# M412 Assignment 8 Solutions, due Friday November 11

1. [10 points] Finish our proof of Fourier's Theorem by showing that

$$\lim_{N\to\infty} \frac{1}{2L} \int_x^{x+L} f(y)\Big(1 + 2\sum_{n=1}^N \cos(\frac{n\pi}{L}(y-x))\Big)dy = \frac{1}{2}f(x^+).$$

**Solution.** Proceeding similarly as we did in class, set $z = y - x$ to obtain the integration

$$\lim_{N\to\infty} \frac{1}{2L} \int_0^L f(x+z)\Big(1 + 2\sum_{n=1}^N \cos(\frac{n\pi}{L}z)\Big)dz.$$

Next, introduce the limit

$$\lim_{y\to x^+} f(y) = f(x^+)$$

by re-writing this integral as

$$\lim_{N\to\infty} \frac{1}{2L} \int_0^L (f(x+z) - f(x^+))\Big(1 + 2\sum_{n=1}^N \cos(\frac{n\pi}{L}z)\Big)dz + \lim_{N\to\infty} \frac{1}{2L} \int_0^L f(x^+)\Big(1 + 2\sum_{n=1}^N \cos(\frac{n\pi}{L}z)\Big)dz.$$

Now evaluate these limits one at a time, beginning with the second, for which we can integrate directly,

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{2L} \int_0^L f(x^+)\Big(1 + 2\sum_{n=1}^N \cos(\frac{n\pi}{L}z)\Big)dz &= \lim_{N\to\infty} \Big[\frac{1}{2L}\int_0^L f(x^+)dz + \frac{1}{L}\sum_{n=1}^N \int_0^L \cos(\frac{n\pi}{L}z)dz\Big] \\ &= \frac{1}{2}f(x^+),\end{aligned}$$

where we have observed that each integration over cosine gives 0 here. Finally, we evaluate the first integral in our decomposition by using the trigonometric identity

$$1 + 2\sum_{n=1}^N \cos\frac{n\pi x}{L} = \frac{\sin[(N+\frac{1}{2})\frac{\pi}{L}x]}{\sin(\frac{\pi}{2L}x)}.$$

We find

$$\lim_{N\to\infty} \frac{1}{2L} \int_0^L (f(x+z) - f(x^+))\Big(\frac{\sin[(N+\frac{1}{2})\frac{\pi}{L}z]}{\sin(\frac{\pi}{2L}z)}\Big)dz = \lim_{N\to\infty} \frac{1}{2L} \int_0^L \frac{(f(x+z) - f(x^+))}{\sin(\frac{\pi}{2L}z)}\Big(\sin[(N+\frac{1}{2})\frac{\pi}{L}z]\Big)dz.$$

The quotient

$$q(z) = \frac{(f(x+z) - f(x^+))}{\sin(\frac{\pi}{2L}z)}$$

is clearly pointwise continuous except possibly near $z = 0$. (This is clear because $f$ is assumed pointwise continuous (in fact, pointwise smooth), and $\sin(\frac{\pi}{2L}z)$ is continuous and non-zero on $z \in [0, L]$, except at the point $z = 0$.) In order to check the behavior as $z \to 0$, we compute

$$\lim_{z\to 0} \frac{(f(x+z) - f(x^+))}{\sin(\frac{\pi}{2L}z)} = \frac{f'(x^+)}{\frac{\pi}{2L}},$$

obtained as in class by L'Hospital's rule. Since $f$ is assumed piecewise smooth, this limit exists, and we can conclude that $q$ is piecewise continuous on $z \in [0, L]$. The Riemann–Lebesgue lemma applies, then, giving that the limit is 0.

2. [5 points] Finish our proof regarding term-by-term integration of the full Fourier series by computing $b_n$.

**Solution.** We have

$$
\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{+L} G(x)\sin\frac{n\pi x}{L}dx = \frac{1}{L}\int_{-L}^{+L}(F(x) - A_0 x)\sin\frac{n\pi x}{L}dx \\
&= \frac{1}{L}\Big[(F(x) - A_0 x)(-\frac{L}{n\pi})\cos\frac{n\pi x}{L}\Big|_{-L}^{+L} + \int_{-L}^{+L}(f(x) - A_0)(\frac{L}{n\pi})\cos\frac{n\pi x}{L}dx\Big] \\
&= \frac{1}{L}\Big[(F(L) - A_0 L)(-\frac{L}{n\pi})\cos n\pi - (F(-L) + A_0 L)(-\frac{L}{n\pi})\cos n\pi + \frac{L}{n\pi}\int_{-L}^{+L} f(x)\cos\frac{n\pi x}{L}dx\Big] \\
&= \frac{1}{L}\Big[(F(L) - F(-L) - 2A_0 L)(-\frac{L}{n\pi})\cos n\pi\Big] + \frac{L}{n\pi}A_n,
\end{aligned}
$$

