

M401 Spring 2010, Practice Problems for Final Exam

The final exam for this class will be on Wednesday, May 12, 8:00-10:00 a.m., in Blocker 161 (the regular classroom). It will cover the method of characteristics (including data given on the x -axis, on a quarter-plane, and on a line through the x - y plane), the wave equation on \mathbb{R} , separation of variables, Fourier series, equilibrium solutions, and eigenfunction expansion. In addition to the problems suggested here, the exercises in Chapter 6 of Constanda are all good, as are Exercises 7.1, 7.2, 7.3, 7.4, and 7.5. Calculators will be allowed, and students can bring a copy of Assignment 4 for trigonometric identities. Finally, the following orthogonality relations will be provided: For integers $m, n = 1, 2, \dots$

$$\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \begin{cases} \frac{L}{2} & m = n \\ 0 & m \neq n \end{cases}$$
$$\int_0^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = \begin{cases} \frac{L}{2} & m = n \\ 0 & m \neq n \end{cases}$$
$$\int_0^L \sin \frac{(n - \frac{1}{2})\pi x}{L} \sin \frac{(m - \frac{1}{2})\pi x}{L} dx = \begin{cases} \frac{L}{2} & m = n \\ 0 & m \neq n \end{cases}$$
$$\int_0^L \cos \frac{(n - \frac{1}{2})\pi x}{L} \cos \frac{(m - \frac{1}{2})\pi x}{L} dx = \begin{cases} \frac{L}{2} & m = n \\ 0 & m \neq n \end{cases}$$

1. Solve the PDE

$$u_t + 5u_x = xu$$
$$u(x, 0) = f(x); \quad x \in (-\infty, +\infty).$$

2. Solve the PDE

$$u_t + tu_x = u$$
$$u(0, t) = f(t); \quad t \geq 0$$
$$u(x, 0) = g(x); \quad x \geq 0,$$

with $f(0) = g(0)$.

3. Solve the PDE

$$2u_x + 3u_y = y^2$$
$$u(x, y) = x^2 + y \quad \text{on the line } y = 1 - x.$$

4. Consider the wave equation

$$u_{tt} = c^2 u_{xx}$$
$$u_x(0, t) = h(t) = H'(t); \quad t \geq 0$$
$$u(x, 0) = 0; \quad x \geq 0$$
$$u_t(x, 0) = 0; \quad x \geq 0.$$

4a. Explain why we should require both $h(0) = 0$ and $h'(0) = 0$ for consistency.

4b. Solve this equation.

5. Consider the PDE

$$\begin{aligned}u_t &= (k(x)u_x)_x \\u(0, t) &= 0; \quad u(L, t) = 0; \quad t \geq 0, \\u(x, 0) &= f(x); \quad x \in [0, L],\end{aligned}$$

where $k(x) > 0$ for all $x \in [0, L]$.

5a. Separate variables for this equation and write down an appropriate ODE for each separation function.

5b. Show that there are no negative eigenvalues for this problem, and that $\lambda = 0$ is not an eigenvalue.

6. We saw in class that the heat equation

$$\begin{aligned}u_t &= ku_{xx} \\u(0, t) &= 0; \quad u(L, t) = 0; \quad t \geq 0 \\u(x, 0) &= f(x); \quad x \in [0, L]\end{aligned}$$

is solved by

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-k \frac{n^2 \pi^2}{L^2} t} \sin \frac{n\pi x}{L},$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

6a. Solve this equation with $k = 2$, $L = 3$, and

$$f(x) = 2 \sin \frac{4\pi x}{3} - 2 \sin \frac{5\pi x}{3}.$$

6b. Solve this equation with $k = 2$, $L = 3$, and

$$f(x) = \cos x.$$

7. Solve the wave equation

$$\begin{aligned}u_{tt} &= c^2 u_{xx} \\u(0, t) &= 0; \quad u_x(L, t) = 0; \quad t \geq 0 \\u(x, 0) &= f(x); \quad x \in [0, L] \\u_t(x, 0) &= g(x); \quad x \in [0, L].\end{aligned}$$

8. Using separation of variables we can solve the heat equation

$$\begin{aligned}u_t &= k u_{xx} \\ u(0, t) &= 0; \quad u_x(L, t) = 0; \quad t \geq 0 \\ u(x, 0) &= f(x); \quad x \in [0, L],\end{aligned}$$

with solution

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-k \frac{(n-\frac{1}{2})^2 \pi^2}{L^2} t} \sin \frac{(n-\frac{1}{2})\pi x}{L},$$

with

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{(n-\frac{1}{2})\pi x}{L} dx.$$

Use this to solve the inhomogeneous problem

$$\begin{aligned}u_t &= k u_{xx} - x - 1 \\ u(0, t) &= -5; \quad u_x(L, t) = 0; \quad t \geq 0 \\ u(x, 0) &= f(x); \quad x \in [0, L].\end{aligned}$$

9. Find a value γ for which the given equation has an equilibrium solution, and find the equilibrium solution. You do not have to solve the PDE.

$$\begin{aligned}u_t &= k u_{xx} + e^{-x} \\ u_x(0, t) &= \gamma; \quad u_x(L, t) = 2; \quad t \geq 0 \\ u(x, 0) &= x; \quad x \in [0, L].\end{aligned}$$

10. Solve the PDE

$$\begin{aligned}u_t &= u_{xx} + t \\ u(0, t) &= 0; \quad u_x(1, t) = 0; \quad t \geq 0 \\ u(x, 0) &= x.\end{aligned}$$

11. Solve the PDE

$$\begin{aligned}u_t &= u_{xx} + e^{-2t} \sin 5x \\ u(0, t) &= 1; \quad u(\pi, t) = 0; \quad t \geq 0 \\ u(x, 0) &= 0; \quad x \in [0, L].\end{aligned}$$

12. Solve the PDE

$$\begin{aligned}u_t &= u_{xx} \\ u(0, t) &= 0; \quad u(L, t) = t; \quad t \geq 0 \\ u(x, 0) &= 0; \quad x \in [0, L].\end{aligned}$$

13. Solve the PDE

$$\begin{aligned}u_t &= ku_{xx} \\u_x(0, t) &= 0; \quad u_x(L, t) = 100; \quad t \geq 0 \\u(x, 0) &= f(x); \quad x \in [0, L].\end{aligned}$$

Note. We saw in class that this problem does not have an equilibrium solution, so we must apply the method of eigenfunction expansion.

