M401 Practice Problems for Midterm Exam

The midterm exam will be Thursday, March 11, 7:00-9:00 p.m. in Blocker 161 (our regular classroom). Calculators will be allowed on the exam, and you can use a copy of Assignment 4. The exam will cover dimensional analysis, perturbation theory for algebraic equations (with regular, non-integer, and singular expansions), and perturbation theory for ordinary differential equations, including regular, Poincare, and two-scale expansions, as well as initial and boundary layer approximations.

1. A simple electric circuit consisting of a voltage source V, a resistance R, an inductance L, and a capacitance C can be modeled by the equation

$$\begin{aligned} L\frac{d^2I}{dt} + R\frac{dI}{dt} + \frac{1}{C}I &= \frac{dV}{dt}, \\ I(0) &= I_0 \\ I'(0) &= J_0, \end{aligned}$$

where $I(t)$ denotes the electric current at time t. Electric current is a dimension, typically denoted E, and $[V] = L^2MT^{-3}E^{-1}$. Use this information to find the dimensions of L, R, and C, and to non-dimensionalize this equation.

2. Find the first two terms in the expansion of each solution to

$$\epsilon x^3 + x^2 + x + \epsilon = 0.$$

3. Determine the first two non-zero terms in the expansion of *one* solution of the equation

$$x^3 + \epsilon^2 x + \epsilon^2 = 0.$$

4. For the ODE

$$\begin{aligned} y'' + k^2 y &= \epsilon y(1-y) \\ y(0) &= 1 \\ y'(0) &= 0, \end{aligned}$$

find a Poincare expansion for the solution with terms $y(t) \approx y_0$ and $\lambda(\epsilon) \approx k + \lambda_1 \epsilon$.

5. Find the first order two-scale approximation for

$$\begin{aligned} y'' + k^2 y &= \epsilon y' y^2 \\ y(0) &= 1 \\ y'(0) &= 0. \end{aligned}$$

6. Find a zeroth order composite approximation for the initial value problem

$$\begin{aligned} \epsilon y'' + (1+t)y' + y &= 0 \\ y(0) &= 0 \\ y'(0) &= \frac{1}{\epsilon}, \end{aligned}$$

for $\epsilon > 0$.

7. Find a first order composite solution for the boundary value problem

$$\begin{aligned} \epsilon y'' + y' + x^3 y &= 0 \\ y(0) &= 0 \\ y(1) &= 1. \end{aligned}$$

8. Find the zeroth order composite approximation for the boundary value problem

$$\begin{aligned} \epsilon y'' + (\cos x)y' + (\sin x)y &= 0 \\ y(0) &= 0 \\ y(\frac{\pi}{4}) &= 1. \end{aligned}$$

9. Find a zeroth order composite approximation for the boundary value problem

$$\begin{aligned} \epsilon y'' - (\cos x)y' + (\sin x)y &= 0 \\ y(0) &= 1 \\ y(\frac{\pi}{4}) &= 0. \end{aligned}$$

(Problems 8 and 9 differ both by a sign change on the y' term and by different boundary conditions.)

Solutions

1. First, $[V'] = L^2MT^{-4}E^{-1}$, so each term in the equation has these dimensions. We have

$$\begin{aligned} [LI''] &= [L]ET^{-2} = L^2MT^{-4}E^{-1} \Rightarrow [L] = L^2MT^{-2}E^{-2} \\ [RI'] &= [R]ET^{-1} = L^2MT^{-4}E^{-1} \Rightarrow [R] = L^2MT^{-3}E^{-2} \\ [C^{-1}I] &= [C]^{-1}E = L^2MT^{-4}E^{-1} \Rightarrow [C] = L^{-2}M^{-1}T^4E^2. \end{aligned}$$

Now set

$$\tau = \frac{t}{A}, \quad i(\tau) = \frac{I(t)}{B},$$

where A denotes a constant with dimension T and B denotes a constant with dimension E. Substituting this into our equation, we have

$$\begin{aligned} L\frac{B}{A^2}i'' + R\frac{B}{A}i' + \frac{B}{C}i &= V' \\ Bi(0) &= I_0 \\ \frac{B}{A}i'(0) &= J_0. \end{aligned}$$