where we have observed that cosine integrates to 0 on $[-L, L]$. Finally, we observe that for

$$F(x) = \int_{-L}^{x} f(y)dy,$$

$F(-L) = 0$ and

$$F(L) = \int_{-L}^{+L} f(y)dy = 2LA_0,$$

so that all terms in the square brackets cancel, and we have

$$b_n = \frac{L}{n\pi}A_n.$$

3. [5 points] Using Fourier's Theorem, prove that the Fourier sine series for a piecewise smooth function $f(x)$ defined on $[0, L]$ converges to $\frac{1}{2}(f(x^-) + f(x^+))$ on $(0, L)$. Under what condition on $f(x)$ does the Fourier sine series definitely not converge at the endpoints $x = 0$ and $x = L$?

**Solution.** Since $f$ is only defined for $x \in [0, L]$, extend it as an odd function to $x \in [-L, L]$. That is, set $f(x) = -f(-x)$ for all $x \in [-L, L]$. According to Fourier's Theorem, the full Fourier series for this extended function converges for all $x \in [-L, L]$. Now, write this Fourier series

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L},$$

and observe that since $f(x)$ is odd,

$$A_0 = 0; \quad A_n = 0 \text{ for all } n = 1, 2, \ldots.$$

Moreover,

$$B_n = \frac{1}{L}\int_{-L}^{+L} f(x)\sin\frac{n\pi x}{L}dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}dx,$$

and this is the Fourier sine series of $f$. It converges on $[-L, L]$ and so it converges on $[0, L]$.

The endpoints are tricky. The point is that since a Fourier sine series consists of a sum of sines, its value must necessarily be 0 for $x = 0$ and for $x = L$. So the series will not converge to $f(x)$ at the endpoints unless $f(0) = 0$ and/or $f(L) = 0$. Observe, however, that the (periodically extended) odd extension of $f$ , denoted here by $f_E$, satisfies

$$f_E(0^-) = -f_E(0^+) \text{ and } f_E(L^-) = -f_E(L^+),$$

and so the series does converge to $\frac{1}{2}(f_E(x^-) + f_E(x^+))$.

4. [10 points] For the heat equation

$$\begin{aligned} u_t &= u_{xx} \\ u(t, 0) &= 0 \\ u(t, L) &= 0 \\ u(0, x) &= f(x), \end{aligned}$$

where $f(x)$ is assumed continuous on $[0, L]$, with $f(0) = f(L) = 0$, and $f'(x)$ is assumed piecewise continuous on $[0, L]$, prove that the infinite series found by the method of separation of variables is a solution. You may use without proving it that under these conditions on $f$ the Fourier sine series associated with $f$ is uniformly convergent.

**Solution.** We are given that the Fourier sine series for $f(x)$ converges,

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}; \quad B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

Our candidate for a full solution to the PDE is then

$$u(t, x) = \sum_{n=1}^{\infty} B_n e^{-\frac{n^2\pi^2}{L^2}t} \sin \frac{n\pi x}{L}.$$

We now use the Weierstrass M-test to show that this series, as well as its term-by-term derivatives, converge uniformly for $t \in [t_0, T]$, for $t_0 > 0$ arbitrarily small and $T$ arbitrarily large. First, the Riemann–Lebesgue lemma implies that

$$\lim_{n\to\infty} B_n = 0,$$

from which we can conclude that there exists some constant $C$ so that

$$|B_n| \leq C$$

for all $n = 1, 2, 3, ....$ We have then

$$|B_n e^{-\frac{n^2\pi^2}{L^2}t} \sin \frac{n\pi x}{L}| \leq C e^{-\frac{n^2\pi^2}{L^2}t_0}.$$

In this way, we can apply the Weierstrass M test with

$$M_n = C e^{-\frac{n^2\pi^2}{L^2}t_0},$$

where it is straightforward to check with the ratio test that

$$\sum_{n=1}^{\infty} M_n \text{ converges.}$$

A slight modification of this argument gives that the series obtained by differentiating our series for $u(t, x)$ term-by-term also converge uniformly, and we can conclude from this and a theorem from class that $u(t, x)$ can be differentiated term-by-term to any order in both $x$ and $t$. Computing directly, we find

$$u_t = \sum_{n=1}^{\infty}(-\frac{n^2\pi^2}{L^2})B_n e^{-\frac{n^2\pi^2}{L^2}t}\sin\frac{n\pi x}{L} = u_{xx}.$$

Finally, since $f(x)$ converges uniformly, Abel's test for uniform convergence gives that the series for $u(t, x)$ converges uniformly for $t \in [0, T]$, which gives continuity down to our initial function $f(x)$.