Solutions

1. We set $U(t) := u(x(t), t)$ so that

$$\frac{dU}{dt} = u_x \frac{dx}{dt} + u_t.$$

We choose $x(t)$ now so that

$$\begin{aligned}\frac{dx}{dt} &= 5; \quad x(0) = x_0 \\ \frac{dU}{dt} &= x(t)U(t); \quad U(0) = f(x_0).\end{aligned}$$

From the first equation we have $x(t) = 5t + x_0$, and so $x_0 = x - 5t$. For the second equation we have

$$\frac{dU}{dt} = (5t + x_0)U,$$

and we solve this by separating variables. We find

$$U(t) = f(x_0)e^{\frac{5}{2}t^2 + x_0t}.$$

We conclude that

$$u(x, t) = f(x - 5t)e^{\frac{5}{2}t^2 + (x-5t)t} = f(x - 5t)e^{(x-\frac{5}{2}t)t}.$$

2. Proceeding as in Problem 1, we obtain the system

$$\begin{aligned}\frac{dx}{dt} &= t; \quad x(0) = x_0 \\ \frac{dU}{dt} &= U; \quad U(0) = g(x_0),\end{aligned}$$

valid so long as $x_0 \geq 0$. From the first equation we have $x(t) = \frac{1}{2}t^2 + x_0$, and so $x_0 = x - \frac{1}{2}t^2$. From the second equation, we have $U(t) = g(x_0)e^t$, and so for $x - \frac{1}{2}t^2 \geq 0$ (i.e., for $x_0 \geq 0$) we have

$$u(x, t) = g(x - \frac{1}{2}t^2)e^t.$$

For $x - \frac{1}{2}t^2 < 0$, we set

$$\begin{aligned}\frac{dx}{dt} &= t; & x(t_0) &= 0 \\ \frac{dU}{dt} &= U; & U(t_0) &= f(t_0),\end{aligned}$$

valid so long as $t_0 \geq 0$. From the first equation we have $x(t) = \frac{1}{2}(t^2 - t_0^2)$, and so

$$t_0 = \pm\sqrt{t^2 - 2x}.$$

We notice first that since $x - \frac{1}{2}t^2 < 0$ here, we are taking the square root of a positive number, and second that since we require $t_0 \geq 0$ we take the positive root. That is, $t_0 = \sqrt{t^2 - 2x}$. From the second equation we have

$$U(t) = f(t_0)e^{t-t_0},$$

and so for $x - \frac{1}{2}t^2 < 0$

$$u(x, t) = f(\sqrt{t^2 - 2x})e^{t-\sqrt{t^2-2x}}.$$

Combining these solutions, we have

$$u(x, t) = \begin{cases} g(x - \frac{1}{2}t^2)e^t & x \geq \frac{1}{2}t^2 \\ f(\sqrt{t^2 - 2x})e^{t-\sqrt{t^2-2x}} & x < \frac{1}{2}t^2. \end{cases}$$

3. First, we divide by 2 to put this problem in the form

$$u_x + \frac{3}{2}u_y = \frac{1}{2}y^2.$$

(Alternatively, we could divide by 3, and proceed with a curve $x(y)$ instead of $y(x)$.) We set $U(x) := u(x, y(x))$, so that

$$\frac{dU}{dx} = u_x + u_y \frac{dy}{dx},$$

and take $y(x)$ so that

$$\begin{aligned}\frac{dy}{dx} &= \frac{3}{2}; & y(x_0) &= 1 - x_0 \\ \frac{dU}{dx} &= \frac{1}{2}y(x)^2; & U(x_0) &= x_0^2 + y_0 = x_0^2 + (1 - x_0).\end{aligned}$$

From the first equation we have

$$y(x) = \frac{3}{2}x + 1 - \frac{5}{2}x_0 \Rightarrow x_0 = \frac{3}{5}x - \frac{2}{5}y + \frac{2}{5}.$$

The second equation is

$$\frac{dU}{dx} = \frac{1}{2}\left(\frac{3}{2}x + 1 - \frac{5}{2}x_0\right)^2; \quad U(x_0) = x_0^2 + (1 - x_0).$$

Integrating, we find

$$U(x) = \frac{1}{9}\left(\frac{3}{2}x + 1 - \frac{5}{2}x_0\right)^3 + C,$$

and the initial condition gives

$$C = x_0^2 + (1 - x_0) - \frac{1}{9}(1 - x_0)^3.$$

We have, then,

$$U(x) = \frac{1}{9}\left(\frac{3}{2}x + 1 - \frac{5}{2}x_0\right)^3 + x_0^2 + (1 - x_0) - \frac{1}{9}(1 - x_0)^3,$$

and finally

$$u(x, y) = \frac{1}{9}y^3 + \left(\frac{3}{5}x - \frac{2}{5}y + \frac{2}{5}\right)^2 + \left(\frac{3}{5} - \frac{3}{5}x + \frac{2}{5}y\right) - \frac{1}{9}\left(\frac{3}{5} - \frac{3}{5}x + \frac{2}{5}y\right)^3.$$

4a. By differentiating the condition $u(x, 0) = 0$ with respect to x we find $u_x(x, 0) = 0$, and so $u_x(0, 0) = 0$. Since $h(0) = u_x(0, 0)$, this sets $h(0) = 0$. Likewise, $h'(t) = u_{xt}(0, t)$, and so $h'(0) = u_{xt}(0, 0)$. But by differentiating the condition $u_t(x, 0) = 0$ with respect to x we find $u_{tx}(x, 0) = 0$, so that $u_{tx}(0, 0) = 0$. If $u_{xt}(x, t)$ and $u_{tx}(x, t)$ are both continuous then they must be equal, and so $h'(0) = 0$.

