M412 Assignment 5 Solutions

1. [10 pts] Haberman 2.5.1 (a).

Solution. Setting $u(x,y) = X(x)Y(y)$, we find that X and Y satisfy

$$\begin{aligned} X'' + \lambda X &= 0 \\ Y'' - \lambda Y &= 0, \end{aligned}$$

with boundary conditions

$$\begin{aligned} X'(0) &= 0 \\ X'(L) &= 0 \\ Y(0) &= 0. \end{aligned}$$

In this case, we find from the X equation that the eigenvalues are $\lambda = 0$ and $\lambda = \frac{n^2\pi^2}{L^2}$, $n = 1, 2, \ldots.$ For $\lambda = 0$, we have

$$Y_0(y) = y,$$

while for $\lambda = \frac{n^2\pi^2}{L^2}$,

$$Y_n(y) = \sinh \frac{n\pi}{L} y.$$

We conclude that the most general form of the solution is

$$u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi}{L} y \cos \frac{n\pi}{L} x.$$

Setting $u(x,H) = f(x)$, we have

$$f(x) = A_0 H + \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi}{L} H \cos \frac{n\pi}{L} x.$$

This is a Fourier cosine series, and we have

$$\begin{aligned} A_0 &= \frac{1}{LH} \int_0^L f(x) dx \\ A_n &= \frac{2}{L \sinh \frac{n\pi}{L} H} \int_0^L f(x) \cos \frac{n\pi}{L} x dx. \end{aligned}$$

This entirely determines the solution.

2. [10 pts] Haberman 2.5.1 (c).

Solution. In this case, separation and a clever choice for the sign of λ give

$$\begin{aligned} X'' - \lambda X &= 0 \\ Y'' + \lambda Y &= 0, \end{aligned}$$

with boundary conditions

$$\begin{aligned} Y(0) &= 0 \\ Y(H) &= 0 \\ X'(0) &= 0. \end{aligned}$$

In this case, the eigenvalues are determined by the Y equation as $\lambda = \frac{n^2\pi^2}{H^2}$, $n = 1, 2, \ldots$. We find that

$$X(x) = \cosh \frac{n\pi}{H} x,$$

and consequently the most general form of the solution is

$$u(x, y) = \sum_{n=1}^{\infty} A_n \cosh \frac{n\pi}{H} x \sin \frac{n\pi}{H} y.$$

Setting $u(L, y) = g(y)$, we have

$$g(y) = \sum_{n=1}^{\infty} A_n \cosh \frac{n\pi}{H} L \sin \frac{n\pi}{H} y.$$

This is a Fourier sine series, and we have

$$A_n = \frac{2}{H \cosh \frac{n\pi}{H} L} \int_0^H g(y) \sin \frac{n\pi y}{H} dy,$$

which entirely determines the solution.

3. [15 pts] Haberman 2.5.2.

Solution.

(a) Since the solution to this equation can be regarded as an equilibrium solution to the heat equation in two dimensions, and since there is no source for heat inside the plate (no Q), the total heat flowing in must be equal to the total heat flowing out. In this case, the heat flow is zero at each boundary except along (x, H), $0 \leq x \leq L$, and so the total flow through this boundary must be zero. This says

$$\int_0^L u_y(x, H) dx = 0 \Rightarrow \int_0^L f(x) dx = 0.$$

(b) Proceeding similarly as in Problem 1 (Haberman Problem 2.5.1(a)), we observe that the X equation gives our eigenvalues again as $\lambda = 0$ and $\lambda = \frac{n^2\pi^2}{L^2}$, $n = 1, 2, \ldots$, with associated solutions to the Y equation

$$\begin{aligned} \lambda = 0 : &\quad Y_0(y) = 1 \\ \lambda = \frac{n^2\pi^2}{L^2} : &\quad Y_n(y) = \cosh \frac{n\pi}{L} y. \end{aligned}$$

The most general form of the solution is consequently

$$u(x, y) = A_0 + \sum_{n=1}^{\infty} A_n \cosh \frac{n\pi}{L} y \cos \frac{n\pi}{L} x.$$

Setting $u_y(x, H) = f(x)$, we have

$$f(x) = \sum_{n=1}^{\infty} A_n \frac{n\pi}{L} \sinh \frac{n\pi}{L} H \cos \frac{n\pi}{L} x. \tag{1}$$

This is a Fourier cosine series and we have

$$A_n = \frac{2}{n\pi \sinh \frac{n\pi}{L}} \int_0^L f(x) \cos \frac{n\pi}{L} x dx.$$

Upon integration of (1) from 0 to L, and observing that $\int_0^L \cos \frac{n\pi}{L} x dx = 0$, we see that

$$\int_0^L f(x)dx = 0,$$

as expected from Part (a).

(c) In order to determine a value for A_0, we integrate the original heat equation over the entire plate. That is,

$$\begin{aligned}
\int_0^L \int_0^H u_t(t,x,y)dxdy &= \int_0^L \int_0^H u_{xx}(t,x,y)dxdy + \int_0^L \int_0^H u_{yy}(t,x,y)dxdy \\
&= \int_0^H u_x(t,L,y) - u_x(t,0,y)dy + \int_0^L u_y(t,x,H) - u_y(t,x,0)dx \\
&= \int_0^H (0-0)dy + \int_0^L (f(x)-0)dx = 0,
\end{aligned}$$

where we have changed order of integration when necessary (without proving we can do it) and where we have assumed that the boundary conditions arose from similar expressions for the full heat equation valid for all t. In this way,

$$\frac{d}{dt}\int_0^L \int_0^H u(t,x,y)dxdy = 0 \Rightarrow \int_0^L \int_0^H u(t,x,y)dxdy = \int_0^L \int_0^H u(0,x,y)dxdy = \int_0^L \int_0^H g(x,y)dxdy.$$