(Here, V' still denotes differentiation with respect to t, but we evaluate the derivative at τA.) Divide by $L\frac{B}{A^2}$ so that the equation becomes

$$i'' + \frac{RA}{L}i' + \frac{A^2}{LC}i = \frac{A^2}{LB}V'.$$

Clearly, it's natural to choose $A = \frac{L}{R}$. The equation doesn't give a clear choice for B, but the initial condition suggests $B = I_0$. With this choice we have

$$\begin{aligned} i'' + i' + \frac{L}{R^2 C}i &= \frac{L}{R^2 I_0}V' \\ i(0) &= 1 \\ i'(0) &= \frac{L}{RI_0}J_0. \end{aligned}$$

Notice that many other choices for A and B are possible.

2. Setting $\epsilon = 0$, we first consider the two regular solutions, which satisfy $x(0)^2 + x(0) = 0 \Rightarrow x(0) = 0, -1$. Since these are not repeated roots, we know the Implicit Function Theorem holds, and we can look for solutions of the form,

$$x(\epsilon) = a_0 + a_1\epsilon + \ldots,$$

for which we obtain

$$\epsilon(a_0 + a_1\epsilon + \ldots)^3 + (a_0 + a_1\epsilon + \ldots)^2 + (a_0 + a_1\epsilon + \ldots) + \epsilon = 0.$$

Equating coefficients of powers of ϵ, we have

$$\begin{aligned} 1 : & a_0^2 + a_0 = 0 \Rightarrow a_0 = 0, -1 \\ \epsilon : & a_0^3 + 2a_0 a_1 + a_1 + 1 = 0 \Rightarrow a_1 = \frac{-1 - a_0^3}{2a_0 + 1} = -1, 0. \end{aligned}$$

We conclude that the two regular roots have expansions

$$x(\epsilon) = \begin{cases} -\epsilon + \mathbf{O}(\epsilon^2) \\ -1 + \mathbf{O}(\epsilon^2). \end{cases}$$

Observe that in each of these, one of the first two terms is 0. *Keep in mind on the exam that these terms count.*

For the singular root we observe that for x large, $\epsilon x^3 + x^2 \cong 0 \Rightarrow x \cong -\frac{1}{\epsilon}$. We try the rescaling $z = \epsilon x$, for which we find

$$\epsilon\frac{z^3}{\epsilon^3} + \frac{z^2}{\epsilon^2} + \frac{z}{\epsilon} + \epsilon = 0 \Rightarrow z^3 + z^2 + \epsilon z + \epsilon^3 = 0.$$

We pose, now, a regular expansion in z,

$$z(\epsilon) = a_0 + a_1\epsilon + \ldots,$$

which gives

$$(a_0 + a_1\epsilon + \dots)^3 + (a_0 + a_1\epsilon + \dots)^2 + \epsilon(a_0 + a_1\epsilon + \dots) + \epsilon^3 = 0.$$

Equating coefficients of powers of ϵ, we find (for the singular root only)

$$\begin{aligned} 1 :& a_0^3 + a_0^2 = 0 \Rightarrow a_0 = -1. \\ \epsilon :& 3a_0^2 a_1 + 2a_0 a_1 + a_0 = 0 \Rightarrow a_1 = -\frac{a_0}{3a_0^2 + 2a_0} = 1, \end{aligned}$$

(Recall that the repeated root near 0 corresponds with the two regular solutions we've already considered.) We conclude that our singular root satisfies

$$x(\epsilon) = -\frac{1}{\epsilon} + 1 + \mathbf{O}(\epsilon).$$

We have

$$x(\epsilon) = \begin{cases} -\epsilon + \mathbf{O}(\epsilon^2) \\ -1 + \mathbf{O}(\epsilon^2) \\ -\frac{1}{\epsilon} + 1 + \mathbf{O}(\epsilon) \end{cases}$$