By the way, it's possible to continue on like this as follows: we have $h'''(t) = u_{xtt}(0, t)$, and from the equation itself we know $u_{tt}(x, t) = c^2 u_{xx}(x, t)$. By differentiating the equation with respect to x and setting $t = 0$ we find $u_{ttx}(x, 0) = c^2 u_{xxx}(x, 0)$. But by differentiating the condition $u_x(x, 0) = 0$ twice with respect to x we find $u_{xxx}(x, 0) = 0$. In this way, $u_{ttx}(0, 0) = c^2 u_{xxx}(0, 0) = 0$, and if we can rearrange the order of differentiation then $h'''(0) = 0$. The reason we don't regard this as a consistency condition is that nothing about our equation suggests that third derivatives should be continuous, or even exist.

4b. For problems like this we always proceed by looking for solutions of the form

$$u(x, t) = F(x - ct) + G(x + ct).$$

Since the d'Alembert solution won't necessarily be given, it's useful to work through the full argument without it. In order to evaluate F and G at positive values $x \geq 0$ we use the initial conditions to write

$$\begin{aligned} 0 &= F(x) + G(x) \\ 0 &= -cF'(x) + cG'(x). \end{aligned}$$

By differentiating the first equation, multiplying it by c and adding and subtracting the second we find $F'(x) = 0$ and $G'(x) = 0$. We have, then, that $F(x)$ and $G(x)$ are both constant functions, say $F(x) = F(0)$ and $G(x) = G(0)$. The condition $0 = F(x) + G(x)$ ensures that $F(0) + G(0) = 0$, and so for $x - ct \geq 0$

$$u(x, t) = F(x - ct) + G(x + ct) = F(0) + G(0) = 0.$$

On the other hand, for $x - ct < 0$ we need to evaluate F at negative values $y < 0$. For this, we use the condition $u_x(0, t) = H'(t)$, which gives

$$H'(t) = F'(-ct) + G'(ct).$$

We now set $y = -ct$, so that

$$H'(-\frac{y}{c}) = F'(y) + G'(-y).$$

Now, for $y < 0$ we integrate on $[y, 0]$, and using z as a dummy variable of integration, we have

$$\int_y^0 H'(-\frac{z}{c})dz = \int_y^0 F'(z)dz + \int_y^0 G'(-z)dz,$$

so that (taking care with the chain rule)

$$-c(H(0) - H(-\frac{y}{c})) = F(0) - F(y) - G(0) + G(-y),$$

and finally

$$F(y) = -cH(-\frac{y}{c}) + cH(0) + F(0),$$

where we have observed that $-G(0) + G(-y) = 0$ by the consideration above. Finally,

$$\begin{aligned} u(x, t) &= F(x - ct) + G(x + ct) \\ &= -cH(-\frac{x - ct}{c}) + cH(0) + F(0) + G(0) \\ &= -cH(t - \frac{x}{c}) + cH(0). \end{aligned}$$

We conclude

$$u(x, t) = \begin{cases} 0 & x \geq ct \\ -cH(t - \frac{x}{c}) + cH(0) & x < ct \end{cases}.$$

5a. We set $u(x, t) = X(x)T(t)$, so that the equation becomes

$$X(x)T'(t) = (k(x)X'(x))'T(t).$$

We divide by $X(x)T(t)$ to find

$$\frac{T'(t)}{T(t)} = \frac{(k(x)X'(x))'}{X(x)} = -\lambda,$$

where we have used the usual argument that the left-hand side depends only on t while the right-hand side depends only on x . The two ODE are

$$\begin{aligned} T'(t) &= -\lambda T(t) \\ (k(x)X'(x))' + \lambda X(x) &= 0, \end{aligned}$$

where the second has boundary conditions $X(0) = 0$ and $X(L) = 0$.

5b. We multiply the eigenvalue problem for $X(x)$ by $X(x)$ and integrate on $[0, L]$ to find

$$\int_0^L X(x)(k(x)X'(x))'dx + \lambda \int_0^L X(x)^2 dx = 0.$$

We integrate the first integral by parts:

$$\int_0^L X(x)(k(x)X'(x))'dx = X(x)k(x)X'(x)\Big|_0^L - \int_0^L k(x)X'(x)^2 dx.$$

Observing that the boundary terms are zero, we conclude

$$\lambda = \frac{\int_0^L k(x)X'(x)^2 dx}{\int_0^L X(x)^2 dx} \geq 0.$$

Moreover, we can only have $\lambda = 0$ if $X'(x) \equiv 0$ so that $X(x)$ is constant, but according to the boundary conditions this constant must be 0. We conclude that $\lambda = 0$ is not an eigenvalue.

6a. In this case we note that

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{3},$$

and so comparison with the given initial conditions gives $b_4 = 2$, $b_5 = -2$, and 0 for the other coefficients. In this way we see immediately that

$$u(x, t) = 2e^{-2\frac{16\pi^2 t}{9}} \sin \frac{4\pi x}{3} - 2e^{-2\frac{25\pi^2 t}{9}} \sin \frac{5\pi x}{3}.$$

6b. In this case we use the general formula

$$\begin{aligned} b_n &= \frac{2}{3} \int_0^3 \cos x \sin \frac{n\pi x}{3} dx \\ &= \frac{2}{3} \int_0^3 \frac{1}{2} [\sin(\frac{n\pi}{3} + 1)x + \sin(\frac{n\pi}{3} - 1)x] dx \\ &= -\frac{1}{3} \left[\frac{1}{\frac{n\pi}{3} + 1} \cos(\frac{n\pi}{3} + 1)x \Big|_0^3 + \frac{1}{\frac{n\pi}{3} - 1} \cos(\frac{n\pi}{3} - 1)x \Big|_0^3 \right] \\ &= -\frac{1}{3} \left[\frac{1}{\frac{n\pi}{3} + 1} (\cos(n\pi + 3) - 1) + \frac{1}{\frac{n\pi}{3} - 1} (\cos(n\pi - 3) - 1) \right] \\ &= -\frac{1}{3} \left[\frac{\frac{2n\pi}{3}}{\frac{n^2\pi^2}{9} - 1} ((-1)^n \cos 3 - 1) \right] \\ &= \frac{2n\pi}{n^2\pi^2 - 9} (1 - (-1)^n \cos 3). \end{aligned}$$

