M611, Fall 2008, Assignment 1

Due Friday, Sept. 5

1. [10 pts] Prove the following theorem: If P and Q are commuting $n \times n$ matrices, then

$$e^P e^Q = e^{P+Q}.$$

Explain in your proof where the assumption that P and Q commute is used. **Hint.** Use Theorem 3.51 from Rudin, stated here for matrices: If the series of $n \times n$ matrices $\sum_{k=0}^{\infty} A_k$, $\sum_{k=0}^{\infty} B_k$, and $\sum_{k=0}^{\infty} C_k$ converge respectively to the matrices $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$, and $C_k = \sum_{j=0}^{k} A_j B_{k-j}$, then $\mathbb{C} = \mathbb{A}\mathbb{B}$.

2. [10 pts] Solve the ODE system

$$\begin{aligned} \frac{dy_1}{dt} &= y_1 + 4y_2; \quad y_1(0) = 0, \\ \frac{dy_2}{dt} &= 5y_1 + 2y_2; \quad y_2(0) = 1. \end{aligned}$$

3. [10 pts] Show that if A is a real-valued 2×2 matrix, and $\lambda \in \mathbb{C} - \mathbb{R}$ is an eigenvalue of A with associated eigenvector $\vec{v}$, then the general solution for

$$\frac{d\vec{y}}{dt} = A\vec{y}; \quad \vec{y}_0 \in \mathbb{R}^2,$$

has the form

$$\vec{y}(t) = \text{Re}\,(C\vec{v}e^{\lambda t}),$$

some some $C \in \mathbb{C}$.

4. [10 pts] Use your result from Problem 3 to solve the ODE system

$$\begin{aligned} \frac{dy_1}{dt} &= 2y_1 + y_2; \quad y_1(0) = 1, \\ \frac{dy_2}{dt} &= -y_1 + 3y_2; \quad y_2(0) = 2. \end{aligned}$$

5. [10 pts] Solve the ODE system

$$\begin{aligned} \frac{dy_1}{dt} &= y_2; \quad y_1(0) = 1 \\ \frac{dy_2}{dt} &= 4y_1 + 3y_2 - 4y_3; \quad y_2(0) = 0 \\ \frac{dy_3}{dt} &= y_1 + 2y_2 - y_3; \quad y_3(0) = 0. \end{aligned}$$