3. Setting $\epsilon = 0$, we find $x(0)^3 = 0$, which indicates that all three roots are near 0. Since solutions to the $\epsilon = 0$ equation are repeated, we expect our scaling to be non-integer and make the substitution $x = \epsilon^p z$, where we suppose that $z(\epsilon)$ does not vanish as $\epsilon \to 0$. We have

$$\epsilon^{3p} z^3 + \epsilon^{2+p} z + \epsilon^2 = 0.$$

Observing that $\epsilon^{2+p} z$ cannot possibly cancel with ϵ^2, we conclude that $\epsilon^{3p} z^3$ and ϵ^2 must cancel, and consequently, we must have $3p = 2$, so that $p = 2/3$. Our equation becomes

$$z^3 + \epsilon^{2/3} z + 1 = 0.$$

Setting $\beta = \epsilon^{2/3}$, we have

$$z^3 + \beta z + 1 = 0.$$

Here, we take the regular expansion,

$$z(\epsilon) = a_0 + a_1\beta + \mathbf{O}(\beta^2),$$

which gives

$$(a_0 + a_1\beta)^3 + \beta(a_0 + a_1\beta) + 1 = 0.$$

Equating coefficients of powers of β, we have

$$\begin{aligned} 1 :& a_0^3 + 1 = 0 \Rightarrow a_0^3 = -1 \Rightarrow a_0 = -1e^{i\frac{2\pi n}{3}}, n = 0, 1, 2 \\ \beta :& 3a_0^2 a_1 + a_0 = 0 \Rightarrow a_1 = -\frac{1}{3a_0} = -\frac{1}{3}(-1e^{-i\frac{2\pi n}{3}}). \end{aligned}$$

Here, we have our choice of which solution to use, so we take the one associated with $n = 0$. We have

$$x(\epsilon) = -\epsilon^{2/3} + \frac{1}{3}\epsilon^{4/3} + \mathbf{O}(\epsilon^2).$$

4. We set $\tau = \lambda(\epsilon)t$ and $Y(\tau) = y(t)$ so that the equation becomes

$$\begin{aligned}\lambda(\epsilon)^2 Y'' + k^2 Y &= \epsilon Y(1-Y)\\ Y(0) &= 1\\ Y'(0) &= 0.\end{aligned}$$

Now look for solutions with

$$\begin{aligned}Y &= Y_0 + \epsilon Y_1 + \dots\\ \lambda(\epsilon) &= k + \lambda_1 \epsilon + \dots,\end{aligned}$$

for which we have

$$\begin{aligned}(k+\lambda_1\epsilon+\dots)^2(Y_0+\epsilon Y_1+\dots)'' + k^2(Y_0+\epsilon Y_1+\dots) &= \epsilon(Y_0+\epsilon Y_1+\dots)(1-(Y_0+\epsilon Y_1+\dots))\\ Y_0(0)+\epsilon Y_1(0)+\dots &= 1\\ Y_0'(0)+\epsilon Y_1'(0)+\dots &= 0.\end{aligned}$$

The ϵ^0 equation is

$$\begin{aligned}k^2 Y_0'' + k^2 Y_0 &= 0\\ Y_0(0) &= 1\\ Y_0'(0) &= 0,\end{aligned}$$

and so $Y_0(\tau) = \cos\tau$. The ϵ^1 equation is

$$k^2 Y_1'' + 2k\lambda_1 Y_0'' + k^2 Y_1 = Y_0(1-Y_0),$$

so that

$$\begin{aligned}k^2 Y_1'' + k^2 Y_1 = &-2k\lambda_1 Y_0'' + Y_0 - Y_0^2 = 2k\lambda_1\cos\tau + \cos\tau - \cos^2\tau\\ &= 2k\lambda_1\cos\tau + \cos\tau - \frac{1}{2} - \frac{1}{2}\cos 2\tau.\end{aligned}$$

We choose $\lambda_1 = -\frac{1}{2k}$, and so t

$$y(t) = \cos[(k - \frac{1}{2k}\epsilon)t] + \mathbf{O}(\epsilon),$$

where the error is uniform on intervals $|t| \le t_1/\epsilon$.

