M401 Spring 2010, Assignment 9 Solutions

1a. [6 pts] Solve the quarter-plane problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx}; \quad (x,t) \in (0,\infty) \times (0,\infty) \\ u_x(0,t) &= 0; \quad t \geq 0 \\ u(x,0) &= f(x); \quad x \geq 0 \\ u_t(x,0) &= g(x); \quad x \geq 0. \end{aligned}$$

Notice that the difference between this problem and the quarter-plane problem we solved in class is the condition $u_x(0,t) = 0$ (replacing $u(0,t) = 0$).

Solution. We look for solutions of the form

$$u(x,t) = F(x - ct) + G(x + ct),$$

and for $x - ct \geq 0$ we obtain the usual d'Alembert solution. In order to evaluate $F(y)$ at $y < 0$, we write

$$u_x(x,t) = F'(x - ct) + G'(x + ct) \Rightarrow u_x(0,t) = F'(-ct) + G'(ct),$$

so that the boundary condition $u_x(0,t) = 0$ implies

$$0 = F'(-ct) + G'(ct).$$

Setting $y = -ct$ and see that

$$F'(y) = -G'(-y),$$

and we integrate both sides from 0 to y:

$$\int_0^y F'(z)dz = -\int_0^y G'(-z)dz \Rightarrow F(y) - F(0) = G(-y) - G(0),$$

so that

$$F(y) = G(-y) + (F(0) - G(0)),$$

and (for $x - ct < 0$)

$$F(x - ct) = G(ct - x) + (F(0) - G(0)).$$

From class, we know that for any $y \geq 0$

$$G(y) = \frac{1}{2}f(y) + \frac{1}{2c}\int_0^y g(z)dz + G(0) - \frac{1}{2}f(0),$$

and so for $x - ct < 0$

$$\begin{aligned} u(x,t) &= F(x - ct) + G(x + ct) \\ &= G(ct - x) + (F(0) - G(0)) + G(x + ct) \\ &= \frac{1}{2}f(ct - x) + \frac{1}{2c}\int_0^{ct-x} g(y)dy + G(0) - \frac{1}{2}f(0) + (F(0) - G(0)) \\ &\quad + \frac{1}{2}f(x + ct) + \frac{1}{2c}\int_0^{x+ct} g(y)dy + G(0) - \frac{1}{2}f(0) \\ &= \frac{1}{2}[f(ct - x) + f(x + ct)] + \frac{1}{2c}\Big(\int_0^{ct-x} g(y)dy + \int_0^{x+ct} g(y)dy\Big), \end{aligned}$$

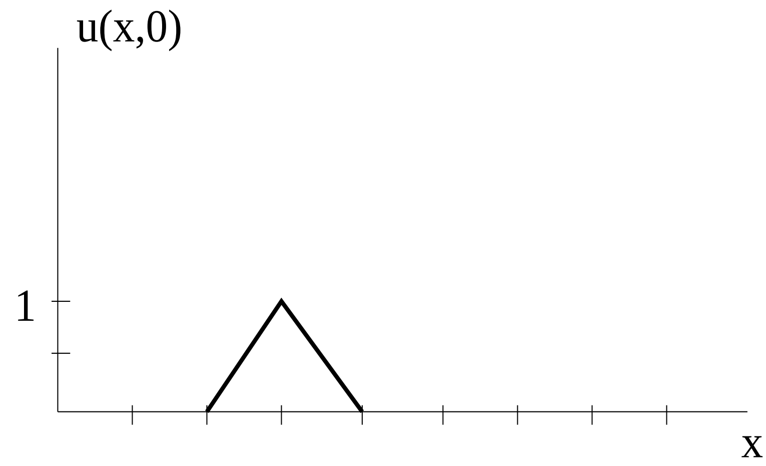

Figure 1: Graph of $u(x,0)$ for Problem 1b.

where we have used $f(0) = F(0) + G(0)$. We conclude

$$u(x,t) = \begin{cases} \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy & x \geq ct \\ \frac{1}{2}[f(ct-x) + f(x+ct)] + \frac{1}{2c}\left(\int_0^{ct-x} g(y)dy + \int_0^{x+ct} g(y)dy\right) & x < ct. \end{cases}$$

1b. [2 pts] Solve the equation from Part (a) with $c = 2$, $g(x) = 0$, and

$$f(x) = \begin{cases} x - 2 & 2 \leq x \leq 3 \\ 4 - x & 3 < x \leq 4 \\ 0 & \text{otherwise} \end{cases}.$$

Sketch graphs of $u(x,0)$, $u(x,1)$, and $u(x,2)$.

