Solution to Problem (5d), Assignment 3

We have $f, g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, and we want to establish

$$f * g(\vec{x}) \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n).$$

First, we clearly have $f * g(\vec{x}) \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ from (5b) and (5c), so we need only show $f * g(\vec{x}) \in C(\mathbb{R}^n)$. To this end, let $\vec{x}_1, \vec{x}_2 \in \mathbb{R}^n$and write

$$\begin{aligned} |f * g(\vec{x}_1) - f * g(\vec{x}_2)| &\leq \int_{\mathbb{R}^n} |f(\vec{x}_1 - \vec{y}) - f(\vec{x}_2 - \vec{y})||g(\vec{y})|d\vec{y} \\ &\leq (\int_{\mathbb{R}^n} |f(\vec{x}_1 - \vec{y}) - f(\vec{x}_2 - \vec{y})|^2 d\vec{y})^{1/2} (\int_{\mathbb{R}^n} |g(\vec{y})|^2 d\vec{y})^{1/2}. \end{aligned}$$

where the second inequality is Cauchy–Schwartz (Holder with $p = q = 2$). Now $g \in L^2(\mathbb{R}^n)$, so we focus on f. For this integral, set $\vec{z} = \vec{x}_2 - \vec{y}$ to get

$$\int_{\mathbb{R}^n} |f(\vec{x}_1 - \vec{y}) - f(\vec{x}_2 - \vec{y})|^2 d\vec{y} = \int_{\mathbb{R}^n} |f(\vec{z} + (\vec{x}_1 - \vec{x}_2)) - f(\vec{z})|^2 d\vec{z}.$$

Now recall the result from class that I referred to as metric continuity: for $1 \leq p < \infty$, and $f \in L^p(\mathbb{R}^n)$, we have

$$\lim_{|\vec{h}| \to 0} \int_{\mathbb{R}^n} |f(\vec{z} + \vec{h}) - f(\vec{z})|^p d\vec{z} = 0.$$

Taking $p = 2$ and $\vec{h} = \vec{x}_1 - \vec{x}_2$ gives the result.

Alternatively, it's possible to proceed by approximating f by functions in $C_c^\infty(\mathbb{R}^n)$ (in fact $C(\mathbb{R})$ is sufficient), but that essentially just reproves metric continuity.