5. We define a slow variable $\tau = \epsilon t$ and look for solutions $Y(\tau, t) = y(t)$, for which this equation becomes

$$\begin{aligned}Y_{tt} + 2\epsilon Y_{t\tau} + \epsilon^2 Y_{\tau\tau} + k^2 Y &= \epsilon(Y_t + \epsilon Y_\tau)Y^2\\ Y(0,0) &= 1\\ Y_t(0,0) + \epsilon Y_\tau(0,0) &= 0.\end{aligned}$$

We now look for solutions

$$Y = Y_0 + \epsilon Y_1 + \dots,$$

for which we have

$$\begin{aligned}(Y_0 + \epsilon Y_1 + \dots)_{tt} + 2\epsilon(Y_0 + \epsilon Y_1 + \dots)_{t\tau} + \epsilon^2(Y_0 + \epsilon Y_1 + \dots)_{\tau\tau} + k^2(Y_0 + \epsilon Y_1 + \dots)\\ = \epsilon((Y_0 + \epsilon Y_1 + \dots)_t + \epsilon(Y_0 + \epsilon Y_1 + \dots)_\tau)(Y_0 + \epsilon Y_1 + \dots)^2\\ Y_0(0,0) + \epsilon Y_1(0,0) + \dots = 1\\ Y_{0_t}(0,0) + \epsilon Y_{1_t}(0,0) + \dots + \epsilon(Y_{0_\tau}(0,0) + \epsilon Y_{1_\tau}(0,0) + \dots) = 0.\end{aligned}$$

The ϵ^0 equation is

$$\begin{aligned}Y_{0_{tt}} + k^2 Y_0 &= 0\\ Y_0(0,0) &= 1\\ Y_{0_t}(0,0) &= 0,\end{aligned}$$

with solution

$$Y_0(t,\tau) = C_1(\tau)\cos kt + C_2(\tau)\sin kt.$$

Using the initial conditions, we find

$$\begin{aligned}C_1(0) &= 1\\ C_2(0) &= 0.\end{aligned}$$

We notice that $f(y, y') = y'y^2$ is not odd in y', so we can take $C_2(\tau) = 0$ for all values of τ. The ϵ^1 equation is

$$Y_{1_{tt}} + 2Y_{0_{t\tau}} + k^2 Y_1 = Y_{0_t} Y_0^2.$$

Here,

$$\begin{aligned}Y_{0_t}(t,\tau) &= -kC_1(\tau)\sin kt\\ Y_{0_{t\tau}}(t,\tau) &= -kC_1'(\tau)\sin kt,\end{aligned}$$

and so we have

$$\begin{aligned}Y_{1_{tt}} + k^2 Y_1 &= -2Y_{0_{t\tau}} + Y_{0_t} Y_0^2\\ &= 2kC_1'(\tau)\sin kt - kC_1(\tau)\sin kt C_1(\tau)^2\cos^2 kt\\ &= 2kC_1'(\tau)\sin kt - kC_1(\tau)^3[\frac{1}{4}\sin kt + \frac{1}{4}\sin 3kt],\end{aligned}$$

where I've used one of the inequalities that you'll have available on the exam (from Assignment 4). In order to eliminate secular terms we choose C_1 so that

$$2kC_1' - \frac{k}{4}C_1^3 = 0,$$

which is

$$\frac{dC_1}{dt} = \frac{1}{8}C_1^3; \quad C_1(0) = 1.$$

Solving this by separating variables we find

$$C_1(\tau) = \frac{2}{\sqrt{4-\tau}}.$$

We conclude that

$$Y_0(t,\tau) = \frac{2}{\sqrt{4-\tau}} \cos kt,$$

so

$$y_0(t) = \frac{2}{\sqrt{4-\epsilon t}} \cos kt.$$

6. For the outer expansion we only require $y = y_0 + \ldots$, and we find

$$(1+t)y_0' + y_0 = 0,$$

for which (by separating variables or using an integrating factor)

$$y_0(t) = \frac{C_1}{1+t}.$$

For the inner expansion, we set $\tau = \frac{t}{\epsilon}$ and $Y(\tau) = y(t)$, for which the equation becomes

$$\begin{aligned} Y'' + (1+\epsilon\tau)Y' + \epsilon Y &= 0 \\ Y(0) &= 0 \\ Y'(0) &= 1. \end{aligned}$$

