Assignment 12, Problem 3 Solution

First, the problem is stated as

$$\begin{aligned}(u_{x_1})^2 + x_2 u_{x_1} &= u \quad \text{in } \mathbb{R} \times \{x_2 > 2\} \\ u(x_1, 2) &= x_1 \quad x_1 \in \mathbb{R},\end{aligned}$$

for which we would define

$$F(\vec{p}, z, \vec{x}) = p_1^2 + x_2 p_1 - z.$$

Computing

$$F_{p_2}(\vec{p}, z, \vec{x}) \equiv 0,$$

we see that this equation is characteristic at every boundary point and consequently that the method of characteristics cannot be applied. In fact, as stated the problem is complete rubbish: direct calculation shows that for $x_2 = 2$ we have $u_{x_1}(x_1, 2) = 1$ and so the equation gives

$$3 = x_1$$

for all $x_1 \in \mathbb{R}$ for any continuous solution. Rubbish.

The problem should have been stated as

$$\begin{aligned}(u_{x_1})^2 + x_2 u_{x_2} &= u \quad \text{in } \mathbb{R} \times \{x_2 > 2\} \\ u(x_1, 2) &= x_1 \quad x_1 \in \mathbb{R},\end{aligned}$$

and in that case we would proceed as follows: $F(\vec{p}, z, \vec{x}) = p_1^2 + x_2 p_2 - z$, so that

$$\begin{aligned}F_z &= -1 \\ D_x F &= (0, p_2) \\ D_p F &= (2p_1, x_2),\end{aligned}$$

and so the characteristic equations become

$$\begin{aligned}\frac{dp^1}{ds} &= p^1; \quad p^1(0) = 1 \Rightarrow p^1(s) = e^s \\ \frac{dp^2}{ds} &= 0; \quad p^2(0) = \frac{x_0^1 - 1}{2} \Rightarrow p^2(s) = \frac{x_0^1 - 1}{2} \\ \frac{dz}{ds} &= 2(p^1)^2 + x^2 p^2; \quad z(0) = x_0^1 \Rightarrow z(s) = e^{2s} + (x_0^1 - 1)e^s \\ \frac{dx^1}{ds} &= 2p^1; \quad x^1(0) = x_0^1 \Rightarrow x^1(s) = 2e^s + (x_0^1 - 2) \\ \frac{dx^2}{ds} &= x^2; \quad x^2(0) = 2; \quad x^2(s) = 2e^s.\end{aligned}$$

Here, we used

$$p^1(0) = u_{x_1}(x_0^1, 2) = 1,$$

and from the PDE

$$(p^1(0))^2 + 2p^2(0) = z(0).$$

Now, given any point $(x_1, x_2) \in \mathbb{R} \times \{x_2 > 0\}$, we invert

$$\begin{aligned} x_1 &= 2e^s + (x_0^1 - 2) \\ x_2 &= 2e^s \end{aligned}$$

for

$$\begin{aligned} e^s &= \frac{x_2}{2} \\ x_0^1 &= x_1 + 2 - x_2. \end{aligned}$$

We conclude

$$u(x_1, x_2) = z(s(x_1, x_2); x_0^1(x_1, x_2)) = (\frac{x_2}{2})^2 + (x_1 + 1 - x_2)\frac{x_2}{2}.$$