5. [10 points] Haberman 3.4.4.

**Solution to Part (a).** If $f(x)$ and $f'(x)$ are both piecewise smooth (on $[0, L]$), $f(x)$ can be expanded as a Fourier sine series and $f'(x)$ can be expanded as a Fourier cosine series,

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$
$$f'(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}.$$

We now compute

$$\begin{aligned} a_n =& \frac{2}{L}\int_0^L f'(x)\cos\frac{n\pi x}{L}dx = \frac{2}{L}\Big[f(x)\cos\frac{n\pi x}{L}\Big|_0^L + \frac{n\pi}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx\Big] \\ =& \frac{2}{L}\Big[f(L)\cos n\pi - f(0)\Big] + \frac{n\pi}{L}B_n. \end{aligned}$$

In order for term-by-term differentiation of the series for $f(x)$ to give the series for $f'(x)$, we require

$$(-1)^n f(L) = f(0),$$

which implies $f(0) = f(L) = 0$.

**Solution to Part (b).** In this case, we have

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}$$
$$f'(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L},$$

and we immediately find

$$b_n = \frac{2}{L}\int_0^L f'(x)\sin\frac{n\pi x}{L}dx = -\frac{n\pi}{L}A_n,$$

which is precisely what we arrive at by differentiating the series for $f(x)$ term by term.

6a. [10 points] Haberman 3.4.11.

**Solution.** Set

$$u(t, x) = \sum_{n=1}^{\infty} c_n(t)\sin\frac{n\pi x}{L},$$

and also (following Haberman's hint)

$$g(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}.$$

Upon substitution, we find

$$\sum_{n=1}^{\infty}(c_n'(t) + k\frac{n^2\pi^2}{L^2}c_n(t))\sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}.$$

Matching coefficients of sine,

$$c_n'(t) + k\frac{n^2\pi^2}{L^2}c_n(t) = B_n.$$

(This equation can also be obtained in the usual way through the trigonometric orthogonality relations.) For initial conditions, we compute

$$f(x) = \sum_{n=1}^{\infty} c_n(0)\sin\frac{n\pi x}{L} \Rightarrow c_n(0) = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx.$$

Solving for $c_n(t)$ with an integrating factor, we find

$$e^{k\frac{n^2\pi^2}{L^2}y}c_n(t) = B_n\int e^{k\frac{n^2\pi^2}{L^2}t}dt = B_n\frac{L^2}{kn^2\pi^2}e^{k\frac{n^2\pi^2}{L^2}t} + C,$$

or

$$c_n(t) = B_n\frac{L^2}{kn^2\pi^2} + Ce^{-k\frac{n^2\pi^2}{L^2}t}.$$

Setting $t = 0$, we have

$$C = c_n(0) - B_n\frac{L^2}{kn^2\pi^2},$$

giving

$$c_n(t) = B_n\frac{L^2}{kn^2\pi^2} + \Big(c_n(0) - B_n\frac{L^2}{kn^2\pi^2}\Big)e^{-k\frac{n^2\pi^2}{L^2}t}.$$

The solution is then

$$u(t,x) = \sum_{n=1}^{\infty}\Big[B_n\frac{L^2}{kn^2\pi^2} + \Big(c_n(0) - B_n\frac{L^2}{kn^2\pi^2}\Big)e^{-k\frac{n^2\pi^2}{L^2}t}\Big]\sin\frac{n\pi x}{L}.$$

6b. [5 points] For the PDE in Haberman 3.4.11, find the equilibrium solution $\bar{u}(x)$ and show that it matches the limit of your full solution as $t \to \infty$.

**Solution.** First, the equilibrium solution satisfies

$$\begin{aligned} k\bar{u}_{xx}(x) &= -\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \\ \bar{u}(0) &= 0 \\ \bar{u}(L) &= 0. \end{aligned}$$

Integrating twice, we have

$$k\bar{u}(x) = \sum_{n=1}^{\infty}\frac{L^2}{n^2\pi^2}B_n\sin\frac{n\pi x}{L} + C_1x + C_2.$$

According to our boundary conditions, $\bar{u}(0) = 0$ implies that $C_2 = 0$, while $\bar{u}(L) = 0$ implies that $C_1 = 0$. Therefore

$$\bar{u}(x) = \sum_{n=1}^{\infty} \frac{L^2}{kn^2\pi^2} B_n \sin \frac{n\pi x}{L}.$$

On the other hand, we see that

$$\lim_{t\to\infty} u(t,x) = \sum_{n=1}^{\infty} B_n \frac{L^2}{kn^2\pi^2} \sin \frac{n\pi x}{L},$$

the same thing.