Solution. From our formula, we have

$$u(x,1) = \begin{cases} \frac{1}{2}[f(x-2) + f(x+2)] & x \geq 2 \\ \frac{1}{2}[f(2-x) + f(x+2)] & x < 2, \end{cases}$$

and

$$u(x,2) = \begin{cases} \frac{1}{2}[f(x-4) + f(x+4)] & x \geq 4 \\ \frac{1}{2}[f(4-x) + f(x+4)] & x < 4. \end{cases}$$

See Figures 1, 2, and 3.

1c. [2 pts] Solve the equation from Part (a) with $c = 2$, $f(x) = 0$ and

$$g(x) = \frac{1}{x^2+1}.$$

Sketch a graph of $u(x,1)$.

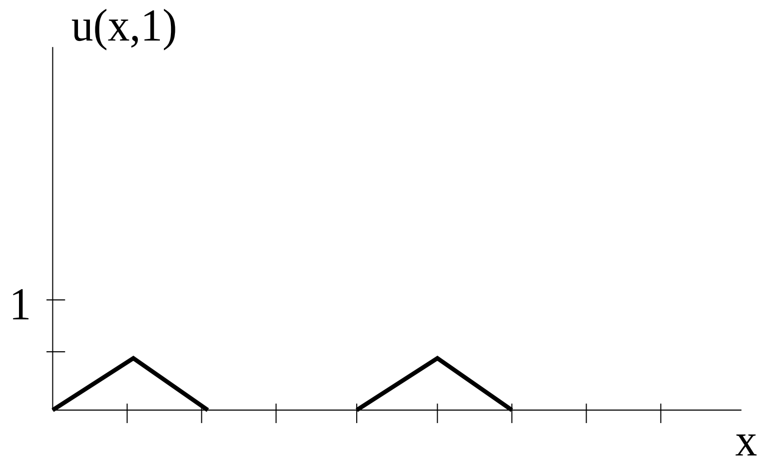

Figure 2: Graph of $u(x, 0)$ for Problem 1b.

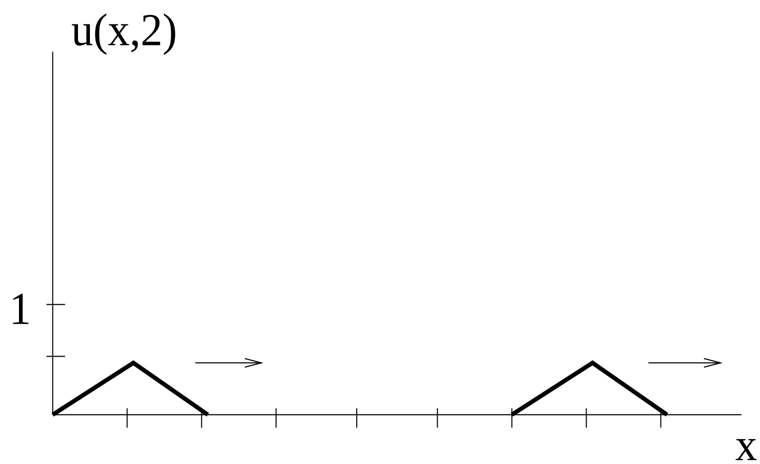

Figure 3: Graph of $u(x, 0)$ for Problem 1b.

