M412 Assignment 7 Solutions, due Friday November 4

1. [15 pts] (Mean Value Property in three space dimensions.) Suppose Ω is an open subset of $\mathbb{R}^3$ and $u \in C^2(\Omega)$ solves the Laplace equation in Ω. Show that if $(x_0, y_0, z_0) \in \Omega$, and $S_r(x_0, y_0, z_0)$ is a sphere centered at (x_0, y_0, z_0) with radius r, contained entirely in Ω, then

$$u(x_0, y_0, z_0) = \frac{1}{4\pi r^2} \int_{\partial S_r(x_0,y_0,z_0)} u(x, y, z) dS.$$

Hints. Laplace's equation in spherical coordinates (r, θ, ϕ) takes the form

$$\sin\phi(r^2 u_r)_r + (\sin\phi u_\phi)_\phi + \frac{1}{\sin\phi} u_{\theta\theta} = 0.$$

(See Haberman p. 28 for a description of spherical coordinates.) The differential surface increment in spherical coordinates is

$$dS = r^2 \sin\phi d\phi d\theta.$$

Solution. Begin by taking the point (x_0, y_0, z_0) as the center point for your spherical coordinates; that is, this point corresponds with $r = 0$. Now, integrate Laplace's equation over the angular variables $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$,

$$\int_0^\pi \int_0^{2\pi} \sin\phi(r^2 u_r)_r d\theta d\phi + \int_0^\pi \int_0^{2\pi} (\sin\phi u_\phi)_\phi d\theta d\phi + \int_0^\pi \int_0^{2\pi} \frac{1}{\sin\phi} u_{\theta\theta} d\theta d\phi = 0.$$

In this case, $u(r, 0, \phi) = u(r, 2\pi, \phi)$, and we have

$$\int_0^\pi \int_0^{2\pi} \frac{1}{\sin\phi} u_{\theta\theta} d\theta d\phi = \int_0^\pi \frac{1}{\sin\phi} \int_0^{2\pi} u_{\theta\theta} d\theta d\phi = \int_0^\pi \frac{1}{\sin\phi} (u_\theta(r, 2\pi, \phi) - u_\theta(r, 0, \phi)) d\phi = 0.$$

In addition, by changing the order of integration on the second summand, we find

$$\int_0^{2\pi} \int_0^\pi (\sin\phi u_\phi)_\phi d\phi d\theta = \int_0^{2\pi} (\sin\pi u_\phi(r, \theta, \pi) - \sin 0 u_\phi(r, \theta, 0)) d\theta = 0.$$

Observing that r is constant with respect to angular integration, we conclude

$$\Big(r^2 (\int_0^\pi \int_0^{2\pi} \sin\phi u d\theta d\phi)_r\Big)_r = 0.$$

Defining

$$I(r) = \int_0^\pi \int_0^{2\pi} u \sin\phi d\theta d\phi,$$

we have

$$(r^2 I')' = 0 \Rightarrow r^2 I' = C_1 \Rightarrow I(r) = -C_1 r^{-1} + C_2.$$

Taking $|I(0)| < \infty$, we conclude that $C_1 = 0$, and so $I(r)$ is constant for all r. Thus

$$I(r) = I(0) = \int_0^\pi \int_0^{2\pi} u(0, \theta, \phi) \sin\phi d\theta d\phi = u(0, \theta, \phi) 2\pi(-1) \cos\phi \Big|_0^\pi = 4\pi u(0, \theta, \phi).$$

Turning this around,

$$u(0,\theta,\phi)=\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}u(r,\theta,\phi)\sin\phi d\theta d\phi=\frac{1}{4\pi r^2}\int_0^{\pi}\int_0^{2\pi}u(r,\theta,\phi)r^2\sin\phi d\theta d\phi,$$

and this is the claim.

2. [10 pts] (Maximum/Minimum principle for the Laplace equation in three space dimensions.) Suppose Ω is a bounded, open, connected subset of $\mathbb{R}^3$ and $u\in C^2(\Omega)\cap C(\bar{\Omega})$ solves the Laplace equation on Ω. Show that u can only attain its maximum or minimum on the interior of Ω if u is constant on the entirety of Ω. (You may use without proof the following fact: If u is constant on any sphere in Ω then u it is constant throughout the entirety of Ω.)

Solution. Suppose that u attains its maximum at a point (x_0,y_0,z_0) on the interior of Ω. For any sphere centered at this point with radius r small enough to that the sphere lies entirely in Ω, the value $u(x_0,y_0,z_0)$ can be computed as an average of values on the surface of the sphere,

$$u(x_0,y_0,z_0)=\frac{1}{4\pi r^2}\int_{\partial S_r(x_0,y_0,z_0)}u(x,y,z)dS.$$

Since $u(x_0,y_0,z_0)\ge u(x,y,z)$ for all $(x,y,z)\in\partial S_r$ (by the assumption that u is maximal at (x_0,y_0,z_0)), this can only possibly hold if $u(x,y,z)=u(x_0,y_0,z_0)$ for all $(x,y,z)\in\partial S_r$. Since this is true for all r small enough for S_r to be contained in Ω, it is true for an entire sphere. By the unproven fact, this means that u must be constant throughout Ω. We conclude that either u does not have a maximum in the interior or Ω, of if it does have a maximum in the interior of Ω it is constant throughout the entirety of Ω.