Here, we can either use trigonometric identity $\cos(A + B) = \cos A \cos B - \sin A \sin B$ to evaluate $\cos(n\pi \pm 3)$, or simply observe that it corresponds with a shift by $n\pi$ of $\cos 3$. We conclude

$$u(x, t) = \sum_{n=1}^{\infty} \frac{2n\pi}{n^2\pi^2 - 9} (1 - (-1)^n \cos 3) e^{-2\frac{n^2\pi^2 t}{9}} \sin \frac{n\pi x}{3}.$$

7. We separate variables with $u(x, t) = X(x)T(t)$ and find

$$X(x)T''(t) = c^2X''(x)T(t).$$

Dividing by $c^2X(x)T(t)$ we obtain

$$\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

where we have argued as usual that both quotients must be constant. We have, then,

$$\begin{aligned} T''(t) + \lambda c^2T(t) &= 0 \\ X''(x) + \lambda X(x) &= 0, \end{aligned}$$

where for the second equation we have boundary conditions $X(0) = 0$ and $X'(L) = 0$. In order to check that there are no negative eigenvalues, we multiply the eigenvalue problem for $X(x)$ by $X(x)$ and integrate on $[0, L]$. We find

$$\int_0^L X(x)X''(x)dx + \lambda \int_0^L X(x)^2dx = 0,$$

and for the first integral we integrate by parts,

$$\int_0^L X(x)X''(x)dx = X(x)X'(x)\Big|_0^L - \int_0^L X'(x)^2dx.$$

The boundary terms are both zero, so we conclude

$$\lambda = \frac{\int_0^L X'(x)^2dx}{\int_0^L X(x)^2dx} \geq 0,$$

and we can only have $\lambda = 0$ if $X(x)$ is constant. But according to the boundary conditions, this constant must be 0, so $\lambda = 0$ cannot be an eigenvalue. For $\lambda > 0$ we look for solutions of the form

$$X(x) = C_1 \cos \sqrt{\lambda}x + C_2 \sin \sqrt{\lambda}x,$$

with

$$X'(x) = -C_1\sqrt{\lambda} \sin \sqrt{\lambda}x + C_2\sqrt{\lambda} \cos \sqrt{\lambda}x.$$

Setting $X(0) = 0$ we find $C_1 = 0$, while setting $X'(L) = 0$ gives

$$C_2\sqrt{\lambda} \cos \sqrt{\lambda}L = 0 \Rightarrow \sqrt{\lambda}L = n\pi - \frac{\pi}{2} \Rightarrow \lambda_n = \frac{(n - \frac{1}{2})^2\pi^2}{L^2}, \quad n = 1, 2, \dots,$$

with associated eigenfunctions

$$X_n(x) = \sin \frac{(n - \frac{1}{2})\pi x}{L}.$$

Returning to the equation for $T(t)$ we have

$$T_n(t) = c_{1n} \cos \frac{(n - \frac{1}{2})\pi ct}{L} + c_{2n} \sin \frac{(n - \frac{1}{2})\pi ct}{L},$$

and so

$$u(x, t) = \sum_{n=1}^{\infty} \left(c_{1n} \cos \frac{(n - \frac{1}{2})\pi ct}{L} + c_{2n} \sin \frac{(n - \frac{1}{2})\pi ct}{L} \right) \sin \frac{(n - \frac{1}{2})\pi x}{L}.$$

Setting $u(x, 0) = f(x)$, we have

$$f(x) = \sum_{n=1}^{\infty} c_{1n} \sin \frac{(n - \frac{1}{2})\pi x}{L},$$

and using the orthogonality relations given on the first page,

$$c_{1n} = \frac{2}{L} \int_0^L f(x) \sin \frac{(n - \frac{1}{2})\pi x}{L} dx.$$

Using $u_t(x, 0) = g(x)$ along with

$$u_t(x, t) = \sum_{n=1}^{\infty} \left(-c_{1n} \frac{(n - \frac{1}{2})\pi c}{L} \sin \frac{(n - \frac{1}{2})\pi ct}{L} + c_{2n} \frac{(n - \frac{1}{2})\pi c}{L} \cos \frac{(n - \frac{1}{2})\pi ct}{L} \right) \sin \frac{(n - \frac{1}{2})\pi x}{L},$$

we have

$$g(x) = \sum_{n=1}^{\infty} c_{2n} \frac{(n - \frac{1}{2})\pi c}{L} \sin \frac{(n - \frac{1}{2})\pi x}{L},$$

and so

$$c_{2n} \frac{(n - \frac{1}{2})\pi c}{L} = \frac{2}{L} \int_0^L g(x) \sin \frac{(n - \frac{1}{2})\pi x}{L} dx,$$

and finally

$$c_{2n} = \frac{2}{(n - \frac{1}{2})\pi c} \int_0^L g(x) \sin \frac{(n - \frac{1}{2})\pi x}{L} dx.$$

The sum for $u(x, t)$ along with the integral expressions for c_{1n} and c_{2n} entirely characterize the solution.