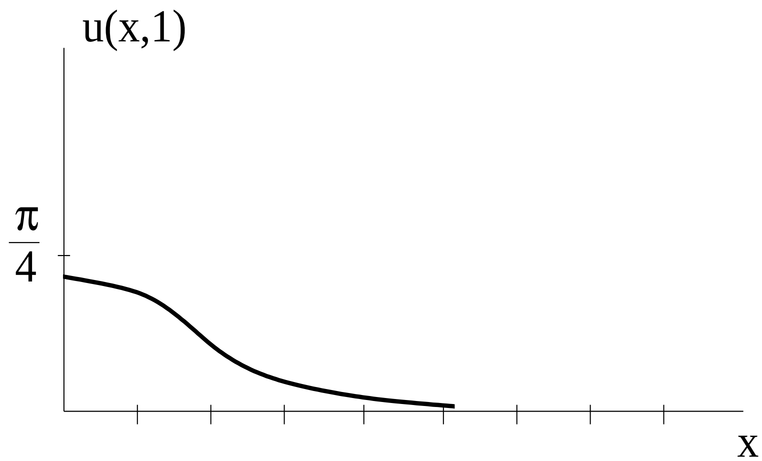

Figure 4: Figure for Problem 1c.

Solution. In this case

$$u(x,t) = \begin{cases} \frac{1}{4}\int_{x-2t}^{x+2t} \frac{1}{y^2+1} dy & x \geq 2t \\ \frac{1}{4}\Big(\int_0^{2t-x} \frac{1}{y^2+1} dy + \int_0^{x+2t} \frac{1}{y^2+1} dy \Big) & x < 2t, \end{cases}$$

and so

$$\begin{aligned} u(x,1) &= \begin{cases} \frac{1}{4}\int_{x-2}^{x+2} \frac{1}{y^2+1} dy & x \geq 2 \\ \frac{1}{4}\Big(\int_0^{2-x} \frac{1}{y^2+1} dy + \int_0^{x+2} \frac{1}{y^2+1} dy \Big) & x < 2 \end{cases} \\ &= \begin{cases} \frac{1}{4}\Big(\tan^{-1}(x+2) - \tan^{-1}(x-2) \Big) & x \geq 2 \\ \frac{1}{4}\Big(\tan^{-1}(2-x) + \tan^{-1}(x+2) \Big) & x < 2. \end{cases} \end{aligned}$$

Since $\tan^{-1} x$ is an odd function, this is really just

$$u(x,1) = \frac{1}{4}\Big(\tan^{-1}(x+2) - \tan^{-1}(x-2) \Big),$$

for all $x \geq 0$. A graph is sketched in Figure 4.

2a. [5 pts] Suppose a chemical is to be combined with a homogeneous fluid such as water in a thin cylindrical tube (i.e., a test tube). For example, you might think of mixing food coloring with water. Let $u(x,t)$ denote the concentration of chemical at time t and distance x along the tube. According to *Fick's law of diffusion*, the flux associated with u is

$$f = -k u_x,$$

where k is referred to as the chemical diffusivity. Explain what Fick's law of diffusion means physically, and use it to derive a PDE for the concentration $u(x,t)$.

Solution. Fick's law states that the chemical will flow from regions in which it has a relatively high concentration to regions in which it has a relatively low concentration. Using the general conservation law

$$u_t + f_x = 0$$

we find that u solves the heat equation

$$u_t = k u_{xx}.$$

That is, the chemical diffuses through the fluid in the same general way that heat diffuses through an object.

2b. [5 pts] Suppose $u(x,t)$ denotes traffic density (number of cars per unit length of road) along a certain stretch of road. In class, we discussed models in which the traffic flux depends only on traffic density u. One drawback of such models is that they do not capture a driver's reaction to what he sees ahead. For example, a driver who sees a higher density of traffic ahead will often slow down, while a driver who sees a lower density of traffic ahead will often speed up. Incorporate this idea to revise our model from class.

