M412 Assignment 9 Solutions

1. [10 pts] Use separation of variables to show that solutions to the quarter-plane problem

$$\begin{aligned} u_t &= u_{xx}; \quad t > 0, 0 < x < \infty \\ u(t,0) &= 0 \\ |u(t,+\infty)| &\text{ bounded} \\ u(0,x) &= f(x), \quad 0 < x < \infty, \end{aligned}$$

can be written in the form

$$u(t,x) = \int_0^\infty C(\omega) e^{-\omega^2 t} \sin \omega x d\omega,$$

for some appropriate constant $C(\omega)$.

Solution. Looking for solutions $u(t,x) = T(t)X(x)$, we find

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda X,$$

which gives the eigenvalue problem

$$\begin{aligned} X''(x) + \lambda X(x) &= 0 \\ X(0) &= 0 \\ \lim_{x \to \infty} |X(x)| &< \infty. \end{aligned}$$

We conclude that the eigenvalues are all positive real numbers $\lambda > 0$, with associated eigenfunctions $X_\lambda(x) = \sin \sqrt{\lambda} x$. In this way, the most general solution is

$$u(t,x) = \int_0^\infty A(\lambda) e^{-\lambda t} \sin \sqrt{\lambda} x d\lambda,$$

where $A(\lambda)$ is to be determined from the initial condition. In order to put this in the form stated in the problem, make the change of variable $\omega = \sqrt{\lambda}$, which gives $d\lambda = 2\omega d\omega$ and consequently

$$u(t,x) = \int_0^\infty A(\omega^2) e^{-\omega^2 t} \sin \omega x (2\omega) d\omega.$$

Upon choosing $C(\omega) = 2\omega A(\omega^2)$, we arrive at the claimed relationship.

2. [20 pts] Show that the coefficient $C(\omega)$ from Problem 1 satisfies

$$C(\omega) = \frac{2}{\pi} \int_0^\infty f(x) \sin \omega x dx.$$

($C(\omega)$ is called the Fourier sine transform of f.)

Solution. First, the initial condition $u(0,x) = f(x)$ gives

$$f(x) = \int_0^\infty C(\omega) \sin \omega x dx.$$

For any real number $k > 0$, multiply both sides of this last expression by $\sin kx$ and integrate x from 0 to ∞. We have

$$\begin{aligned}\int_0^\infty f(x)\sin kx dx &= \int_0^\infty \sin kx \int_0^\infty C(\omega)\sin\omega x d\omega dx \\ &= \lim_{L\to\infty}\lim_{N\to\infty}\int_0^N \sin kx \int_0^L C(\omega)\sin\omega x d\omega dx \\ &= \lim_{L\to\infty}\lim_{N\to\infty}\int_0^L C(\omega)\int_0^N \sin kx\sin\omega x dx d\omega,\end{aligned}$$

where in the second step we have switched the order of integration. (In general, this step requires some assumptions on $C(\omega)$, but since $C(\omega)$ is what we are searching for, we assume the steps are valid, keeping in mind that in principle they should later be verified. Though we won't carry out the verification here.) We now employ the trigonometric relationship

$$\sin A\sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B),$$

to write this last integral as

$$\lim_{L\to\infty}\lim_{N\to\infty}\int_0^L C(\omega)\int_0^N \frac{1}{2}\cos[(k-\omega)x] - \frac{1}{2}\cos[(k+\omega)x]dx d\omega.$$

Upon integration over x, this last expression becomes

$$\lim_{L\to\infty}\lim_{N\to\infty}\int_0^L C(\omega)\Big[\frac{\sin[(k-\omega)N]}{2(k-\omega)} - \frac{\sin[(k+\omega)N]}{2(k+\omega)}\Big]d\omega.$$

For the integration involving

$$\frac{\sin[(k+\omega)N]}{2(k+\omega)},$$

we observe that with $k > 0$ we never have the issue of division by 0, and hence the Riemann–Lebesgue lemma gives that the limit as $N \to \infty$ of this integral is 0. For the remaining term, we make the change of variable, $z = k - \omega$, from which we have

$$\begin{aligned}\lim_{L\to\infty}\lim_{N\to\infty}\int_0^L C(\omega)\frac{\sin[(k-\omega)N]}{2(k-\omega)}d\omega &= \lim_{L\to\infty}\lim_{N\to\infty}\int_k^{k-L} C(k-z)\frac{\sin[zN]}{2z}(-dz) \\ &= \lim_{L\to\infty}\lim_{N\to\infty}\int_{k-L}^k C(k-z)\frac{\sin[zN]}{2z}dz \\ &= \lim_{L\to\infty}\lim_{N\to\infty}\int_{k-L}^k \Big(C(k-z) - C(k)\Big)\frac{\sin[zN]}{2z}dz \\ &+ \lim_{L\to\infty}\lim_{N\to\infty}\int_{k-L}^k C(k)\frac{\sin[zN]}{2z}dz.\end{aligned}$$

