M412 Assignment 2, due Friday September 9

1. [10 pts] Use the method of diagonalization to determine a general solution for the ODE system

$$\begin{aligned} y_1' &= -\,y_1 + \frac{3}{4} y_2 \\ y_2' &= -\,5y_1 + 3y_2. \end{aligned}$$

Solution. In matrix form, this equation has the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} -1 & \frac{3}{4} \\ -5 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

or $y' = Ay$, with

$$A = \begin{pmatrix} -1 & \frac{3}{4} \\ -5 & 3 \end{pmatrix}.$$

The eigenvalues of A can be computed from the relation

$$\det \begin{pmatrix} -1-\mu & \frac{3}{4} \\ -5 & 3-\mu \end{pmatrix} = 0 \Rightarrow \mu_1 = \frac{1}{2}, \mu_2 = \frac{3}{2}.$$

For the eigenvalue μ_1, we find the associated eigenvector $V_1 = \begin{pmatrix} a \\ b \end{pmatrix}$ by solving

$$\begin{pmatrix} -1 & \frac{3}{4} \\ -5 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{2} \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow -a + \frac{3}{4} b = \frac{1}{2} a.$$

Setting $a = 1$, we have $b = 2$. Proceeding similarly for the eigenvalue $\mu_2 = 3/2$, we conclude

$$V_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \quad V_2 = \begin{pmatrix} 1 \\ \frac{10}{3} \end{pmatrix}.$$

Setting

$$P = \begin{pmatrix} 1 & 1 \\ 2 & \frac{10}{3} \end{pmatrix},$$

we make the change of variables $y = Pz$, for which

$$Pz' = APz \Rightarrow z' = P^{-1}APz = Dz,$$

where

$$D = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}.$$

This diagonal system for z is trivial to solve, and we find

$$\begin{aligned} z_1(t) &= C_1 e^{\frac{1}{2}t} \\ z_2(t) &= C_2 e^{\frac{3}{2}t}. \end{aligned}$$

Finally, computing $y = Pz$, we determine

$$\begin{aligned} y_1(t) &= C_1 e^{\frac{1}{2}t} + C_2 e^{\frac{3}{2}t} \\ y_2(t) &= 2C_1 e^{\frac{1}{2}t} + \frac{10}{3} e^{\frac{3}{2}t}. \end{aligned}$$

2. [10 pts] Use the method of characteristics to solve the PDE

$$u_t + t^2 u_x = 0$$
$$u(0,x) = e^{-x^2}.$$

Solution. Taking $U(t) = u(t, x(t))$ with $x'(t) = t^2$ and initial condition $x(0) = x_0$, we have

$$\frac{d}{dt}U(t) = 0; \quad U(0) = u(0, x_0) = e^{-x_0^2} \Rightarrow u(t, x(t)) = e^{-x_0^2}.$$

Solving for $x(t)$, we have $x(t) = \frac{1}{3}t^3 + x_0$, so that $x_0 = x - \frac{1}{3}t^3$. We conclude

$$u(t,x) = e^{-(x-\frac{1}{3}t^3)^2}.$$

3. [10 pts] Use the method of characteristics to solve the PDE

$$u_t + 2u_x = t$$
$$u(0,x) = 1 - x.$$

Solution. Taking $x'(t) = 2$, we have

$$\frac{d}{dt}U(t) = t \Rightarrow U(t) = \frac{t^2}{2} + u(0, x_0).$$

Solving $x' = 2;\ x(0) = x_0$, we have, $x(t) = 2t + x_0$, so that $x_0 = x - 2t$, and consequently

$$u(t,x) = \frac{t^2}{2} + 1 - (x - 2t).$$

4. [10 pts] Use the method of characterstics to solve the PDE

$$u_t + u_x + 2tu^2 = 0$$
$$u(0,x) = e^{-x}.$$

Solution. Taking $x'(t) = 1$, we have

$$\frac{d}{dt}U(t) = -2tU^2 \Rightarrow \int \frac{dU}{U^2} = -\int 2t dt \Rightarrow -\frac{1}{U} = -t^2 + C \Rightarrow U(t) = \frac{1}{t^2 + u(0,x_0)^{-1}}.$$

Solving $x' = 1,\ x(0) = x_0$, we have $x(t) = t + x_0$, so that $x_0 = x - t$, and we conclude

$$u(t,x) = \frac{1}{t^2 + e^{x-t}}.$$

5. [10 pts] Use the method of characteristics to solve the PDE

$$u_t + 2u_x = x^2$$
$$u(t,0) = t^2, \quad t > 0$$
$$u(0,x) = x, \quad x > 0.$$

Solution. Taking $x'(t) = 2$, we see that the analysis divides in the two cases: (1) $x \geq 2t$ (which uses $u(0, x)$) and (2) $x \leq 2t$ (which uses $u(t, 0)$). For $x \geq 2t$, we have

$$\begin{aligned} x'(t) &= 2; \quad x(0) = x_0 \Rightarrow x(t) = 2t + x_0 \\ U'(t) &= x(t)^2 = (2t + x_0)^2; \quad U(0) = x_0 \Rightarrow U(t) = \frac{1}{6}(2t + x_0)^3 - \frac{1}{6}(x_0^3 - 6x_0). \end{aligned}$$

Substituting $x_0 = x - 2t$, we conclude

$$u(t, x) = \frac{1}{6}x^3 - \frac{1}{6}(x - 2t)^3 + (x - 2t), \quad x \geq 2t.$$

On the other hand, for $x \leq 2t$, we have

$$\begin{aligned} x'(t) &= 2; \quad x(t_0) = 0 \Rightarrow x(t) = 2t - 2t_0 \\ U'(t) &= x(t)^2 = 4(t - t_0)^2; \quad U(t_0) = t_0^2 \Rightarrow U(t) = \frac{4}{3}(t - t_0)^3 + t_0^2. \end{aligned}$$

Substituting $t_0 = t - x/2$, we conclude

$$u(t, x) = \frac{4}{3}(\frac{x}{2})^3 + (t - \frac{x}{2})^2.$$

Altogether,

$$u(t, x) = \begin{cases} \frac{1}{6}x^3 + (t - \frac{x}{2})^2, & x \leq 2t \\ \frac{1}{6}x^3 - \frac{1}{6}(x - 2t)^3 + (x - 2t), & x \geq 2t. \end{cases}$$