Solution. First, our general traffic flow model from class had the form

$$u_t + f(u)_x = 0,$$

and the specific model we considered was the Gompertz-Greenberg model with

$$f = -cu \ln(\frac{u}{u_{\max}}).$$

Now, if traffic has higher density ahead of a driver then $u_x > 0$, while if traffic has a lower density ahead of a driver then $u_x < 0$. (We are thinking of these derivatives as evaluated at the driver's current position.) If drivers who see high density ahead begin to slow down the effect will be a shift of density to the left, while if drivers who see lower density ahead begin to speed up the effect will be a shift of density to the right. That is, we replace f with a revised flux

$$\mathcal{F} = f(u) - k u_x.$$

Our general model becomes

$$u_t + f(u)_x = k u_{xx},$$

and the Gompertz-Greenberg model becomes

$$u_t - c\Big(u \ln(\frac{u}{u_{\max}})\Big)_x = k u_{xx}.$$

3. For the PDE

$$\begin{aligned} u_t + V(x)u &= k u_{xx} \\ u(0,t) &= 0; \quad u(L,t) = 0 \\ u(x,0) &= f(x), \end{aligned}$$

suppose $V(x) \geq 0$ for all $x \in [0,L]$.

3a. [5 pts] Write down equations for $X(x)$ and $T(t)$ under the separation assumption $u(x,t) = X(x)T(t)$.

Solution. Upon substituting $u(x,t) = X(x)T(t)$ into our equation, we find

$$X(x)T'(t) + V(x)X(x)T(t) = kX''(x)T(t).$$

We divide by $X(x)T(t)$ to get

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} - \frac{1}{k}V(x).$$

As discussed in class, since the left hand side depends only on t and the right hand side depends only on x both sides are equal to the same constant, $-\lambda$. This gives two equations

$$\begin{aligned} T' &= -k\lambda T \\ X'' - \frac{1}{k}V(x)X(x) + \lambda X &= 0, \end{aligned}$$

where the second has boundary conditions

$$\begin{aligned} X(0) &= 0 \\ X(L) &= 0. \end{aligned}$$

3b. [5 pts] Show that the eigenvalue problem for $X(x)$ has no negative eigenvalues and that 0 is not an eigenvalue.

Solution. We multiply the equation for X by X and integrate on $[0, L]$:

$$\int_0^L X''(x)X(x)dx - \frac{1}{k}\int_0^L V(x)X(x)^2dx = -\lambda\int_0^L X(x)^2dx.$$

For the first integral, we integrate by parts to find

$$\int_0^L X''(x)X(x)dx = -\int_0^L X'(x)^2dx,$$

so that

$$\lambda = \frac{\int_0^L X'(x)^2 + \frac{1}{k}V(x)X(x)^2dx}{\int_0^L X(x)^2dx}.$$

Clearly, $\lambda \geq 0$, and if $\lambda = 0$ then $X(x)$ must be constant, so that $X(0) = 0$ ensures $X(x) \equiv 0$.

3c. [4 pts] Show that if λ_1 and λ_2 are two different eigenvalues for this problem, and $X_1(x)$ and $X_2(x)$ are the associated eigenfunctions, then $X_1(x)$ and $X_2(x)$ are *orthogonal* in the following sense:

$$\int_0^L X_1(x)X_2(x)dx = 0.$$

Hint. Write out the eigenvalue equation for X_1 and multiply it by X_2, then write out the eigenvalue problem for X_2 and multiply it by X_1. Now subtract and integrate.

Solution. Following the hint we have

$$\begin{aligned} X_2X_1'' - \frac{1}{k}VX_1X_2 + \lambda_1X_1X_2 &= 0 \\ X_1X_2'' - \frac{1}{k}VX_1X_2 + \lambda_2X_1X_2 &= 0, \end{aligned}$$

so that (upon subtraction)

$$(X_2X_1'' - X_1X_2'') + (\lambda_1 - \lambda_2)X_1X_2 = 0.$$

Integrating on $[0, L]$ we have

$$\int_0^L (X_2X_1'' - X_1X_2'')dx + (\lambda_1 - \lambda_2)\int_0^L X_1X_2dx = 0,$$

and after integration by parts

$$-\int_0^L X_2'X_1' - X_1'X_2'dx + (\lambda_1 - \lambda_2)\int_0^L X_1X_2dx = 0.$$

The first integral is clearly 0, and since $\lambda_1 \neq \lambda_2$ this gives orthogonality.