The Riemann–Lebesgue lemma asserts that the first of these two integrals goes to 0 in the limit as $N \to \infty$. For the second, we make the change of variable $y = zN$, from which we observe,

$$\begin{aligned}\lim_{L\to\infty}\lim_{N\to\infty}\int_{k-L}^k C(k)\frac{\sin[zN]}{2z}dz &= \lim_{L\to\infty}\lim_{N\to\infty}\int_{(k-L)N}^{kN} C(k)\frac{\sin[y]}{2\frac{y}{N}}\frac{1}{N}dy \\ &= C(k)\frac{1}{2}\int_{-\infty}^{+\infty}\frac{\sin y}{y}dy = C(k)\frac{\pi}{2},\end{aligned}$$

where in this final equality we have used the integration

$$\int_{-\infty}^{+\infty} \frac{\sin y}{y} dy = \pi.$$

Solving for $C(k)$, we conclude

$$C(k) = \frac{2}{\pi} \int_0^{\infty} f(x) \sin kx dx,$$

which is precisely the relationship we set out to establish, with the independent variable denoted k rather than ω.

3. [5 pts] Haberman 10.3.3.

Solution. First recall the relationship

$$(e^{i\theta})^* = e^{-i\theta},$$

where $*$ denote complex conjugate. (That is, $(\alpha + i\beta)^* = \alpha - i\beta$. This relationship is straightforward to prove: $(e^{i\theta})^* = (\cos\theta + i\sin\theta)^* = (\cos\theta - i\sin\theta) = e^{-i\theta}$.) The Fourier transform of $f(x)$ is

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega x} f(x) dx.$$

The complex conjugate of this is

$$F^*(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega x} f(x) dx = F(-\omega).$$

If you're uncertain about bringing the complex conjugate inside the integration, consider it in the following way.

$$\begin{aligned}
F^*(\omega) &= \Big(\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega x} f(x) dx\Big)^* \\
&= \Big(\frac{1}{2\pi} \int_{-\infty}^{+\infty} (\cos\omega x + i\sin\omega x) f(x) dx\Big)^* \\
&= \Big(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \cos\omega x f(x) dx + i\frac{1}{2\pi} \int_{-\infty}^{+\infty} \sin\omega x f(x) dx\Big)^* \\
&= \Big(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \cos\omega x f(x) dx - i\frac{1}{2\pi} \int_{-\infty}^{+\infty} \sin\omega x f(x) dx\Big)^* \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega x} f(x) dx.
\end{aligned}$$

4. [5 pts] Haberman 10.3.7.

Solution. In this case, compute directly,

$$\begin{aligned}
\mathcal{F}^{-1}[F(\omega)] &= \int_{-\infty}^{+\infty} e^{i\omega x} e^{-\alpha|\omega|} d\omega \\
&= \int_{-\infty}^{0} e^{(ix+\alpha)\omega} d\omega + \int_0^{\infty} e^{(ix-\alpha)\omega} d\omega \\
&= \frac{1}{ix+\alpha} - \frac{1}{ix-\alpha} = \frac{(ix-\alpha)-(ix+\alpha)}{(ix+\alpha)(ix-\alpha)} \\
&= \frac{-2\alpha}{-x^2-\alpha^2} = \frac{2\alpha}{\alpha^2+x^2}.
\end{aligned}$$

5. In this problem, we will combine three problems from Haberman to solve the PDE

$$\begin{aligned} u_t &= k u_{xxx} \\ u(0,x) &= f(x). \end{aligned}$$

5a. [5 pts] Haberman 10.3.8.

Solution. Recalling that

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega x} f(x) dx,$$

we differentiate with respect to ω to find

$$\frac{dF}{d\omega} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} ix e^{i\omega x} f(x) dx.$$

The claimed relationship follows from multiplication on both sides of this last expression by i.

5b. [10 pts] Haberman 10.4.6. Proceed here by taking a Fourier transform of the equation for $y(x)$ and using (5a).

