M611, Fall 2008, Assignment 12

Due Monday Dec. 1

1. [10 pts]

1a. Use our solution from class to

$$\begin{aligned} u_t + \vec{b} \cdot Du &= 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+ \\ u(\vec{x}, 0) &= g(\vec{x}) \quad \vec{x} \in \mathbb{R}^n \end{aligned}$$

to give a reasonable definition for the operator $e^{-t\vec{b}\cdot D}$. Verify that under sufficiently strong assumptions on g your definition is equivalent to

$$e^{-t\vec{b}\cdot D} := \sum_{k=0}^{\infty} \frac{(-t\vec{b} \cdot D)^k}{k!}.$$

(Here, I'm using the notation $\vec{b} \cdot D := b_1\partial_{x_1} + b_2\partial_{x_2} + \cdots + b_n\partial_{x_n}$.)

1b. Use (1a) to derive a solution to

$$\begin{aligned} u_t + \vec{b} \cdot Du + cu &= f \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+ \\ u(\vec{x}, 0) &= g(\vec{x}) \quad \vec{x} \in \mathbb{R}^n \end{aligned}$$

(This is a second approach to solving the same problem we solved earlier by the method of characteristics.)

1c. Use (1a) to solve

$$\begin{aligned} u_t + \vec{b} \cdot Du + cu - \Delta u &= f \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+ \\ u(\vec{x}, 0) &= g(\vec{x}) \quad \vec{x} \in \mathbb{R}^n. \end{aligned}$$

1d. Use our solution to Poisson's equation to give a reasonable definition for the operator $(-\Delta)^{-1}$.

1e. Use our solutions to the wave equation in dimensions $n = 1, 2, 3$ to develop reasonable definitions in these dimensions for the operator $\cos(\sqrt{-\Delta}t)$.

2. [10 pts] Evans 3.5.3.

3. [10 pts] Solve the nonlinear PDE

$$\begin{aligned} (u_{x_1})^2 + x_2 u_{x_1} &= u \quad \text{in } \mathbb{R} \times \{x_2 > 2\} \\ u(x_1, 2) &= x_1 \quad x_1 \in \mathbb{R}. \end{aligned}$$

(The method also gives a solution in $\mathbb{R} \times \{x_2 < 2\}$.)

4. [10 pts] Consider the initial value problem for *Burgers' equation* in $\mathbb{R}$

$$\begin{aligned} u_t + uu_x &= 0 \quad \textit{in } \mathbb{R} \times \mathbb{R}_+ \\ u(x, 0) &= f(x), \quad x \in \mathbb{R}. \end{aligned}$$

Show that if $f \in C^1(\mathbb{R})$ and $u \in C^1(\mathbb{R} \times \mathbb{R}_+)$ then f must be non-decreasing on $\mathbb{R}$.

5. [10 pts] Let u be a C^1 solution of

$$\vec{b}(\vec{x}) \cdot Du(\vec{x}) = -u$$

in a neighborhood containing $B(0,1) \subset \mathbb{R}^2$. Show that if

$$\vec{b}(\vec{x}) \cdot \vec{x} \geq 0 \quad \text{on } \partial B(0,1)$$

then u must be identically 0 in $B(0,1)$.