8. We begin by finding the equilibrium solution $\bar{u}(x)$, which solves

$$\begin{aligned} 0 &= k\bar{u}'' - x - 1 \\ \bar{u}(0) &= -5 \\ \bar{u}'(L) &= 0. \end{aligned}$$

Integrating once, we have $k\bar{u}'(x) = \frac{x^2}{2} + x + C_1$, and $\bar{u}'(L) = 0$ implies $0 = \frac{1}{2}L^2 + L + C_1$, so that

$$C_1 = -\frac{1}{2}L^2 - L.$$

Integrating again, we have

$$k\bar{u}(x) = \frac{1}{6}x^3 + \frac{1}{2}x^2 - \left(\frac{1}{2}L^2 + L\right)x + C_2,$$

and setting $\bar{u}(0) = -5$ we find $C_2 = -5k$. We conclude

$$\bar{u}(x) = \frac{1}{k} \left(\frac{1}{6}x^3 + \frac{1}{2}x^2 - \left(\frac{1}{2}L^2 + L\right)x - 5k \right).$$

In order to solve the ODE we set $v = u - \bar{u}$, and observe that v solves the homogeneous problem

$$\begin{aligned} v_t &= kv_{xx} \\ v(0, t) &= 0; \quad v_x(L, t) = 0; \quad t \geq 0 \\ v(x, 0) &= f(x) - \bar{u}(x); \quad x \in [0, L]. \end{aligned}$$

As is given in the problem statement, the solution to this problem is

$$v(x, t) = \sum_{n=1}^{\infty} b_n e^{-k \frac{(n-\frac{1}{2})^2 \pi^2}{L^2} t} \sin \frac{(n-\frac{1}{2})\pi x}{L},$$

with

$$b_n = \frac{2}{L} \int_0^L (f(x) - \bar{u}(x)) \sin \frac{(n-\frac{1}{2})\pi x}{L} dx,$$

and finally

$$u(x, t) = \frac{1}{k} \left(\frac{1}{6}x^3 + \frac{1}{2}x^2 - \left(\frac{1}{2}L^2 + L\right)x - 5k \right) + \sum_{n=1}^{\infty} b_n e^{-k \frac{(n-\frac{1}{2})^2 \pi^2}{L^2} t} \sin \frac{(n-\frac{1}{2})\pi x}{L}.$$

9. The equilibrium solution $\bar{u}(x)$ will solve

$$\begin{aligned} 0 &= k\bar{u}'' + e^{-x} \\ \bar{u}'(0) &= \gamma \\ \bar{u}'(L) &= 2. \end{aligned}$$

Integrating once, we have $k\bar{u}'(x) = e^{-x} + C_1$. The condition $\bar{u}'(0) = \gamma$ gives $k\gamma = 1 + C_1$, while the condition $\bar{u}'(L) = 2$ gives $2k = e^{-L} + C_1$. From the second of these we have $C_1 = 2k - e^{-L}$, and so from the first

$$\gamma = \frac{1 + 2k - e^{-L}}{k}.$$

Integrating again now,

$$k\bar{u}(x) = -e^{-x} + (2k - e^{-L})x + C_2.$$

In order to get a value for C_2 we use the heat conservation relation

$$\int_0^L f(x) dx = \int_0^L \bar{u}(x) dx,$$

with

$$\begin{aligned}\int_0^L \bar{u}(x)dx &= \frac{1}{k} \int_0^L (-e^{-x} + (2k - e^{-L})x + C_2)dx \\ &= \frac{1}{k} (e^{-x} + \frac{1}{2}(2k - e^{-L})x^2 + C_2x) \Big|_0^L \\ &= \frac{1}{k} (e^{-L} + \frac{1}{2}(2k - e^{-L})L^2 + C_2L - 1),\end{aligned}$$

while

$$\int_0^L f(x)dx = \int_0^L xdx = \frac{1}{2}L^2.$$

Solving for C_2 , we find

$$C_2 = \frac{1}{L} \left(k\frac{1}{2}L^2 - e^{-L} - \frac{1}{2}(2k - e^{-L})L^2 + 1 \right),$$

and so finally

$$\bar{u}(x) = \frac{1}{k} \left[-e^{-x} + (2k - e^{-L})x + \frac{1}{L} \left(k\frac{1}{2}L^2 - e^{-L} - \frac{1}{2}(2k - e^{-L})L^2 + 1 \right) \right].$$

10. First, we need to find the eigenvalues and eigenvectors for the separation equation for $X(x)$, which in this case is

$$X''(x) + \lambda X(x) = 0,$$

with boundary conditions $X(0) = 0$ and $X'(1) = 0$. We can establish that there are no negative eigenvalues and that $\lambda = 0$ is not an eigenvalue in the usual way, and for $\lambda > 0$ we look for solutions of the form

$$X(x) = C_1 \cos \sqrt{\lambda}x + C_2 \sin \sqrt{\lambda}x.$$

Proceeding as in this step of the solution to Problem 7, we find that the eigenfunctions are

$$X_n(x) = \sin\left(n - \frac{1}{2}\right)\pi x, \quad n = 1, 2, \dots$$

We look for solutions of the form

$$u(x, t) = \sum_{n=1}^{\infty} c_n(t) \sin\left(n - \frac{1}{2}\right)\pi x,$$

for which we have

$$\sum_{n=1}^{\infty} c'_n(t) \sin\left(n - \frac{1}{2}\right)\pi x = - \sum_{n=1}^{\infty} c_n(t) \left(n - \frac{1}{2}\right)^2 \pi^2 \sin\left(n - \frac{1}{2}\right)\pi x + t,$$

so that

$$\sum_{n=1}^{\infty} (c'_n(t) + (n - \frac{1}{2})^2 \pi^2 c_n(t)) \sin\left(n - \frac{1}{2}\right)\pi x = t.$$

Using the trigonometric relations on the first page, we find

$$c'_n(t) + (n - \frac{1}{2})^2 \pi^2 c_n(t) = 2t \int_0^1 \sin(n - \frac{1}{2})\pi x dx = -\frac{2t}{(n - \frac{1}{2})\pi} \cos(n - \frac{1}{2})\pi x \Big|_0^1,$$

and this gives

$$c'_n(t) + (n - \frac{1}{2})^2 \pi^2 c_n(t) = \frac{2t}{(n - \frac{1}{2})\pi}.$$

