M412 Assignment 7, due Friday November 4

1. [15 pts] (Mean Value Property in three space dimensions.) Suppose Ω is an open subset of $\mathbb{R}^3$ and $u \in C^2(\Omega)$ solves the Laplace equation in Ω. Show that if $(x_0, y_0, z_0) \in \Omega$, and $S_r(x_0, y_0, z_0)$ is a sphere centered at (x_0, y_0, z_0) with radius r, contained entirely in Ω, then

$$u(x_0, y_0, z_0) = \frac{1}{4\pi r^2} \int_{\partial S_r(x_0,y_0,z_0)} u(x, y, z) dS.$$

Hints. Laplace's equation in spherical coordinates (r, θ, ϕ) takes the form

$$\sin\phi (r^2 u_r)_r + (\sin\phi u_\phi)_\phi + \frac{1}{\sin\phi} u_{\theta\theta} = 0.$$

(See Haberman p. 28 for a description of spherical coordinates.) The differential surface increment in spherical coordinates is

$$dS = r^2 \sin\phi d\phi d\theta.$$

2. [10 pts] (Maximum/Minimum principle for the Laplace equation in three space dimensions.) Suppose Ω is a bounded, open, connected subset of $\mathbb{R}^3$ and $u \in C^2(\Omega) \cap C(\bar{\Omega})$ solves the Laplace equation on Ω. Show that u can only attain its maximum or minimum on the interior of Ω if u is constant on the entirety of Ω. (You may use without proof the following fact: If u is constant on any sphere in Ω then u it is constant throughout the entirety of Ω.)

3. [5 pts] (Uniqueness of solutions to the Laplace equation in three space dimensions.) Suppose Ω is a bounded, open, connected subset of $\mathbb{R}^3$. Show that solutions $u \in C^2(\Omega) \cap C(\bar{\Omega})$ to the Laplace equation

$$\begin{aligned} \triangle u &= 0; \quad \Omega \\ u &= f; \quad \partial\Omega \end{aligned}$$

are unique.

4. [5 pts] (Stability of solutions to the Laplace equation in three space dimensions.) Suppose Ω is a bounded, open, connected subset of $\mathbb{R}^3$. Show that solutions $u \in C^2(\Omega) \cap C(\bar{\Omega})$ to the Laplace equation

$$\begin{aligned} \triangle u &= 0; \quad \Omega \\ u &= f; \quad \partial\Omega \end{aligned}$$

are stable with respect to small changes in the boundary data f.

5. [5 pts] Show that a necessary condition for solutions to the Laplace equation on Ω to exist is

$$\int_{\partial\Omega} \nabla u \cdot \vec{n} dS = 0.$$

What does this condition correspond with physically.

6. [10 pts] Establish the trigonometric identity

$$1 + 2\sum_{n=1}^{N} \cos\frac{n\pi x}{L} = \frac{\sin[(N + \frac{1}{2})\frac{\pi}{L}x]}{\sin(\frac{\pi}{2L}x)}.$$

Hint. Set $y = \frac{\pi x}{L}$ and use Euler's formula

$$e^{\pm iny} = \cos ny \pm i \sin ny$$

to show that

$$2\cos ny = e^{iny} + e^{-iny}.$$

Then find a way to employ the relation

$$\sum_{n=1}^{N} x^n = \frac{x - x^{N+1}}{1 - x}.$$