M412 Practice Problems for Exam 3

Qualititative properties of the Laplace equation

1. For the Laplace equation

$$\begin{aligned} u_{xx} + u_{yy} &= 0; \quad 0 \le x \le 1, \quad 0 \le y \le 1, \\ u(x,0) = 10x(1-x); &\quad u(x,1) = 1-x \\ u(0,y) = 10y(1-y); &\quad u(1,y) = 1-y, \end{aligned}$$

find upper and lower bounds on $u(x,y)$ in the square $(x,y) \in [0,1] \times [0,1]$.

2. For the Laplace equation

$$\begin{aligned} u_{xx} + u_{yy} &= 0; \quad 0 \le x \le \pi, \quad 0 \le y \le \pi \\ u_y(x,0) = \sin x; &\quad u_y(x,1) = \sin x \\ u(0,y) = 1; &\quad u(1,y) = 0, \end{aligned}$$

compute

$$\int_0^\pi \int_0^\pi u(x,y) dx dy.$$

Well-posedness

3. Show that solutions to the heat equation

$$\begin{aligned} u_t &= u_{xx} \\ u(t,0) &= 0 \\ u(t,L) &= 0 \\ u(0,x) &= f(x) \end{aligned}$$

are *stable* with respect to small changes in $f(x)$. Specify any assumptions you are making on $f(x)$ and small changes to $f(x)$.

Convergence, integration, and differentiation of Fourier series

4. Show that if $f(x)$ is piecewise continuous on the interval $[-L, L]$, then

$$\int_{-L}^{+L} f(x)^2 dx = 2LA_0^2 + L \sum_{n=1}^{\infty} (A_n^2 + B_n^2),$$

where

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L}.$$

5. Suppose $f(x)$ is piecewise smooth on the interval $[0, L]$. Prove that the Fourier sine series of $f(x)$ can be integrated term-by-term on $y \in [0, x]$ for all $x \in [0, L]$.

The method of eigenfunction expansion

6. Haberman 8.3.6.

7. Haberman 8.3.7.

8. Use the method of eigenfunction expansion to solve the wave equation

$$\begin{aligned} u_{tt} &= u_{xx}, \quad 0 < x < 1, t > 0 \\ u_x(t,0) &= 0 \\ u_x(t,1) &= 0 \\ u(0,x) &= f(x) \\ u_t(0,x) &= g(x). \end{aligned}$$

(This problem was on the previous practice exam as well, and can be solved by separation of variables. Solve it here with the method of eigenfunction expansion.)

9. Solve the Laplace equation

$$\begin{aligned} u_{xx} + u_{yy} = &- \pi^2 \sin \pi x; \quad 0 \le x \le 1, \quad 0 \le y \le 2, \\ u(x,0) &= 0; \quad u(x,2) = 0 \\ u(0,y) &= 2\sin \pi y; \quad u(1,y) = \sin \frac{\pi y}{2}. \end{aligned}$$

Fourier Transforms

10. Show that solutions to the quarter-plane problem

$$\begin{aligned} u_t &= u_{xx} \\ u(t,0) &= 0 \\ |u(t,\infty)| &\text{ bounded} \\ u(0,x) &= f(x), \end{aligned}$$

can be written in the form

$$u(t,x) = \int_0^\infty 2\omega C(\omega) e^{-\omega^2 t} \sin \omega x d\omega,$$

for some appropriate constant $C(\omega)$.

Solutions.

1. By the maximum principle, $0 \le u(x,y) \le \frac{5}{2}$.

2. $\int_0^\pi \int_0^\pi u(x,y) dxdy = \pi^2 - \frac{\pi^3}{2}$.

3. Let $w(t,x)$ solve the same problem with slightly different changed $f(x)$: $w(0,x) = f(x) + \epsilon h(x)$, where $f(x)$ and $h(x)$ are both assumed continuous with piecewise continuous first derivatives (this will guarantee

not only that we have solutions, but that they are continuous as $t \to 0$). The variable $v = w - u$ solves

$$\begin{aligned} v_t &= v_{xx} \\ v(t,0) &= 0 \\ v(t,L) &= 0 \\ v(0,x) &= \epsilon h(x). \end{aligned}$$

Solving for $v(t,x)$, we have the Fourier sine series

$$v(t,x) = \sum_{n=1}^{\infty} B_n e^{-\frac{n^2\pi^2}{L^2}t} \sin\frac{n\pi x}{L},$$

where

$$B_n = \frac{2}{L}\int_0^L \epsilon h(x) \sin\frac{n\pi x}{L} dx.$$

Setting

$$\bar{B}_n = \frac{2}{L}\int_0^L h(x) \sin\frac{n\pi x}{L} dx,$$

we see that

$$v(t,x) = \epsilon \sum_{n=1}^{\infty} \bar{B}_n e^{-\frac{n^2\pi^2}{L^2}t} \sin\frac{n\pi x}{L},$$

where according to our assumptions this sum converges. Therefore the smaller we take ϵ to be, the smaller $v(t,x)$ will be.

4. Compute directly,

$$\int_{-L}^{+L} f(x)^2 dx = \int_{-L}^{+L} (A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L})^2 dx,$$

and use the orthogonality relations for integrals over sine and cosine.

5. Set $F(x) = \int_0^x f(y)dy$ and show that the Fourier cosine series of $F(x)$ is precisely the term-by-term anti-derivative of $f(y)$.

6.

$$u(t,x) = (1 - \frac{x}{\pi}) + \sum_{n=1}^{\infty} c_n(t) \sin nx,$$

where

$$c_n(t) = \begin{cases} -\frac{2}{n\pi} e^{-n^2 t}, & n \neq 5 \\ \frac{1}{23} e^{-2t} - (\frac{1}{23} + \frac{2}{5\pi}) e^{-25t}, & n = 5. \end{cases}$$

7.

$$u(t,x) = \frac{x}{L}t + \sum_{n=1}^{\infty} (-1)^n \frac{2L^2}{n^3\pi^3} (1 - e^{-\frac{n^2\pi^2}{L^2}t}) \sin\frac{n\pi x}{L}.$$

8.

$$u(t,x) = A_0 + B_0 t + \sum_{n=1}^{\infty} (A_n \cos n\pi t + B_n \sin n\pi t) \cos n\pi x,$$

where

$$A_0 = \int_0^1 f(x)dx$$
$$A_n = 2\int_0^1 f(x)\cos n\pi x dx$$
$$B_0 = \int_0^1 g(x)dx$$
$$B_n = \frac{2}{n\pi}\int_0^1 g(x)\cos n\pi x dx.$$

9.

$$u(x,y) = \frac{\sinh \frac{\pi x}{2}}{\sinh \frac{\pi}{2}} \sin \frac{\pi y}{2} + \left(1 + \cosh \pi x - (\frac{1}{\sinh \pi} + \frac{\cosh \pi}{\sinh \pi}) \sinh \pi x\right) \sin \pi y.$$

10. The solution is in the statement of the problem.