Again, to zeroth order, an expansion only requires $Y = Y_0 + \ldots$, and we find

$$\begin{aligned} Y_0'' + Y_0' &= 0 \\ Y_0(0) &= 0 \\ Y_0'(0) &= 1, \end{aligned}$$

so that

$$Y_0(\tau) = 1 - e^{-\tau}.$$

The outer solution in the inner variable is

$$y = \frac{C_1}{1+\epsilon\tau} = C_1 + \mathbf{O}(\epsilon),$$

while the inner solution in the outer variable is

$$Y = 1 - e^{-\frac{t}{\epsilon}} = 1 + \mathbf{O}(\epsilon),$$

since $e^{-\frac{t}{\epsilon}}$ is transcendentally small for t away from the initial layer. Matching, we have $C_1 = 1$, so the zeroth order composite solution is

$$y_c(t) = \frac{1}{1+t} + (1 - e^{-\frac{t}{\epsilon}}) - 1 = \frac{1}{1+t} - e^{-\frac{t}{\epsilon}}.$$

7. First, we note that $b(x) = 1 > 0$, so the boundary layer will be on the left, near $x = 0$. For the outer expansion we have

$$y = y_0 + \epsilon y_1 + \dots,$$

so that

$$\begin{aligned} \epsilon(y_0 + \epsilon y_1 + \dots)'' + (y_0 + \epsilon y_1 + \dots)' + x^3(y_0 + \epsilon y_1 + \dots) &= 0 \\ y_0(1) + \epsilon y_1(1) + \dots &= 1. \end{aligned}$$

The ϵ^0 equation is

$$\begin{aligned} y_0' + x^3 y_0 &= 0 \\ y_0(1) &= 1, \end{aligned}$$

with solution

$$y_0(x) = e^{\frac{1}{4}} e^{-\frac{x^4}{4}}.$$

The ϵ^1 equation is

$$\begin{aligned} y_0'' + y_1' + x^3 y_1 &= 0 \\ y_1(1) &= 0, \end{aligned}$$

and we have

$$y_0''(x) = e^{\frac{1}{4}} e^{-\frac{x^4}{4}} x^6 - 3e^{\frac{1}{4}} x^2 e^{-\frac{x^4}{4}}.$$

Our equation becomes

$$y_1' + x^3 y_1 = -e^{\frac{1}{4}} e^{-\frac{x^4}{4}} x^6 + 3e^{\frac{1}{4}} x^2 e^{-\frac{x^4}{4}},$$

and so using an integrating factor

$$(e^{\frac{x^4}{4}} y_1)' = -e^{\frac{1}{4}} x^6 + 3e^{\frac{1}{4}} x^2,$$

and so

$$y_1(x) = e^{\frac{1}{4}}(-\frac{x^7}{7} + x^3)e^{-\frac{x^4}{4}} + C_2 e^{-\frac{x^4}{4}}.$$

Setting $y_1(1) = 0$ we have

$$0 = \frac{6}{7} + C_2 e^{-\frac{1}{4}} \Rightarrow C_2 = -\frac{6}{7} e^{\frac{1}{4}}.$$

We see that

$$y_1(x) = e^{\frac{1}{4}}(x^3 - \frac{1}{7}x^7)e^{-\frac{x^4}{4}} - \frac{6}{7} e^{\frac{1}{4}} e^{-\frac{x^4}{4}},$$

and so the outer solution is

$$y(x) = e^{\frac{1}{4}} e^{-\frac{x^4}{4}} + \epsilon(e^{\frac{1}{4}}(x^3 - \frac{1}{7}x^7)e^{-\frac{x^4}{4}} - \frac{6}{7} e^{\frac{1}{4}} e^{-\frac{x^4}{4}}) + \dots$$

For the inner solution, we set $\xi = \frac{x}{\epsilon}$ and $Y(\xi) = y(x)$, and the equation becomes

$$\begin{aligned} Y'' + Y' + \epsilon^4 \xi^3 Y &= 0 \\ Y(0) &= 0. \end{aligned}$$

We look for solutions

$$Y = Y_0 + \epsilon Y_1 + \dots,$$

for which we have

$$\begin{aligned} (Y_0 + \epsilon Y_1 + \dots)'' + (Y_0 + \epsilon Y_1 + \dots)' + \epsilon^4 \xi^3 (Y_0 + \epsilon Y_1 + \dots) &= 0 \\ Y_0(0) + \epsilon Y_1(0) + \dots &= 0. \end{aligned}$$