7a. [10 points] Haberman 3.4.12.

**Solution.** In this case we look for solutions of the form

$$u(t,x) = C_0(t) + \sum_{n=1}^{\infty} C_n(t) \cos \frac{n\pi x}{L}. \tag{1}$$

Upon substitution into the equation, we have

$$C_0'(t) + \sum_{n=1}^{\infty} \left( C_n'(t) + \frac{n^2\pi^2}{L^2} C_n(t) \right) \cos \frac{n\pi x}{L} = e^{-t} + e^{-2t} \cos \frac{3\pi x}{L},$$

from which we have three cases:

$$\begin{aligned}
n = 0: &\quad C_0'(t) = e^{-t}; \quad C_0(0) = \frac{1}{L} \int_0^L f(x)dx \\
n = 3: &\quad C_3'(t) + k\frac{9\pi^2}{L^2} C_3(t) = e^{-2t}; \quad C_3(0) = \frac{2}{L} \int_0^L f(x) \cos \frac{3\pi x}{L} dx \\
n \neq 0, 3: &\quad C_n'(t) + k\frac{n^2\pi^2}{L^2} C_n(t) = 0; \quad C_n(0) = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx.
\end{aligned}$$

Solving these, we find

$$\begin{aligned}
C_0(t) &= (1 - e^{-t}) + C_0(0) \\
C_3(t) &= (\frac{9\pi^2 k}{L^2} - 2)^{-1} (e^{-2t} - e^{-k\frac{9\pi^2 t}{L^2}}) + C_3(0) e^{-k\frac{9\pi^2 t}{L^2}} \\
C_n(t) &= C_n(0) e^{-k\frac{n^2\pi^2}{L^2} t}; \quad n \neq 0, 3,
\end{aligned}$$

which can be substituted into (1) to get

$$\begin{aligned}
u(t,x) = (1 - e^{-t}) + C_0(0) + &\left( (\frac{9\pi^2 k}{L^2} - 2)^{-1} (e^{-2t} - e^{-k\frac{9\pi^2 t}{L^2}}) + C_3(0) e^{-k\frac{9\pi^2 t}{L^2}} \right) \cos \frac{3\pi x}{L} \\
&+ \sum_{n\neq 0,3} C_n(0) e^{-k\frac{n^2\pi^2}{L^2} t} \cos \frac{n\pi x}{L}.
\end{aligned}$$

7b. [5 points] For the PDE in Haberman 3.4.12, find the equilibrium solution $\bar{u}(x)$ and show that it matches the limit of your full solution as $t \to \infty$.

**Solution.** First, the equilibrium solution should satisfy

$$\begin{aligned} k\bar{u}_{xx} &= 0 \\ \bar{u}_x(0) &= 0 \\ \bar{u}_x(L) &= 0, \end{aligned}$$

from which we can only conclude that

$$\bar{u}(x) = C,$$

for some constant $C$. In order to determine the value of $C$, we integrate the entire equation,

$$\int_0^L u_t dx = k\int_0^L u_{xx} dx + \int_0^L e^{-t} dx + e^{-2t}\int_0^L e^{-2t}\cos\frac{3\pi x}{L} dx,$$

from which we find

$$\frac{d}{dt}\int_0^L u dx = Le^{-t} \Rightarrow \int_0^L u dx = -Le^{-t} + K.$$

Setting $t = 0$, this becomes

$$\int_0^L f(x) dx = -L + K \Rightarrow K = L + \int_0^L f(x) dx.$$

Finally, taking the limit as $t$ approached $\infty$,

$$\int_0^L \bar{u} dx = L + \int_0^L f(x) dx \Rightarrow CL = L + \int_0^L f(x) dx \Rightarrow C = 1 + \frac{1}{L}\int_0^L f(x) dx.$$

Thus

$$\bar{u}(x) = 1 + \frac{1}{L}\int_0^L f(x) dx.$$

Finally, taking the limit as $t \to \infty$ in the solution from Part (a), we have

$$\lim_{t\to\infty} u(t,x) = 1 + C_0(0) = 1 + \frac{1}{L}\int_0^L f(x) dx,$$

which is indeed the equilibrium solution.