For initial values, we use $u(x, 0) = x$, which gives

$$x = \sum_{n=1}^{\infty} c_n(0) \sin(n - \frac{1}{2})\pi x,$$

which gives

$$\begin{aligned} c_n(0) &= 2 \int_0^1 x \sin(n - \frac{1}{2})\pi x dx \\ &= 2 \left[-\frac{x}{(n - \frac{1}{2})\pi} \cos(n - \frac{1}{2})\pi x \Big|_0^1 + \frac{1}{(n - \frac{1}{2})\pi} \int_0^1 \cos(n - \frac{1}{2})\pi x dx \right] \\ &= 2 \left[\frac{1}{(n - \frac{1}{2})^2 \pi^2} \sin(n - \frac{1}{2})\pi x \Big|_0^1 \right] = \frac{2}{(n - \frac{1}{2})^2 \pi^2} \sin(n - \frac{1}{2})\pi \\ &= \frac{2}{(n - \frac{1}{2})^2 \pi^2} (-1)^{n+1}. \end{aligned}$$

We have, then, that for each $n = 1, 2, \dots$, we need to solve the ODE

$$\begin{aligned} c'_n(t) + (n - \frac{1}{2})^2 \pi^2 c_n(t) &= \frac{2t}{(n - \frac{1}{2})\pi} \\ c_n(0) &= \frac{2}{(n - \frac{1}{2})^2 \pi^2} (-1)^{n+1}. \end{aligned}$$

We proceed with an integrating factor

$$(e^{(n - \frac{1}{2})^2 \pi^2 t} c_n)' = \frac{2}{(n - \frac{1}{2})\pi} t e^{(n - \frac{1}{2})^2 \pi^2 t},$$

so that

$$\begin{aligned} e^{(n - \frac{1}{2})^2 \pi^2 t} c_n &= \frac{2}{(n - \frac{1}{2})\pi} \int t e^{(n - \frac{1}{2})^2 \pi^2 t} dt \\ &= \frac{2}{(n - \frac{1}{2})\pi} \left[\frac{t}{(n - \frac{1}{2})^2 \pi^2} e^{(n - \frac{1}{2})^2 \pi^2 t} - \frac{1}{(n - \frac{1}{2})^2 \pi^2} \int e^{(n - \frac{1}{2})^2 \pi^2 t} dt \right] \\ &= \frac{2}{(n - \frac{1}{2})\pi} \left[\frac{t}{(n - \frac{1}{2})^2 \pi^2} e^{(n - \frac{1}{2})^2 \pi^2 t} - \frac{1}{(n - \frac{1}{2})^4 \pi^4} e^{(n - \frac{1}{2})^2 \pi^2 t} + K \right], \end{aligned}$$

and so

$$c_n(t) = \frac{2t}{(n - \frac{1}{2})^3 \pi^3} - \frac{2}{(n - \frac{1}{2})^5 \pi^5} + K e^{-(n - \frac{1}{2})^2 \pi^2 t}.$$

Now use the initial condition to write

$$\frac{2}{(n - \frac{1}{2})^2 \pi^2} (-1)^{n+1} = -\frac{2}{(n - \frac{1}{2})^5 \pi^5} + K,$$

so that

$$K = \frac{2}{(n - \frac{1}{2})^2 \pi^2} (-1)^{n+1} + \frac{2}{(n - \frac{1}{2})^5 \pi^5}.$$

We have, then,

$$c_n(t) = \frac{2t}{(n - \frac{1}{2})^3 \pi^3} - \frac{2}{(n - \frac{1}{2})^5 \pi^5} + \left(\frac{2}{(n - \frac{1}{2})^2 \pi^2} (-1)^{n+1} + \frac{2}{(n - \frac{1}{2})^5 \pi^5} \right) e^{-(n - \frac{1}{2})^2 \pi^2 t}.$$

Finally,

$$u(x, t) = \sum_{n=1}^{\infty} \left[\frac{2t}{(n - \frac{1}{2})^3 \pi^3} - \frac{2}{(n - \frac{1}{2})^5 \pi^5} + \frac{2}{(n - \frac{1}{2})^2 \pi^2} \left(\frac{1}{(n - \frac{1}{2})^3 \pi^3} + (-1)^{n+1} \right) e^{-(n - \frac{1}{2})^2 \pi^2 t} \right] \sin\left(n - \frac{1}{2}\right) \pi x.$$

11. First, we need to move the inhomogenous boundary condition up to the equation. We do this by setting

$$p(x, t) := 1 - \frac{x}{\pi},$$

where of course the t dependence in p is trivial. We now set $v = u - p$, so that v solves

$$\begin{aligned} v_t &= v_{xx} + e^{-2t} \sin 5x \\ v(0, t) &= 0; \quad v(\pi, t) = 0; \quad t \geq 0 \\ v(x, 0) &= \frac{x}{\pi} - 1. \end{aligned}$$

We now solve this equation for v by the method of eigenfunction expansion. In this case the eigenfunctions are those for the heat equation with $L = \pi$ and temperatures zero at the boundaries,

$$X_n(x) = \sin nx; \quad n = 1, 2, \dots$$

Accordingly, we look for solutions

$$v(x, t) = \sum_{n=1}^{\infty} c_n(t) \sin nx,$$

for which the equation gives

$$\sum_{n=1}^{\infty} c'_n(t) \sin nx = - \sum_{n=1}^{\infty} c_n(t) n^2 \sin nx + e^{-2t} \sin 5x,$$

or equivalently

$$e^{-2t} \sin 5x = \sum_{n=1}^{\infty} (c'_n(t) + n^2 c_n(t)) \sin nx.$$

Simply by matching terms, we see

$$c'_n(t) + n^2 c_n(t) = \begin{cases} 0 & n \neq 5 \\ e^{-2t} & n = 5 \end{cases}.$$

