Solution to Problem 5, Assignment 6

First, v is clearly a solution to Poisson's equation

$$\begin{aligned} -\Delta v &= M_f(e^{x_1+1}) \quad \text{in } B^o(0,1) \\ v &= M_g + M_f(e^2 - e^{x_1+1}) \quad \text{on } \partial B(0,1). \end{aligned}$$

Upon setting $w = u - v$, we find

$$\begin{aligned} -\Delta w &= f - M_f(e^{x_1+1}) \leq 0 \quad \text{in } B^o(0,1) \\ w &= g - M_g - M_f(e^2 - e^{x_1+1}) \leq 0 \quad \text{on } \partial B(0,1). \end{aligned}$$

According, then, to Problem 2.5.4 $w \in C^2(B(0,1))$ is subharmonic in $B^o(0,1)$. We conclude from Part (b) of Problem 2.5.4 that w achieves its maximum on the boundary, and so

$$w(\vec{x}) \leq 0 \quad \text{for all } \vec{x} \in B(0,1).$$

This gives

$$u(\vec{x}) \leq v(\vec{x}) \leq M_g + (e^2 - e^{-1})M_f,$$

for all $\vec{x} \in B(0,1)$. Likewise, $w = u + v$ satisfies

$$\begin{aligned} -\Delta w &= f + M_f(e^{x_1+1}) \geq 0 \quad \text{in } B^o(0,1) \\ w &= g + M_g + M_f(e^2 - e^{x_1+1}) \geq 0 \quad \text{on } \partial B(0,1), \end{aligned}$$

and according to Problem 2.5.4 (with v replaced by $-v$) w achieves its minimum on the boundary, whence $w(\vec{x}) \geq 0$ and

$$u(\vec{x}) \geq -v(\vec{x}) \geq -M_g - (e^2 - e^{-1})M_f.$$

Combining these upper and lower estimates, we obtain the claim.

Note. Evans allows the constant C to depend on the dimension n, but it doesn't need to. He probably took a different approach, possibly using the Green's function for $B(0,1)$.