M412 Assignment 3 Solutions

1. [10 pts] Use the method of characteristics to solve the PDE

$$\begin{aligned} u_x - u_y + 2y &= 0 \\ u(x,y) &= xy \text{ on the line } x + 2y = 1. \end{aligned}$$

Solution. In this case, set $U(t) = u(x(t), y(t))$ and choose

$$\begin{aligned} \frac{dx}{dt} &= 1; \quad x(0) = x_0 \Rightarrow x = t + x_0 \\ \frac{dy}{dt} &= -1; \quad y(0) = y_0 \Rightarrow y = -t + y_0 \\ \frac{dU}{dt} &= -2y(t) = 2(t - y_0); \quad U(0) = x_0 y_0 \Rightarrow U(t) = (t - y_0)^2 + x_0 y_0 - y_0^2. \end{aligned}$$

Using $x_0 + 2y_0 = 1$ and the expressions above to eliminate t, x_0, and y_0, we find

$$u(x,y) = (1 - x - y)(-2 + 3x + 3y) + y^2.$$

2. [10 pts] For the PDE

$$\begin{aligned} u_t + f(u)_x &= 0 \\ u(0,x) &= g(x), \end{aligned}$$

use the method of characteristics to show that solutions satisfy the implicit relationship

$$u(t,x) = g(x - f'(u(t,x))t).$$

Solution. First, use the chain rule to re-write this in quasilinear form,

$$u_t + f'(u)u_x = 0.$$

Next, set $U(t) = u(t, x(t))$ and choose

$$\begin{aligned} \frac{dx}{dt} &= f'(U); \quad x(0) = x_0 \\ \frac{dU}{dt} &= 0; \quad U(0) = g(x_0). \end{aligned}$$

From the second of these equations, we see that $U(t)$ is constant, with $U(t) = g(x_0)$ for all t. Substituting this back into the first equation, we have

$$x(t) = f'(g(x_0))t + x_0 \Rightarrow x_0 = x - f'(g(x_0))t = x - f'(U(t))t.$$

We conclude

$$u(t,x) = g(x_0) = g(x - f'(u(t,x))t).$$

3. [20 pts] Use the methods of characteristics and diagonalization to solve the PDE system

$$\begin{aligned} u_{1_t} - u_{1_x} - u_{2_x} &= 0, \quad u_1(0,x) = f(x) \\ u_{2_t} - u_{1_x} &= 0, \quad u_2(0,x) = g(x). \end{aligned}$$

Solution. We write this in system notation

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_x.$$

The matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

has the eigenvalue-eigenvector pairs

$$\frac{1-\sqrt{5}}{2}, \begin{pmatrix} 1 \\ \frac{2}{1-\sqrt{5}} \end{pmatrix} \quad \text{and} \quad \frac{1+\sqrt{5}}{2}, \begin{pmatrix} 1 \\ \frac{2}{1+\sqrt{5}} \end{pmatrix}.$$

In order to diagonalize the system, we construct

$$P = \begin{pmatrix} 1 & 1 \\ \frac{2}{1-\sqrt{5}} & \frac{2}{1+\sqrt{5}} \end{pmatrix}.$$

Introducing the variable transformation

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = P \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \tag{1}$$

we find

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}_t = \begin{pmatrix} \frac{2}{1-\sqrt{5}} & 0 \\ 0 & \frac{2}{1+\sqrt{5}} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}_x,$$

from which we have

$$\begin{aligned} w_{1_t} - \frac{1-\sqrt{5}}{2} w_{1_x} &= 0; \quad w_1(0,x) = h(x) \\ w_{2_t} - \frac{1+\sqrt{5}}{2} w_{2_x} &= 0; \quad w_2(0,x) = k(x), \end{aligned}$$

where the functions $h(x)$ and $k(x)$ will be determined in terms of $f(x)$ and $g(x)$. Solving each of these equations by the method of characteristics, we find

$$\begin{aligned} w_1(t,x) &= h(x + \frac{1-\sqrt{5}}{2} t) \\ w_2(t,x) &= k(x + \frac{1+\sqrt{5}}{2}). \end{aligned}$$

Returning through (1) to variables u_1 and u_2, we conclude

$$\begin{aligned} u_1(t,x) &= h(x + \frac{1-\sqrt{5}}{2} t) + k(x + \frac{1+\sqrt{5}}{2}) \\ u_2(t,x) &= \frac{2}{1-\sqrt{5}} h(x + \frac{1-\sqrt{5}}{2} t) + \frac{2}{1+\sqrt{5}} k(x + \frac{1+\sqrt{5}}{2}). \end{aligned}$$

Finally, we solve for $h(x)$ and $k(x)$ in terms of $f(x)$ and $g(x)$, with

$$\begin{aligned} f(x) &= h(x) + k(x) \\ g(x) &= \frac{2}{1-\sqrt{5}} h(x) + \frac{2}{1+\sqrt{5}} k(x). \end{aligned}$$