For the initial values we have

$$\frac{x}{\pi} - 1 = \sum_{n=1}^{\infty} c_n(0) \sin nx,$$

which is a Fourier sine series for $\frac{x}{\pi} - 1$ on the interval $[0, \pi]$. We have, then,

$$\begin{aligned} c_n(0) &= \frac{2}{\pi} \int_0^{\pi} \left(\frac{x}{\pi} - 1\right) \sin nxdx \\ &\stackrel{\text{parts}}{=} \frac{2}{\pi} \left[-\left(\frac{x}{\pi} - 1\right) \frac{1}{n} \cos nx \Big|_0^{\pi} + \frac{1}{n\pi} \int_0^{\pi} \cos nxdx \right] \\ &= \frac{2}{\pi} \left[-\frac{1}{n} + \frac{1}{n^2\pi} \sin nx \Big|_0^{\pi} \right] \\ &= -\frac{2}{n\pi}. \end{aligned}$$

For $n \neq 5$, we solve

$$\begin{aligned} c'_n(t) + n^2 c_n(t) &= 0 \\ c_n(0) &= -\frac{2}{n\pi}, \end{aligned}$$

to find

$$c_n(t) = -\frac{2}{n\pi} e^{-n^2 t}; \quad n \neq 5,$$

while for $n = 5$ we solve

$$\begin{aligned} c'_5(t) + 25c_5(t) &= e^{-2t} \\ c_5(0) &= -\frac{2}{5\pi}. \end{aligned}$$

Using an integrating factor, we obtain

$$(e^{25t} c_5)' = e^{23t} \Rightarrow e^{25t} c_5 = \frac{1}{23} e^{23t} + K \Rightarrow c_5(t) = \frac{1}{23} e^{-2t} + K e^{-25t}.$$

Setting $c_5(0) = -\frac{2}{5\pi}$, we have $-\frac{2}{5\pi} = \frac{1}{23} + K$, so that $K = -\frac{2}{5\pi} - \frac{1}{23}$. In this way, we find

$$c_5(t) = \frac{1}{23} e^{-2t} - \left(\frac{2}{5\pi} + \frac{1}{23}\right) e^{-25t}.$$

We conclude

$$v(x, t) = \left(\frac{1}{23} e^{-2t} - \left(\frac{2}{5\pi} + \frac{1}{23}\right) e^{-25t}\right) \sin 5x + \sum_{\substack{n=1 \\ n \neq 5}}^{\infty} -\frac{2}{n\pi} e^{-n^2 t} \sin nx,$$

and so

$$u(x, t) = 1 - \frac{x}{\pi} + v(x, t).$$

12. We begin by shifting the boundary inhomogeneity up to the equation. We accomplish this by setting

$$p(x, t) = \frac{xt}{L},$$

and then using $v = u - p$, which solves

$$\begin{aligned} v_t &= v_{xx} - \frac{x}{L} \\ v(0, t) &= 0; \quad v(L, t) = 0; \quad t \geq 0 \\ v(x, 0) &= 0; \quad x \in [0, L]. \end{aligned}$$

This equation can now be solved either by using an equilibrium solution or with eigenfunction expansion. We notice, however, that the equilibrium solution will be cubic in x (after integrating $\frac{x}{L}$ twice), and this suggests that evaluating the expansion coefficients in that case would require integrating by parts three successive times, and so we will use the eigenfunction expansion method. Observing that the eigenfunctions of the associated separation function for the x variable are $X_n(x) = \sin \frac{n\pi x}{L}$, we look for solutions of the form

$$v(x, t) = \sum_{n=1}^{\infty} c_n(t) \sin \frac{n\pi x}{L}.$$

Substituting this into our equation, we have

$$\sum_{n=1}^{\infty} c'_n(t) \sin \frac{n\pi x}{L} = - \sum_{n=1}^{\infty} c_n(t) \frac{n^2\pi^2}{L^2} \sin \frac{n\pi x}{L} - \frac{x}{L},$$

or equivalently

$$-\frac{x}{L} = \sum_{n=1}^{\infty} (c'_n(t) + \frac{n^2\pi^2}{L^2}c_n(t)) \sin \frac{n\pi x}{L}.$$

This is a Fourier sine series for the function $-\frac{x}{L}$ with coefficients $b_n(t) = c'_n(t) + \frac{n^2\pi^2}{L^2}c_n(t)$, and we have

$$c'_n(t) + \frac{n^2\pi^2}{L^2}c_n(t) = -\frac{2}{L^2} \int_0^L x \sin \frac{n\pi x}{L} dx.$$

Integrating the right-hand side by parts, we get

$$c'_n(t) + \frac{n^2\pi^2}{L^2}c_n(t) = (-1)^n \frac{2}{n\pi}.$$

For the initial values we have

$$0 = \sum_{n=1}^{\infty} c_n(0) \sin \frac{n\pi x}{L},$$

and this clearly implies $c_n(0) = 0$ for all $n = 1, 2, \dots$. In this way we need to solve the ODE

$$\begin{aligned}c'_n(t) + \frac{n^2\pi^2}{L^2}c_n(t) &= (-1)^n \frac{2}{n\pi} \\c_n(0) &= 0.\end{aligned}$$

Using an integrating factor, we have

$$(e^{\frac{n^2\pi^2}{L^2}t}c_n)' = (-1)^n \frac{2}{n\pi} e^{\frac{n^2\pi^2}{L^2}t} \Rightarrow e^{\frac{n^2\pi^2}{L^2}t}c_n(t) = (-1)^n \frac{2L^2}{n^3\pi^3} e^{\frac{n^2\pi^2}{L^2}t} + K,$$

so that

$$c_n(t) = (-1)^n \frac{2L^2}{n^3\pi^3} + Ke^{-\frac{n^2\pi^2}{L^2}t}.$$

Using $c_n(0) = 0$, we find

$$K = -(-1)^n \frac{2L^2}{n^3\pi^3},$$

and so

$$c_n(t) = (-1)^n \frac{2L^2}{n^3\pi^3} - (-1)^n \frac{2L^2}{n^3\pi^3} e^{-\frac{n^2\pi^2}{L^2}t} = (-1)^n \frac{2L^2}{n^3\pi^3} \left(1 - e^{-\frac{n^2\pi^2}{L^2}t}\right).$$

