M611, Fall 2008, Assignment 3

Due Friday, Sept. 19

First, let's clean up some old business from Assignment 2. One assumption of the Schauder Fixed Point Theorem is that Ω, a subset of a Banach space, is compact. Here are some important definitions.

Definitions. Let X denote a metric space with metric ρ.

(i) We say that $A \subset X$ is open if for every element $x_0 \in A$ there exists some $r > 0$ so that if $x \in X$ and $\rho(x, x_0) < r$ then $x \in A$.

(ii) By an open cover of a set $A \subset X$ we mean a (possibly uncountable) collection $\{U_k\}$ of open subsets of X so that

$$A \subset \bigcup_k U_k.$$

(iii) A subset $K \subset X$ is said to be compact provided every open cover of K contains a finite subcover. (This is the general, topological, definition of a compact set.)

Lemma. A metric space X is compact (as in (iii) above) if and only if every sequence $\{x_k\}_{k=1}^{\infty} \subset X$ has a convergent subsequence.

Okay, now we know what is meant by a compact subset of a metric space (and therefore of a Banach space, since a Banach space is a metric space), but there's still a problem. In general it's difficult to establish that a set is compact. No worries, we turn to a slight modification of the Schauder Theorem.

Schauder's Revenge. Let Ω be a closed convex set in a Banach space X, and suppose $T : \Omega \to \Omega$ is continuous and such that $\overline{T(\Omega)}$ is compact in X. Then T has a fixed point in Ω.

Notice in particular that we have removed the requirement that Ω be compact, though to be fair we have added the requirement that $\overline{T(\Omega)}$ be compact. But in many cases, this latter condition is a bit easier to verify. Let's verify it in the case of Peano's Theorem. We'll set $J = [t_0 - \tau, t_0 + \tau]$ and

$$\Omega = \{\vec{y} \in C(J) : \vec{y}(t_0) = \vec{y}_0, \max_{t \in J} |\vec{y}(t) - \vec{y}_0| \leq \beta\},$$

and

$$\rho(\vec{y}_1, \vec{y}_2) := \max_{t \in J} |\vec{y}_1(t) - \vec{y}_2(t)|.$$

In order to show that $\overline{T(\Omega)}$ is compact in X we need to show that given any sequence $\{\phi_k\}_{k=1}^{\infty} \subset \overline{T(\Omega)}$ there exists a subsequence $\{\phi_{k_j}\}_{j=1}^{\infty}$ so that

$$\phi_{k_j} \to \phi$$

for some $\phi \in \overline{T(\Omega)}$.

In order to accomplish this, we're going to use the Arzela–Ascoli Theorem, and before I state that I need to specify what an equicontinuous sequence of functions is, so here goes.

Definition. Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of real-valued functions defined on $U \subset \mathbb{R}^n$, and fix $\vec{x}_0 \in U$. We say that $\{f_k\}_{k=1}^{\infty}$ is equicontinuous at $\vec{x}_0$ if for every $\epsilon > 0$ there exists a $\delta > 0$, independent of k, so that for all $\vec{x} \in U$ such that $|\vec{x} - \vec{x}_0| < \delta$ we have

$$|f_k(\vec{x}) - f_k(\vec{x}_0)| < \epsilon.$$

Arzela–Ascoli Theorem. Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of real-valued functions defined on a compact subset $K \subset \mathbb{R}^n$, that there exists a value $M > 0$ so that $|f_k(\vec{x})| \leq M$ for all $k = 1, 2, ...$ and all $\vec{x} \in U$ (we say the sequence is uniformly bounded), and that the sequence is equicontinuous at every $\vec{x} \in U$. Then there exists a subsequence $\{f_{k_j}\}_{j=1}^{\infty}$ that converges uniformly on K.

Now let's apply this to our problem. Consider any sequence $\{\vec{\phi}_k\}_{k=1}^{\infty} \subset T(\Omega)$ (dropping the closure for a moment), and note that $\vec{\phi}_k \in T(\Omega)$ implies there exists $\vec{y}_k \in \Omega$ so that $\vec{\phi}_k = T\vec{y}_k$:

$$\vec{\phi}_k(t) = \vec{y}_0 + \int_{t_0}^{t} \vec{f}(\vec{y}_k(s), s)ds.$$

We have, then, for (w.l.g.) $t_2 > t_1$

$$|\vec{\phi}_k(t_1) - \vec{\phi}_k(t_2)| \leq \int_{t_1}^{t_2} |\vec{f}(\vec{y}_k(s), s)|ds \leq M|t_2 - t_1|,$$

for all $t_1, t_2 \in J$. Since this is independent of k we conclude that $\{\vec{\phi}_k\}_{k=1}^{\infty}$ is equicontinuous on J. By a similar calculation we see that $\{\vec{\phi}_k\}_{k=1}^{\infty}$ is uniformly bounded, and so we can conclude from the Arzela–Ascoli Theorem that there exists a uniformly convergent subsequence, which must by definition converge to some $\vec{\phi} \in \overline{T(\Omega)}$. Finally, for the case $\{\vec{\phi}_k\}_{k=1}^{\infty} \subset \overline{T(\Omega)}$ (i.e., we bring back the closure) we note that each $\vec{\phi}_k$ is the (uniform, by passing to a subsequence) limit of a sequence in $T(\Omega)$, and so boundedness and equicontinuity can be recovered by taking a limit in the above inequalities. This gives that $\overline{T(\Omega)}$ is indeed compact.