We find

$$h(x) = \frac{3-\sqrt{5}}{5-\sqrt{5}}f(x) - \frac{1}{\sqrt{5}}g(x)$$
$$k(x) = \frac{2}{5-\sqrt{5}}f(x) + \frac{1}{\sqrt{5}}g(x).$$

Finally,

$$u_1(t,x) = \frac{3-\sqrt{5}}{5-\sqrt{5}}f(x+\frac{1-\sqrt{5}}{2}t) - \frac{1}{\sqrt{5}}g(x+\frac{1-\sqrt{5}}{2}t) + \frac{2}{5-\sqrt{5}}f(x+\frac{1+\sqrt{5}}{2}t) + \frac{1}{\sqrt{5}}g(x+\frac{1+\sqrt{5}}{2}t)$$
$$u_2(t,x) = \frac{2}{1-\sqrt{5}}\Big[\frac{3-\sqrt{5}}{5-\sqrt{5}}f(x+\frac{1-\sqrt{5}}{2}t) - \frac{1}{\sqrt{5}}g(x+\frac{1-\sqrt{5}}{2}t)\Big]$$
$$+ \frac{2}{1+\sqrt{5}}\Big[\frac{2}{5-\sqrt{5}}f(x+\frac{1+\sqrt{5}}{2}t) + \frac{1}{\sqrt{5}}g(x+\frac{1+\sqrt{5}}{2}t)\Big].$$

4. [10 pts] Haberman Problem 12.4.4.

Solution.

(a) Proceeding as in class, we know that solutions must have the form

$$u(t,x) = F(x-ct) + G(x+ct)$$

for some functions $F(x)$ and $G(x)$. For $x > 0$, we can proceed as in class to find

$$F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(y)dy$$
$$G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(y)dy, \tag{2}$$

where we have dropped off the constants of integration that cancel upon adding F and G. The issue is to determine the behavior of $F(x)$ and $G(x)$ for $x < 0$. We accomplish this by observing that our boundary condition $u_x(t,0) = 0$ gives that for $x < 0$

$$F'(-ct) + G'(ct) = 0 \Rightarrow F'(x) = -G'(-x).$$

In this way,

$$\int_0^x F'(y)dy = -\int_0^x G'(-x)dx \Rightarrow F(x) - F(0) = G(-x) - G(0),$$

or

$$F(x) = G(-x).$$

(It's clear from (2) that $F(0)$ and $G(0)$ cancel.) For $x - ct > 0$, the arguments of F and G are both positive, and we have

$$u(t,x) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy,$$

that is, the usual d'Alembert solution. For $x - ct < 0$, we have, rather,

$$\begin{aligned}
u(t,x) &= F(x-ct) + G(x+ct) \\
&= G(ct-x) + G(x+ct) \\
&= \frac{1}{2}f(ct-x) + \frac{1}{2c}\int_0^{ct-x} g(y)dy + \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(y)dy \\
&= \frac{1}{2}[f(ct-x) + f(x+ct)] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(y)dy.
\end{aligned}$$

Altogether, we have

$$u(t,x) = \begin{cases} \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy, & x - ct > 0 \\ \frac{1}{2}[f(ct-x) + f(x+ct)] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(y)dy, & x - ct < 0. \end{cases}$$

(b) If we take

$$u(0,x) = \begin{cases} f(x), & x > 0 \\ f(-x), & x < 0 \end{cases}$$

and similarly

$$u_t(0,x) = \begin{cases} g(x), & x > 0 \\ g(-x), & x < 0, \end{cases}$$

d'Alembert's solution becomes precisely $u(t,x)$ from (a).

In this case, the block

$$u(0,x) = \begin{cases} 1, & 4 < x < 5 \\ 0, & \text{otherwise} \end{cases}$$

splits into two pieces, each with height 1/2, one moving to the right and the other moving toward the origin. The piece moving toward the origin bounces off the t-axis in the characteristic plane and after that continues to move away from the origin.

5. [10 pts] **Haberman Problem 12.4.6.** (The solution to this one is in the back.) The solution process is almost precisely as in Problem 12.4.4, except that the condition $u_x(t,0) = h(t)$ gives

$$F'(-ct) + G'(ct) = h(t),$$

from which we see that for $x < 0$,

$$F'(x) = -G'(-x) + h(-\frac{x}{c}).$$

Integrating from 0 to x, we have

$$\int_0^x F'(y)dy = -\int_0^x G'(-y)dy + \int_0^x h(-\frac{y}{c})dt \Rightarrow F(x) = \int_0^x h(-\frac{y}{c})dy,$$

where we have used that $G(-x)$ is 0 for $x > 0$. In order to get the text's form, set $\bar{t} = -\frac{y}{c}$.