The ϵ^0 equation is

$$\begin{aligned} Y_0'' + Y_0' &= 0 \\ Y_0(0) &= 0, \end{aligned}$$

with solution

$$Y_0(\xi) = C_1 - C_1 e^{-\xi}.$$

The ϵ^1 equation is exactly the same, with the same solution, except with a possibly different constant

$$Y_1(\xi) = C_2 - C_2 e^{-\xi}.$$

The inner solution is

$$Y(\xi) = C_1 - C_1 e^{-\xi} + \epsilon(C_2 - C_2 e^{-\xi}) + \dots.$$

Now, we write the outer expansion in the inner variable

$$\begin{aligned} y &= e^{\frac{1}{4}} e^{-\frac{\epsilon^4 \xi^4}{4}} + \epsilon(e^{\frac{1}{4}}(\epsilon^3 \xi^3 - \frac{\epsilon^7 \xi^7}{7}) e^{-\frac{\epsilon^4 \xi^4}{4}} - \frac{6}{7} e^{\frac{1}{4}} e^{-\frac{\epsilon^4 \xi^4}{4}}) \\ &= e^{\frac{1}{4}} - \epsilon \frac{6}{7} e^{\frac{1}{4}} + \dots, \end{aligned}$$

where I've used the Taylor expansion

$$e^{-\frac{\epsilon^4 \xi^4}{4}} = 1 - \frac{\epsilon^4 \xi^4}{4} + \mathbf{O}(\xi^8),$$

and we write the inner expansion in the outer variable

$$\begin{aligned} Y &= C_1 - C_1 e^{-\frac{x}{\epsilon}} + \epsilon(C_2 - C_2 e^{-\frac{x}{\epsilon}}) + \dots \\ &= C_1 + \epsilon C_2 + \dots, \end{aligned}$$

where we have observed that the terms $e^{-\frac{x}{\epsilon}}$ are transcendentally small away from the boundary layer. Matching, we find

$$C_1 = e^{\frac{1}{4}}; \quad C_2 = -\frac{6}{7} e^{\frac{1}{4}},$$

and so the composite solution is

$$\begin{aligned} y^c(x) &= e^{\frac{1}{4}}e^{-\frac{x^4}{4}} + \epsilon(e^{\frac{1}{4}}(x^3 - \frac{x^7}{7})e^{-\frac{x^4}{4}} - \frac{6}{7}e^{\frac{1}{4}}e^{-\frac{x^4}{4}}) \\ &+ e^{\frac{1}{4}} - e^{\frac{1}{4}}e^{-\frac{x}{\epsilon}} + \epsilon(-\frac{6}{7}e^{\frac{1}{4}} + \frac{6}{7}e^{\frac{1}{4}}e^{-\frac{x}{\epsilon}}) - e^{\frac{1}{4}} + \epsilon\frac{6}{7}e^{\frac{1}{4}} \\ &= e^{\frac{1}{4}}e^{-\frac{x^4}{4}} - e^{\frac{1}{4}}e^{-\frac{x}{\epsilon}} \\ &+ \epsilon\Big(e^{\frac{1}{4}}(x^3 - \frac{x^7}{7})e^{-\frac{x^4}{4}} - \frac{6}{7}e^{\frac{1}{4}}e^{-\frac{x^4}{4}} + \frac{6}{7}e^{\frac{1}{4}}e^{-\frac{x}{\epsilon}}\Big). \end{aligned}$$

8. First, $b(x) = \cos x > 0$ on this interval, so the boundary layer will be on the left, at $x = 0$. Since we only need the zeroth approximation we have simply $y = y_0 + \ldots$ for the outer solution, and so

$$\begin{aligned} (\cos x)y_0' + (\sin x)y &= 0 \\ y_0(\frac{\pi}{4}) &= 1, \end{aligned}$$

with solution (by separation of variables or an integrating factor) $y_0(x) = \sqrt{2}\cos x$. For the inner solution, we scale with $\xi = \frac{x}{\epsilon}$, and our equation becomes

$$\begin{aligned} Y'' + (\cos\epsilon\xi)Y' + \epsilon(\sin\epsilon\xi)Y &= 0 \\ Y(0) &= 0. \end{aligned}$$

