \leq 4 \\
    0 & \text{otherwise} 
\end{cases} \]

Sketch graphs of \( u(x, 0) \), \( u(x, 1) \), and \( u(x, 2) \).

1c. [2 pts] Solve the equation from Part (a) with \( c = 2 \), \( f(x) = 0 \) and

\[ g(x) = \frac{1}{x^2 + 1}. \]

Sketch a graph of \( u(x, 1) \).

2a. [5 pts] Suppose a chemical is to be combined with a homogeneous fluid such as water in a thin cylindrical tube (i.e., a test tube). For example, you might think of mixing food coloring with water. Let \( u(x, t) \) denote the concentration of chemical at time \( t \) and distance \( x \) along the tube. According to Fick’s law of diffusion, the flux associated with \( u \) is

\[ f = -ku_x, \]

where \( k \) is referred to as the chemical diffusivity. Explain what Fick’s law of diffusion means physically, and use it to derive a PDE for the concentration \( u(x, t) \).

2b. [5 pts] Suppose \( u(x, t) \) denotes traffic density (number of cars per unit length of road) along a certain stretch of road. In class, we discussed models in which the traffic flux depends only on traffic density \( u \). One drawback of such models is that they do not capture a driver’s reaction to what he sees ahead. For example, a driver who sees a higher density of traffic ahead will often slow down, while a driver who sees a lower density of traffic ahead will often speed up. Incorporate this idea to revise our model from class.

3. For the PDE

\[ u_t - V(x)u = ku_{xx} \]

\[ u(0, t) = 0; \quad u(L, t) = 0 \]

\[ u(x, 0) = f(x). \]
suppose $V(x) \geq 0$ for all $x \in [0, L]$.

3a. [3 pts] Write down equations for $X(x)$ and $T(t)$ under the separation assumption $u(x, t) = X(x)T(t)$.

3b. [3 pts] Show that the eigenvalue problem for $X(x)$ has no negative eigenvalues and that 0 is not an eigenvalue.

3c. [4 pts] Show that if $\lambda_1$ and $\lambda_2$ are two different eigenvalues for this problem, and $X_1(x)$ and $X_2(x)$ are the associated eigenfunctions, then $X_1(x)$ and $X_2(x)$ are orthogonal in the following sense:

$$\int_0^L X_1(x)X_2(x)dx = 0.$$

**Hint.** Write out the eigenvalue equation for $X_1$ and multiply it by $X_2$, then write out the eigenvalue problem for $X_2$ and multiply it by $X_1$. Now subtract and integrate.

4. Consider the fourth order eigenvalue problem

$$X''' - \lambda X = 0$$

$$X(0) = 0; \quad X(L) = 0$$

$$X''(0) = 0; \quad X''(L) = 0.$$

4a. [5 pts] Show that there are no negative eigenvalues for this problem, and that $\lambda = 0$ is not an eigenvalue.

4b. [5 pts] Find the eigenvalues and eigenfunctions for this problem.

5. Exercise 5.1 in Constanda, Parts (i) and (iii).