3. [5 pts] (Uniqueness of solutions to the Laplace equation in three space dimensions.) Suppose Ω is a bounded, open, connected subset of $\mathbb{R}^3$. Show that solutions $u\in C^2(\Omega)\cap C(\bar{\Omega})$ to the Laplace equation

$$\begin{aligned}\triangle u&=0;\quad \Omega\\ u&=f;\quad \partial\Omega\end{aligned}$$

are unique.

Solution. Let u_1 and u_2 both solve this equation, and set $v=u_1-u_2$. Then

$$\begin{aligned}\triangle v&=0;\quad \Omega\\ v&=0;\quad \partial\Omega,\end{aligned}$$

and by the maximum/minimum principle of Problem 2, $v\equiv 0$.

4. [5 pts] (Stability of solutions to the Laplace equation in three space dimensions.) Suppose Ω is a bounded, open, connected subset of $\mathbb{R}^3$. Show that solutions $u\in C^2(\Omega)\cap C(\bar{\Omega})$ to the Laplace equation

$$\begin{aligned}\triangle u&=0;\quad \Omega\\ u&=f;\quad \partial\Omega\end{aligned}$$

are stable with respect to small changes in the boundary data f.

Solution. Let w solve the slightly altered equation

$$\begin{aligned}\triangle w&=0;\quad \Omega\\ w&=f+s;\quad \partial\Omega,\end{aligned}$$

where s is assumed to be some small function defined on $(x, y, z) \in \partial\Omega$. Taking $v = w - u$, we have

$$\begin{aligned} \triangle v &= 0; \quad \Omega \\ v &= s; \quad \partial\Omega, \end{aligned}$$

for which the maximum/minimum principle asserts

$$\inf_{(x,y,z)\in\partial\Omega} s(x, y) \leq v(x, y) \leq \sup_{(x,y,z)\in\partial\Omega} s(x, y).$$

Since small changes in the boundary data lead to only small changes in the solution, we say that solutions to this equation are stable.

5. [5 pts] Show that a necessary condition for solutions to the Laplace equation on Ω to exist is

$$\int_{\partial\Omega} \nabla u \cdot \vec{n} dS = 0.$$

What does this condition correspond with physically.

Solution. Noting that $\triangle u = \nabla \cdot (\nabla u)$, we see that if $\triangle u = 0$, there holds

$$0 = \int_{\Omega} \triangle u dV = \int_{\Omega} \nabla \cdot (\nabla u) dV = \int_{\partial\Omega} \nabla u \cdot \vec{n} dS.$$

This asserts that the total heat flow through the boundary must be 0 for a heat equation to have an equilibrium solution.

6. [10 pts] Establish the trigonometric identity

$$1 + 2\sum_{n=1}^{N} \cos\frac{n\pi x}{L} = \frac{\sin[(N + \frac{1}{2})\frac{\pi}{L}x]}{\sin(\frac{\pi}{2L}x)}.$$

Hint. Set $y = \frac{\pi x}{L}$ and use Euler's formula

$$e^{\pm iny} = \cos ny \pm i \sin ny$$

to show that

$$2\cos ny = e^{iny} + e^{-iny}.$$

Then find a way to employ the relation

$$\sum_{n=1}^{N} x^n = \frac{x - x^{N+1}}{1 - x}.$$

Solution. Beginning with $2\cos ny = e^{iny} + e^{-iny}$, write

$$2\sum_{n=1}^{N} \cos ny = \sum_{n=1}^{N}(e^{iny} + e^{-iny}) = \sum_{n=1}^{N}((e^{iy})^n + (e^{-iy})^n),$$

from which we can employ the suggested relation to find

$$\begin{aligned}
\sum_{n=1}^{N}((e^{iy})^n + (e^{-iy})^n) &= \frac{e^{iy} - e^{iy(N+1)}}{1 - e^{iy}} + \frac{e^{-iy} - e^{-iy(N+1)}}{1 - e^{-iy}} \\
&= \frac{e^{i\frac{y}{2}} - e^{iy(N+\frac{1}{2})}}{e^{-i\frac{y}{2}} - e^{i\frac{y}{2}}} + \frac{e^{-i\frac{y}{2}} - e^{-iy(N+\frac{1}{2})}}{e^{i\frac{y}{2}} - e^{-i\frac{y}{2}}} \\
&= \frac{e^{i\frac{y}{2}} - e^{-i\frac{y}{2}}}{e^{-i\frac{y}{2}} - e^{i\frac{y}{2}}} + \frac{e^{iy(N+\frac{1}{2})} - e^{-iy(N+\frac{1}{2})}}{e^{i\frac{y}{2}} - e^{-i\frac{y}{2}}} \\
&= -1 + \frac{\sin[(N+\frac{1}{2})y]}{\sin[\frac{y}{2}]},
\end{aligned}$$

which is the claimed identity.