M412 Assignment 9, due Monday December 5

1. [10 pts] Use separation of variables to show that solutions to the quarter-plane problem

$$\begin{aligned} u_t &= u_{xx}; \quad t > 0, 0 < x < \infty \\ u(t,0) &= 0 \\ |u(t,+\infty)| &\text{ bounded} \\ u(0,x) &= f(x), \quad 0 < x < \infty, \end{aligned}$$

can be written in the form

$$u(t,x) = \int_0^\infty C(\omega) e^{-\omega^2 t} \sin \omega x d\omega,$$

for some appropriate constant $C(\omega)$.

2. [20 pts] Show that the coefficient $C(\omega)$ from Problem 10 satisfies

$$C(\omega) = \frac{2}{\pi} \int_0^\infty f(x) \sin \omega x dx.$$

($C(\omega)$ is called the Fourier sine transform of f.)

3. [5 pts] Haberman 10.3.3.

4. [5 pts] Haberman 10.3.7.

5. In this problem, we will combine three problems from Haberman to solve the PDE

$$\begin{aligned} u_t &= k u_{xxx} \\ u(0,x) &= f(x). \end{aligned}$$

5a. [5 pts] Haberman 10.3.8.

5b. [10 pts] Haberman 10.4.6. Proceed here by taking a Fourier transform of the equation for $y(x)$ and using (5a).

5c. [10 pts] Haberman 10.4.7, Parts (a), (b), and (c). In Part (a), Haberman is only asking that you show

$$u(t,x) = \mathcal{F}^{-1}[\hat{f}(\omega) e^{ik\omega^3 t}].$$

In Part (b), you will write this as a convolution, while in Part (c) you will need to compute

$$\mathcal{F}^{-1}[e^{ik\omega^3 t}]$$

in terms of the Airy function $Ai(x)$ from Part (5b). In this last calculation, you will want to make the change of variables

$$z = -(3kt)^{1/3} \omega.$$

You can check your final answer in the back of Haberman.