Finally, upon integration of $u(x,y)$ from Part (b), and keeping in mind that $u(x,y)$ from Part (b) is still a solution to the full heat equation (the limit of it as $t \to \infty$), we have

$$\int_0^L \int_0^H u(x,y)dxdy = \int_0^L \int_0^H g(x,y)dxdy \Rightarrow A_0 = \frac{1}{LH}\int_0^L \int_0^H g(x,y)dxdy.$$

4. [10 pts] Show that in polar dimensions (r,θ) the Laplace equation in two space dimensions takes the form

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0.$$

Hint: Recall that the relationship between cartesian and polar coordinates is $x = r\cos\theta$ and $y = r\sin\theta$. Set

$$u(r,\theta) := v(x(r,\theta), y(r,\theta)),$$

where $v(x,y)$ satisfies Laplace's equation in cartesian coordinates,

$$v_{xx} + v_{yy} = 0.$$

Solution. (For an alternative approach see Haberman Problem 1.5.3.) According to the chain rule,

$$\begin{aligned}
u_r &= v_x \frac{\partial x}{\partial r} + v_y \frac{\partial y}{\partial r} \\
u_{rr} &= v_{xx}(\frac{\partial x}{\partial r})^2 + v_{xy}\frac{\partial x}{\partial r}\frac{\partial y}{\partial r} + v_x \frac{\partial^2 x}{\partial r^2} + v_{yx}\frac{\partial x}{\partial r}\frac{\partial y}{\partial r} + v_{yy}(\frac{\partial y}{\partial r})^2 + v_y \frac{\partial^2 y}{\partial r^2}.
\end{aligned}$$

Keeping in mind the relations

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial y}{\partial r} = \sin\theta,$$

we have

$$u_{rr} = v_{xx}\cos^2\theta + v_{yy}\sin^2\theta + 2v_{xy}\sin\theta\cos\theta.$$

Similarly,

$$u_{\theta\theta} = v_{xx}r^2\sin^2\theta - 2v_{xy}r^2\sin\theta\cos\theta - v_x r\cos\theta + v_{yy}r^2\cos^2\theta - v_y r\sin\theta.$$

Combining these,

$$u_{rr} + \frac{1}{r^2}u_{\theta\theta} = v_{xx} + v_{yy} - \frac{1}{r}u_r = -\frac{1}{r}u_r,$$

which is precisely the claimed relationship.

5. [10 pts] Haberman 2.5.3 (a).

Solution. Taking the separation assumption $u(r,\theta) = R(r)T(\theta)$, we have

$$\frac{r^2R'' + rR'}{R} = -\frac{T''}{T} = \lambda,$$

from which we obtain

$$\begin{aligned} r^2R'' + rR' - \lambda R &= 0 \\ T'' + \lambda T &= 0, \end{aligned}$$

with boundary conditions

$$\begin{aligned} \lim_{r\to\infty} R(r) &< \infty \\ T(-\pi) &= T(\pi) \\ T'(-\pi) &= T'(\pi). \end{aligned}$$

In this case, it is the T equation that determines the eigenvalues, and we find $\lambda = 0$ and $\lambda = n^2$, $n = 1, 2, 3...$, with associated eigenfunctions

$$\begin{aligned} \lambda = 0: &\quad T_0(\theta) = 1 \\ \lambda = n^2: &\quad T_{1_n}(\theta) = \cos n\theta, \quad \text{and} \quad T_{2_n}(\theta) = \sin n\theta. \end{aligned}$$

For the $R(r)$ equation, we have

$$\begin{aligned} \lambda = 0: &\quad R_0(r) = 1 \\ \lambda = n^2: &\quad R_n(\theta) = r^{-n}. \end{aligned}$$

The most general solution is

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^{-n}(A_n\cos n\theta + B_n\sin n\theta).$$

Setting $u(a,\theta) = \ln 2 + 4\cos 3\theta$, we have

$$\ln 2 + 4\cos 3\theta = A_0 + \sum_{n=1}^{\infty} a^{-n}(A_n\cos n\theta + B_n\sin n\theta),$$

for which we can match terms to find

$$A_0 = \ln 2$$
$$A_3 = 4a^3,$$

with every other coefficient 0.

6. [10 pts] Haberman 2.5.5 (a).

Solution. Separating variables as in the previous problem, we find that the eigenvalues are determined by

$$T'' + \lambda T = 0$$
$$T'(0) = 0$$
$$T(\frac{\pi}{2}) = 0,$$

from which we find that the eigenvalues are $\lambda = (2n-1)^2$, $n = 1, 2, ...$, with eigenfunctions $T(\theta) = \cos(2n-1)\theta$. Using in this case the additional restriction that $R(0)$ is bounded, we have

$$R(r) = r^{2n-1},$$

and so that most general form of solution is

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{2n-1} \cos(2n-1)\theta.$$

Setting $u(1,\theta) = f(\theta)$, we have

$$f(\theta) = \sum_{n=1}^{\infty} A_n \cos(2n-1)\theta.$$

Upon multiplication of both sides by $\cos(2m-1)\theta$ and integration on $0 \leq \theta \leq \frac{\pi}{2}$, we conclude

$$A_n = \frac{4}{\pi} \int_0^{\frac{\pi}{2}} f(\theta) \cos(2n-1)\theta d\theta,$$

which entirely determines the solution.