The actual assignment

1. [10 pts] Use the method of Lebesgue to evaluate the integral

$$\int_0^1 \sqrt{x}dx.$$

2. [10 pts] In this problem I want to collect several important differentiation formulas. First, recall that for a function $\vec{f}(\vec{x})$, with $\vec{x} \in \mathbb{R}^n$ and $\vec{f} \in \mathbb{R}^m$, Evans uses the notation

$$D_x\vec{f}(\vec{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}.$$

Two important special cases are $n = 1$ and $m = 1$. First, for $m = 1$ we obtain the gradient vector

$$Df(\vec{x}) = (f_{x_1}, f_{x_2}, \ldots, f_{x_n}).$$

(The gradient vector is often regarded as a column vector, and in certain contexts this is important, so I may not typically refer to this as the gradient vector.) For $n = 1$ we obtain

$$D\vec{f}(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x} \\ \frac{\partial f_2}{\partial x} \\ \vdots \\ \frac{\partial f_m}{\partial x} \end{pmatrix} = \frac{d\vec{f}}{dx}.$$

Theorem. Suppose $\vec{y} : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\vec{x}_0 \in \mathbb{R}^n$ and $\vec{f} : \mathbb{R}^m \to \mathbb{R}^k$ is differentiable at $\vec{y}(\vec{x}_0) \in \mathbb{R}^m$. Then $\vec{h} = \vec{f}(\vec{y}(\vec{x}))$ is differentiable at $\vec{x} = \vec{x}_0$ and

$$D_x\vec{h}(\vec{x}_0) = D_y\vec{f}(\vec{y}(\vec{x}_0))D_x\vec{y}(\vec{x}_0).$$

In particular, notice that this is the multiplication of a $k \times m$ matrix by a $m \times n$ matrix, resulting in a $k \times m$ matrix.

For the following problems assume all functions are as differentiable as necessary.

(2a) Show that if $f : \mathbb{R}^n \to \mathbb{R}$ and $\vec{x} : \mathbb{R} \to \mathbb{R}^n$ then

$$\frac{d}{dt} f(\vec{x}(t)) = D_x f \cdot \frac{d\vec{x}}{dt}.$$

(2b) Show that if $\vec{x} \in \mathbb{R}^n$ and $f : [0, \infty) \to \mathbb{R}$ then

$$D_x f(|\vec{x}|) = f'(|\vec{x}|)\frac{\vec{x}}{|\vec{x}|},$$

and in particular

$$D_x(|\vec{x}|^r) = r|\vec{x}|^{r-2}\vec{x}.$$

(2c) Show that if $\vec{x} \in \mathbb{R}^n$ and $f : [0, \infty) \to \mathbb{R}$ then

$$\Delta f(|\vec{x}|) = f''(|\vec{x}|) + \frac{n-1}{r} f'(|\vec{x}|).$$

Here, $\Delta := \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$.

(2d) Show that if $u, v : \mathbb{R}^n \to \mathbb{R}$ then

$$\Delta(uv) = v\Delta u + 2Du \cdot Dv + u\Delta v.$$

(2e) Show that if $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$ and $\vec{g} : \mathbb{R}^n \to \mathbb{R}^m$ are regarded as row vectors then

$$D_x(\vec{f}(\vec{x}) \cdot \vec{g}(\vec{x})) = \vec{g}(\vec{x})D_x\vec{f}(\vec{x}) + \vec{f}(\vec{x})D_x\vec{g}(\vec{x}).$$

(2f) Show that if $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ then

$$D_x(f(\vec{x})\vec{g}(\vec{x})) = \vec{g}(\vec{x}) \otimes D_x f(\vec{x}) + f(\vec{x}) D_x \vec{g}(\vec{x}),$$

where $\otimes$ denotes the $m \times n$ tensor product matrix

$$\vec{g}(\vec{x}) \otimes D_x f(\vec{x}) = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_m \end{pmatrix} (f_{x_1}, f_{x_2}, \dots, f_{x_n}).$$

3. [10 pts] Prove the following theorem:

Theorem. Suppose $U \subset \mathbb{R}^n$ is open and bounded. Then for any $1 \leq p^* \leq p$

$$\|u\|_{L^{p^*}(U)} \leq C\|u\|_{L^p(U)}.$$

4. [10 pts] Prove Holder's inequality for three functions,

$$\int_U |uvw| d\vec{x} \leq \|u\|_{L^p(U)} \|v\|_{L^q(U)} \|w\|_{L^r(U)},$$

where $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$.

5. [10 pts] We define the convolution of two functions $f, g : \mathbb{R}^n \to \mathbb{R}$ as

$$f * g(\vec{x}) := \int_{\mathbb{R}^n} f(\vec{x} - \vec{y}) g(\vec{y}) d\vec{y}.$$

Establish the following properties:

(5a) $f * g(\vec{x}) = g * f(\vec{x})$.

(5b) For $f, g \in L^2(\mathbb{R}^n)$ we have $f * g \in L^\infty(\mathbb{R}^n)$ with the estimate

$$\|f * g\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}.$$

(5c) For $f, g \in L^1(\mathbb{R}^n)$ we have $f * g \in L^1(\mathbb{R}^n)$ with the estimate

$$\|f * g\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)}.$$

(5d) For $f, g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ we have

$$f * g \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n).$$