Here, we need to remember that the Taylor polynomials for cosine and sine are

$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2} + \ldots \\ \sin x &= x - \frac{x^3}{6} + \ldots, \end{aligned}$$

and so

$$\begin{aligned} \cos(\epsilon\xi) &= 1 + \mathbf{O}(\epsilon^2) \\ \sin(\epsilon\xi) &= \mathbf{O}(\epsilon). \end{aligned}$$

Note the following shortcut: For any function with a continuous first derivative,

$$f(\epsilon\xi) = f(0) + \mathbf{O}(\epsilon),$$

and this is actually all that's required here.

Setting $Y = Y_0 + \ldots$, we see that Y_0 solves

$$\begin{aligned} Y_0'' + Y_0' &= 0 \\ Y_0(0) &= 0, \end{aligned}$$

with solution $Y_0(\xi) = C_1 - C_1 e^{-\xi}$. In this case, the outer solution in the inner variable is

$$y = \sqrt{2}\cos\epsilon\xi = \sqrt{2} + \ldots,$$

while the inner solution in the outer variable is

$$Y = C_1 - C_1 e^{-\frac{x}{\epsilon}} = C_1 + \dots.$$

Clearly, $C_1 = \sqrt{2}$, and our zeroth order composite approximation is

$$y^c(x) = \sqrt{2}\cos x + \sqrt{2} - \sqrt{2}e^{-\frac{x}{\epsilon}} - \sqrt{2} = \sqrt{2}(\cos x - e^{-\frac{x}{\epsilon}}).$$

9. In this case $b(x) = -\cos x < 0$ on this interval, so the boundary layer will be on the right, at $x = \frac{\pi}{4}$. The zeroth order outer solution $y_0(x)$ solves

$$\begin{aligned} -(\cos x)y_0' + (\sin x)y_0 &= 0 \\ y_0(0) &= 1, \end{aligned}$$

(keeping in mind that $x = 0$ is now the outer boundary) with solution $y_0(x) = \frac{1}{\cos x}$.

For the inner solution, we use the scaled variable $\xi = \frac{\frac{\pi}{4} - x}{\epsilon}$ and set $Y(\xi) = y(x)$, so that (recall the sign change in Y')

$$\begin{aligned} Y'' + \cos(\frac{\pi}{4} - \epsilon\xi)Y' + \epsilon\sin(\frac{\pi}{4} - \epsilon\xi)Y &= 0 \\ Y(0) &= 0 \end{aligned}$$

(keeping in mind that $Y(0) = y(\frac{\pi}{4})$). Recalling the shortcut from the solution to Problem 8, we use

$$\cos(\frac{\pi}{4} - \epsilon\xi) = \cos\frac{\pi}{4} + \mathbf{O}(\epsilon),$$

and so Y_0 solves

$$\begin{aligned} Y_0'' + \frac{1}{\sqrt{2}}Y_0' &= 0 \\ Y_0(0) &= 0, \end{aligned}$$

with solution

$$Y_0(\xi) = C_1 - C_1 e^{-\frac{1}{\sqrt{2}}\xi}.$$

The outer solution in the inner variable is

$$y = \frac{1}{\cos(\frac{\pi}{4} - \epsilon\xi)} = \sqrt{2} + \mathbf{O}(\epsilon),$$

while the inner solution in the outer variable is

$$Y = C_1 - C_1 e^{-\frac{1}{\sqrt{2}}\frac{\frac{\pi}{4} - x}{\epsilon}} = C_1 + \mathbf{O}(\epsilon).$$

Clearly, $C_1 = \sqrt{2}$, and the zeroth order composite solution is

$$y^c(x) = \frac{1}{\cos x} + \sqrt{2} - \sqrt{2}e^{-\frac{1}{\sqrt{2}}\frac{\frac{\pi}{4} - x}{\epsilon}} - \sqrt{2} = \frac{1}{\cos x} - \sqrt{2}e^{-\frac{1}{\sqrt{2}}\frac{\frac{\pi}{4} - x}{\epsilon}}.$$