Solution. Taking the Fourier transform of

$$\frac{d^2 y}{dx^2} - xy = 0,$$

we have

$$\mathcal{F}[\frac{d^2 y}{dx^2}] - \mathcal{F}[xy] = 0.$$

According to (5a), this gives

$$-\omega^2 \hat{y} + i\hat{y}'(\omega) = 0,$$

which can be solved by separation of variables for

$$\hat{y}(\omega) = Ce^{-i\frac{1}{3}\omega^3},$$

for some constant of integration C. Inverting, we have

$$y(x) = \int_{-\infty}^{+\infty} e^{-i\omega x} C e^{-i\frac{1}{3}\omega^3} d\omega.$$

In order to put this in the form Haberman recommends, use Euler's formula to get

$$\begin{aligned} y(x) &= C \int_{-\infty}^{+\infty} e^{-i\omega x - i\frac{1}{3}\omega^3} d\omega \\ &= C \int_{-\infty}^{+\infty} \cos[\omega x + \frac{1}{3}\omega^3] - i \sin[\omega x + \frac{1}{3}\omega^3] d\omega \\ &= 2C \int_0^{\infty} \cos[\omega x + \frac{1}{3}\omega^3] d\omega, \end{aligned}$$

where in the last equality we have observed the integration properties of even and odd functions. Finally, according to Haberman, we have the initial condition

$$y(0) = \frac{1}{\pi} \int_0^{\infty} \cos[\frac{1}{3}\omega^3] d\omega,$$

which provides

$$C = \frac{1}{2\pi}.$$

We conclude

$$y(x) = \frac{1}{\pi}\int_0^\infty \cos[\omega x + \frac{1}{3}\omega^3]d\omega,$$

typically referred to as the Airy function and denoted $Ai(x)$.

5c. [10 pts] Haberman 10.4.7, Parts (a), (b), and (c). In Part (a), Haberman is only asking that you show

$$u(t,x) = \mathcal{F}^{-1}[\hat{f}(\omega)e^{ik\omega^3 t}].$$

In Part (b), you will write this as a convolution, while in Part (c) you will need to compute

$$\mathcal{F}^{-1}[e^{ik\omega^3 t}]$$

in terms of the Airy function $Ai(x)$ from Part (5b). In this last calculation, you will want to make the change of variables

$$z = (3kt)^{1/3}\omega.$$

You can check your final answer in the back of Haberman.

Solution. First, taking a Fourier transform of the equation and its initial conditions, we find

$$\begin{aligned}\hat{u}_t &= i\omega^3 k\hat{u}\\ \hat{u}(0,\omega) &= \hat{f}(\omega),\end{aligned}$$

which is solved by

$$\hat{u}(t,\omega) = \hat{f}(\omega)e^{ik\omega^3 t}.$$

By the Fourier Inversion Theorem, this gives

$$u(t,x) = \mathcal{F}^{-1}[\hat{f}(\omega)e^{ik\omega^3 t}].$$

For Part (b), we note that by the Convolution Theorem,

$$u(t,x) = \mathcal{F}^{-1}[\hat{f}(\omega)] * \mathcal{F}^{-1}[e^{ik\omega^3 t}] = f(x) * \mathcal{F}^{-1}[e^{ik\omega^3 t}].$$

For Part (c), the key is to recognize how $\mathcal{F}^{-1}[e^{ik\omega^3 t}]$ can be written in terms of the Airy equation. We have

$$\mathcal{F}^{-1}[e^{ik\omega^3 t}] = \int_{-\infty}^{+\infty} e^{-i\omega x + ik\omega^3 t}d\omega,$$

in which we make the suggested change of variables $z = -(3kt)^{1/3}\omega$, giving

$$\begin{aligned}\mathcal{F}^{-1}[e^{ik\omega^3}] &= \int_{-\infty}^{+\infty} e^{i\frac{x}{(3kt)^{1/3}}z - i\frac{1}{3}z^3}\frac{dz}{(3kt)^{1/3}}\\ &= \frac{2}{(3kt)^{1/3}}\int_0^\infty \cos[\frac{-x}{(3kt)^{1/3}}z + \frac{1}{3}z^3]dz\\ &= \frac{2\pi}{(3kt)^{1/2}}Ai(\frac{-x}{(3kt)^{1/3}}),\end{aligned}$$

where $Ai(\cdot)$ is as in Haberman 10.4.6. We conclude

$$u(t,x) = f(x) * \frac{2\pi}{(3kt)^{1/2}} Ai\left(\frac{-x}{(3kt)^{1/3}}\right) = \frac{1}{(3kt)^{1/2}} \int_{-\infty}^{+\infty} Ai\left(\frac{-(x-y)}{(3kt)^{1/3}}\right) f(y) dy$$
$$= \frac{1}{(3kt)^{1/2}} \int_{-\infty}^{+\infty} Ai\left(\frac{y-x}{(3kt)^{1/3}}\right) f(y) dy.$$