# Assignment 1, Solutions

1. [5 pts] We solve this problem by separation of variables. We have

$$\frac{dy}{y^2+1} = 3x^2dx \Rightarrow \int \frac{dy}{y^2+1} = \int 3x^2dx \Rightarrow \tan^{-1} y = x^3 + C \Rightarrow y(x) = \tan(x^3 + C).$$

Using the initial condition $y(0) = 1$, we have $1 = \tan(C)$, so that $C = \frac{\pi}{4} \pm k\pi$, $k = 1, 2, ....$ By the periodicity of $\tan(x)$, each value of $C$ gives the same representation, so without loss of generality we take $C = \frac{\pi}{4}$. Hence,

$$y(x) = \tan(x^3 + \frac{\pi}{4}).$$

The domain of $\tan(x)$ containing $x = 0$ is $(-\frac{\pi}{2}, \frac{\pi}{2})$, so our domain is described by the inequality

$$-\frac{\pi}{2} < x^3 + \frac{\pi}{4} < \frac{\pi}{2} \Rightarrow -\frac{3\pi}{4} < x^3 < \frac{\pi}{4} \Rightarrow (-\frac{3\pi}{4})^{1/3} < x < (\frac{\pi}{4})^{1/3}.$$

2. [5 pts] We solve this problem by the integrating factor method. For

$$\mu(x) = e^{\int \frac{3x}{x^2+1} dx} = (x^2 + 1)^{3/2},$$

we have

$$(x^2 + 1)^{3/2} y(x) = \int 6x\sqrt{x^2 + 1}dx = 2(x^2 + 1)^{3/2} + C \Rightarrow y(x) = \frac{C}{(x^2 + 1)^{3/2}} + 2.$$

3. [5 pts] The general solution for $y''(x) + 3y(x) = 0$ is $y(x) = C_1 \sin \sqrt{3}x + C_2 \cos \sqrt{3}x$. Setting $y(0) = 0$, we have $0 = C_2$, while setting $y(\pi) = 0$, we have $0 = C_1 \sin \sqrt{3}\pi$. Since $\sin \sqrt{3}\pi \neq 0$, we must have $C_1 = 0$. Consequently, $C_1$ and $C_2$ are both 0, and $y(x) \equiv 0$ is the only possible solution.

4. [5 pts] The general solution for $y''(x) + 4y = 0$ is $y(x) = C_1 \sin 2x + C_2 \cos 2x$. Setting $y(0) = 0$, we find that $C_2 = 0$, but setting $y(\pi) = 0$, we have $0 = C_1 \sin 2\pi$, which is satisfied for any value of $C_1$. Consequently, we have an infinite number of solutions, $y(x) = C_1 \sin 2x$, for any $C_1$.

5. [10 pts] For $\lambda > 0$, the general solution for $y''(x) + \lambda y(x) = 0$ is $y(x) = C_1 \sin \sqrt{\lambda}x + C_2 \cos \sqrt{\lambda}x$. Setting $y(0) = 0$, we have $0 = C_2$, while setting $y(\pi) = 0$, we have $0 = C_1 \sin \sqrt{\lambda}\pi$. Hence, we have a nontrivial solution if and only if $\sin \sqrt{\lambda}\pi = 0$, or $\sqrt{\lambda}\pi = n\pi$, for $n = 1, 2, ....$ Therefore, the positive eigenvalues are $\lambda = n^2$, $n = 1, 2, 3, ....$

For $\lambda = 0$, the general solution is $y(x) = C_1x + C_2$, for which $y(0) = 0 \Rightarrow C_2 = 0$ and $y(\pi) = 0 \Rightarrow C_1 = 0$. So, $\lambda = 0$ is *not* an eigenvalue.

For $\lambda < 0$, the general solution is $y(x) = C_1e^{\sqrt{-\lambda}x} + C_2e^{\sqrt{\lambda}x}$. Again, we find $C_1 = C_2 = 0$, so that no $\lambda < 0$ are eigenvalues.