4. Consider the fourth order eigenvalue problem

$$\begin{aligned} X'''' - \lambda X &= 0 \\ X(0) = 0; &\quad X(L) = 0 \\ X''(0) = 0; &\quad X''(L) = 0. \end{aligned}$$

4a. [5 pts] Show that there are no negative eigenvalues for this problem, and that $\lambda = 0$ is not an eigenvalue.

Solution. We begin by multiplying the equation by $X''(x)$ and integrating on $[0, L]$. That is,

$$\int_0^L X''''(x)X''(x)dx - \lambda\int_0^L X''(x)X(x)dx = 0.$$

Integrating each of these by parts once, we find

$$-\int_0^L X'''(x)^2dx + \lambda\int_0^L X'(x)^2dx = 0,$$

so that

$$\lambda = \frac{\int_0^L X'''(x)^2dx}{\int_0^L X'(x)^2dx} \geq 0.$$

Clearly, we cannot have a solution with $X'(x) \equiv 0$, because this would be constant, and according to the boundary conditions would be 0. So we are not dividing by 0 here. We can only get $\lambda = 0$ is $X'''(x) \equiv 0$, which means

$$X(x) = C_1x^2 + C_2x + C_3,$$

for some constants C_1, C_2, and C_3. But $X(0) = 0$ implies $C_3 = 0$, and $X''(0) = 0$ implies $C_1 = 0$. This leaves $0 = X(L) = C_2L$, which implies $C_2 = 0$. So $X(x) \equiv 0$, and $\lambda = 0$ is not an eigenfunction.

A student pointed out that this can also be accomplished by multiplying by $X(x)$ (instead of $X''(x)$) if we integrate by parts twice. That is,

$$\int_0^L X''''(x)X(x)dx - \lambda \int_0^L X(x)^2 dx = 0,$$

and for the first integral we compute

$$\begin{aligned}\int_0^L X''''(x)X(x)dx &= X'''(x)X(x)\Big|_0^L - \int_0^L X'''(x)X'(x)dx \\ &= -\left[X''(x)X'(x)\Big|_0^L - \int_0^L X''(x)^2 dx\right] \\ &= \int_0^L X''(x)^2 dx,\end{aligned}$$

where the boundary conditions have been used to eliminate all boundary terms. We conclude

$$\lambda = \frac{\int_0^L X''(x)^2 dx}{\int_0^L X(x)^2 dx} \geq 0.$$

Notice that this establishes the curious identity

$$\frac{\int_0^L X'''(x)^2 dx}{\int_0^L X'(x)^2 dx} = \frac{\int_0^L X''(x)^2 dx}{\int_0^L X(x)^2 dx}.$$

4b. [5 pts] Find the eigenvalues and eigenfunctions for this problem.

Solution. We take $\lambda > 0$ so that the auxiliary equation for his ODE is

$$r^4 = \lambda \Rightarrow r = \lambda^{\frac{1}{4}},$$

and we must recall that this has four values. Let $\sqrt[4]{\lambda}$ denote the positive real fourth root of λ. (Notice the difference, in this notation, between $\lambda^{\frac{1}{4}}$ and $\sqrt[4]{\lambda}$.) Then

$$\lambda^{\frac{1}{4}} = \sqrt[4]{\lambda} e^{i\frac{2\pi n}{4}} = \begin{cases} \sqrt[4]{\lambda} & n = 0 \\ i\sqrt[4]{\lambda} & n = 1 \\ -\sqrt[4]{\lambda} & n = 2 \\ -i\sqrt[4]{\lambda} & n = 3 \end{cases}.$$

The associated general solution is

$$X(x) = C_1 e^{-\sqrt[4]{\lambda}x} + C_2 e^{\sqrt[4]{\lambda}x} + C_3 \cos(\sqrt[4]{\lambda}x) + C_4 \sin(\sqrt[4]{\lambda}x),$$

with

$$X''(x) = C_1\sqrt{\lambda} e^{-\sqrt[4]{\lambda}x} + C_2\sqrt{\lambda} e^{\sqrt[4]{\lambda}x} - C_3\sqrt{\lambda}\cos(\sqrt[4]{\lambda}x) - C_4\sqrt{\lambda}\sin(\sqrt[4]{\lambda}x).$$

