M412 Assignment 6 Solutions, due Friday October 28

1. [10 pts] Show that for the eigenvalue problem

$$(p(x)u_x)_x + q(x)u + \lambda\sigma(x)u = 0; \quad a \le x \le b,$$

eigenvalues λ are related to their eigenfunctions u by the *Rayleigh quotient*

$$\lambda = \frac{\int_a^b (p(x)u_x^2 - q(x)u^2)dx + (p(a)u(a)u_x(a) - p(b)u(b)u_x(b))}{\int_a^b \sigma(x)u(x)^2 dx}.$$

Solution. Multiply the eigenvalue equation by $u(x)$ and integrate x from a to b,

$$\int_a^b u(p(x)u_x)_x dx + \int_a^b q(x)u^2 dx + \lambda \int_a^b \sigma(x)u^2 dx = 0.$$

Integrating the first term by parts, we find

$$p(x)uu_x\Big|_a^b - \int_a^b p(x)u_x^2 dx + \int_a^b q(x)u^2 dx + \lambda \int_a^b \sigma(x)u^2 dx = 0.$$

Solving for λ gives the claimed relationship.

2. [10 pts] Haberman 2.5.10.

Solution. Let u_1 and u_2 both be solutions to the problem

$$\begin{aligned} \triangle u &= g(\mathbf{x}) \\ u &= f(\mathbf{x}) \text{ on the boundary.} \end{aligned}$$

Then the variable $v = u_1 - u_2$ solves

$$\begin{aligned} \triangle v &= 0 \\ v &= 0 \text{ on the boundary.} \end{aligned}$$

According to the maximum principle, this gives

$$v \equiv 0.$$

3. [10 pts] Haberman 2.5.14.

Solution. Let $u(t,x)$ solve the equation with $u(0,x) = f(x)$ and let $w(t,x)$ solve the equation with the perturbed initial condition $f(x) + \frac{1}{n}\sin\frac{n\pi x}{L}$, and define $v(t,x) = w(t,x) - u(t,x)$ as the error between the two. Solve for v by separation of variables, taking $v(t,x) = T(t)X(x)$, so that

$$-\frac{T'}{kT} = \frac{X''}{X} = -\lambda; \quad X(0) = 0; \quad X(L) = 0.$$

Proceeding as usual, we find

$$\lambda = \frac{m^2\pi^2}{L^2}; \quad X_m(x) = \sin\frac{m\pi x}{L}; \quad m = 1, 2, ...,$$

where I've used the index m because the problem uses an index n in its statement. We have

$$v(t,x) = \sum_{m=1}^{\infty} A_m e^{+k\frac{m^2\pi^2}{L^2}t} \sin \frac{m\pi x}{L},$$

from which we can already see the main point, that we now have exponential growth in t rather than exponential decay. Setting $v(0,x) = \frac{1}{n} \sin \frac{n\pi x}{L}$, we immediately see that

$$A_m = \begin{cases} 0, & m \neq n \\ \frac{1}{n}, & m = n, \end{cases}$$

so that

$$v(t,x) = \frac{1}{n} e^{+k\frac{n^2\pi^2}{L^2}t} \sin \frac{n\pi x}{L}.$$

(We say that the n^{th} eigenmode has been *excited*.) The main observation to make here is that the smaller n is the faster our exponential growth in t will be. So even for arbitrarily small perturbations (changes to initial data), errors grow exponentially in time.