We conclude that

$$v(x, t) = \sum_{n=1}^{\infty} (-1)^n \frac{2L^2}{n^3\pi^3} \left(1 - e^{-\frac{n^2\pi^2}{L^2}t}\right) \sin \frac{n\pi x}{L},$$

and so

$$u(x, t) = \frac{xt}{L} + \sum_{n=1}^{\infty} (-1)^n \frac{2L^2}{n^3\pi^3} \left(1 - e^{-\frac{n^2\pi^2}{L^2}t}\right) \sin \frac{n\pi x}{L}.$$

13. First, we must more the inhomogeneity from the boundary conditions to the equation. In order to do this we will choose a function $p(x)$ so that $p'(0) = 0$ and $p'(L) = 100$, and subtract this function from u . The easiest way to construct such a function is to set

$$p'(x) = C_1x + C_2,$$

and use the boundary conditions to get

$$p'(x) = \frac{100}{L}x \Rightarrow p(x) = \frac{50}{L}x^2 + C_3.$$

Since any choice of p that satisfies the boundary conditions will work, we can take $C_3 = 0$, but in order to see that the choice simply doesn't matter, we'll carry it through the calculation. We set $v = u - p$ and observe that v solves

$$\begin{aligned}v_t &= kv_{xx} + \frac{100k}{L} \\v_x(0, t) &= 0; \quad v_x(L, t) = 0; \quad t \geq 0 \\v(x, 0) &= f(x) - \frac{50}{L}x^2 - C_3.\end{aligned}$$

Noting that the eigenfunctions associated with separation solutions for this problem are $X_0(x) = 1$ and $X_n(x) = \cos \frac{n\pi x}{L}$ (see our second class example on separation of variables), we look for solutions of the form

$$v(x, t) = c_0(t) + \sum_{n=1}^{\infty} c_n(t) \cos \frac{n\pi x}{L},$$

for which we find

$$c'_0(t) + \sum_{n=1}^{\infty} c'_n(t) \cos \frac{n\pi x}{L} = - \sum_{n=1}^{\infty} c_n(t) k \frac{n^2 \pi^2}{L^2} \cos \frac{n\pi x}{L} + \frac{100k}{L},$$

or equivalently

$$\frac{100k}{L} = c'_0(t) + \sum_{n=1}^{\infty} (c'_n(t) + c_n(t) k \frac{n^2 \pi^2}{L^2}) \cos \frac{n\pi x}{L},$$

so that the right-hand side is a Fourier cosine series for $\frac{100k}{L}$ with coefficients $a_0(t) = c'_0(t)$ and $a_n(t) = c'_n(t) + c_n(t) k \frac{n^2 \pi^2}{L^2}$. We have, then,

$$c'_0(t) = \frac{1}{L} \int_0^L \frac{100k}{L} dx = \frac{100k}{L},$$

and

$$c'_n(t) + c_n(t) k \frac{n^2 \pi^2}{L^2} = \frac{2}{L} \int_0^L \frac{100k}{L} \cos \frac{n\pi x}{L} dx = 0.$$

For the initial conditions we have

$$f(x) - \frac{50}{L} x^2 - C_3 = c_0(0) + \sum_{n=1}^{\infty} c_n(0) \cos \frac{n\pi x}{L},$$

and so

$$\begin{aligned} c_0(0) &= \frac{1}{L} \int_0^L f(x) - \frac{50}{L} x^2 - C_3 dx \\ c_n(0) &= \frac{2}{L} \int_0^L (f(x) - \frac{50}{L} x^2 - C_3) \cos \frac{n\pi x}{L} dx. \end{aligned}$$

Since $f(x)$ is unspecified, we won't evaluate these integrals, but in order to understand what happens with the constant C_3 , we note that the terms involving C_3 are easily evaluated, and we find

$$\begin{aligned} c_0(0) &= \frac{1}{L} \int_0^L (f(x) - \frac{50}{L} x^2) dx - C_3 \\ c_n(0) &= \frac{2}{L} \int_0^L (f(x) - \frac{50}{L} x^2) \cos \frac{n\pi x}{L} dx. \end{aligned}$$

For $c_0(t)$ we have the ODE

$$c_0' = \frac{100k}{L}$$

$$c_0(0) = \frac{1}{L} \int_0^L (f(x) - \frac{50}{L}x^2)dx - C_3,$$

with solution

$$c_0(t) = \frac{100k}{L}t + \frac{1}{L} \int_0^L (f(x) - \frac{50}{L}x^2)dx - C_3,$$

while for $c_n(t)$ we have the ODE

$$c_n'(t) + c_n(t)k \frac{n^2\pi^2}{L^2} = 0$$

$$c_n(0) = \frac{2}{L} \int_0^L (f(x) - \frac{50}{L}x^2) \cos \frac{n\pi x}{L} dx,$$

with solution

$$c_n(t) = \left[\frac{2}{L} \int_0^L (f(x) - \frac{50}{L}x^2) \cos \frac{n\pi x}{L} dx \right] e^{-k \frac{n^2\pi^2}{L^2} t}.$$

We have, then,

$$v(x, t) = \frac{100k}{L}t + \frac{1}{L} \int_0^L (f(x) - \frac{50}{L}x^2)dx - C_3 + \sum_{n=1}^{\infty} \left[\frac{2}{L} \int_0^L (f(x) - \frac{50}{L}x^2) \cos \frac{n\pi x}{L} dx \right] e^{-k \frac{n^2\pi^2}{L^2} t} \cos \frac{n\pi x}{L},$$

and so

$$u(x, t) = v(x, t) + p(x)$$

$$= \frac{100k}{L}t + \frac{50}{L}x^2 + \frac{1}{L} \int_0^L (f(x) - \frac{50}{L}x^2)dx$$

$$+ \sum_{n=1}^{\infty} \left[\frac{2}{L} \int_0^L (f(x) - \frac{50}{L}x^2) \cos \frac{n\pi x}{L} dx \right] e^{-k \frac{n^2\pi^2}{L^2} t} \cos \frac{n\pi x}{L},$$

where in particular we note that the C_3 in v has canceled with the C_3 in $p(x)$.