The conditions at $x = 0$ give

$$C_1 + C_2 + C_3 = 0$$
$$\sqrt{\lambda}(C_1 + C_2 - C_3) = 0.$$

If we multiply the first by $\sqrt{\lambda}$ and subtract the second, we find $C_3 = 0$. This leaves us with the equation

$$C_1 + C_2 = 0.$$

Likewise, the two equations at $x = L$ give

$$C_1 e^{-\sqrt[4]{\lambda}L} + C_2 e^{\sqrt[4]{\lambda}L} + C_4 \sin(\sqrt[4]{\lambda}L) = 0$$
$$\sqrt{\lambda}\Big(C_1 e^{-\sqrt[4]{\lambda}L} + C_2 e^{\sqrt[4]{\lambda}L} - C_4 \sin(\sqrt[4]{\lambda}L)\Big) = 0.$$

Again, if we multiply the first equation by $\sqrt{\lambda}$ and subtract off the second equation, we find

$$2\sqrt{\lambda}C_4 \sin(\sqrt[4]{\lambda}L) = 0 \Rightarrow \lambda = \frac{n^4\pi^4}{L^4}.$$

(If $C_4 = 0$ the, as we show below $C_1 = C_2 = 0$ and we do not have an eigenvalue.) Finally, since this term is 0, we have, for C_1 and C_2, the system

$$C_1 + C_2 = 0$$
$$C_1 e^{-\sqrt[4]{\lambda}L} + C_2 e^{\sqrt[4]{\lambda}L} = 0.$$

If we multiply the first by $e^{-\sqrt[4]{\lambda}L}$ and subtract the second, we find

$$C_2(e^{-\sqrt[4]{\lambda}L} - e^{\sqrt[4]{\lambda}L}) = 0 \Rightarrow C_2 = 0 \Rightarrow C_1 = 0.$$

So the eigenvalues are

$$\lambda_n = \frac{n^4\pi^4}{L^4},$$

and the eigenfunctions are

$$X_n(x) = \sin(\frac{n\pi x}{L}).$$

5. Exercise 5.1 in Constanda, Parts (i) and (iii).

Solution to (i). In this case $k = 1$ and $L = 1$, so the general solution to the heat equation is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin(n\pi x),$$

so that

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x).$$

For (i) we have

$$f(x) = \sin(2\pi x) - 3\sin(6\pi x),$$

and we see simply by comparing terms that

$$b_2 = 1; \quad b_6 = -3.$$

We conclude

$$u(x,t) = e^{-4\pi^2 t}\sin(2\pi x) - 3e^{-36\pi^2 t}\sin(6\pi x).$$

Solution to (iii). In this case, we use the general coefficient formula

$$b_n = 2\int_0^1 f(x)\sin(n\pi x)dx,$$

with $f(x) = 2x + 1$. This gives

$$\begin{aligned}
b_n &= 4\int_0^1 x\sin(n\pi x)dx + 2\int_0^1 \sin(n\pi x)dx \\
&= 4\Big(-\frac{x}{n\pi}\cos(n\pi x)\Big|_0^1 + \int_0^1 \frac{1}{n\pi}\cos(n\pi x)\Big) - \frac{2}{n\pi}\cos(n\pi x)\Big|_0^1 \\
&= -\frac{4}{n\pi}\cos(n\pi) + \frac{4}{n^2\pi^2}\sin(n\pi x)\Big|_0^1 - \frac{2}{n\pi}(\cos(n\pi) - 1) \\
&= -\frac{6}{n\pi}\cos(n\pi) + \frac{2}{n\pi}.
\end{aligned}$$

We observe that $\cos(n\pi) = (-1)^n$, so that

$$b_n = \frac{2}{n\pi}\Big(1 - 3(-1)^n\Big).$$

The solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\Big(1 - 3(-1)^n\Big)e^{-n^2\pi^2 t}\sin(n\pi x).$$