M611, Fall 2008, Assignments 8-9

I'll be out of town Friday, Oct. 31, so we won't have class that day and homework won't be due. Instead, this long assignment will be due Friday, Nov. 7.

1. [10 pts] Evans 2.5.6.

2. [10 pts] Evans 2.5.7.

3. [10 pts] Let $T : \mathcal{D}(0,1) \to \mathbb{R}^n$ be given by

$$T(\varphi) := \sum_{k=1}^{\infty} \frac{\varphi(x) \sin k\pi x}{x} dx.$$

Decide whether or not T is a distribution, and prove your answer. What is the order of this distribution?

4. [10 pts] Compute the distributional derivative of

$$u(x) = \log |x|,$$

for $U = \mathbb{R}$; i.e., compute the derivative of the distribution T_u.

5. [10 pts] Let $u(\vec{x}) = \frac{\vec{x}}{|\vec{x}|^3}$ for $\vec{x} \in \mathbb{R}^3 - \{0\}$.

5a. Compute $\nabla \cdot u(\vec{x})$ for $\vec{x} \neq 0$.

5b. Compute $\nabla \cdot T_u$; i.e., the distributional derivative of u.

6. [10 pts] Suppose $\psi \in \mathcal{D}(U)$ and $T \in \mathcal{D}'(U)$. Show that

$$\partial_{x_i}(\psi T) = (\partial_{x_i}\psi)T + \psi \partial_{x_i} T.$$

7. [10 pts] Prove Theorem 4 from our section on distributions.

8. [10 pts] (This problem fills the first gap we left in our proof of Theorem 5 in our section on distributions.)

Show that if $\varphi \in \mathcal{D}(\mathbb{R}^n)$ then for any standard unit vector $\hat{e}_i$

$$\frac{\varphi(\vec{x} + \epsilon \hat{e}_i) - \varphi(\vec{x})}{\epsilon} \overset{\epsilon \to 0}{\to} \varphi_{x_i}(\vec{x}) \quad \text{in } \mathcal{D}(\mathbb{R}^n).$$

9. [10 pts] (This problem fills the second gap we left in our proof of Theorem 5 in our section on distributions.)

Show that if $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$ then

$$\sum_{\vec{k} \in \mathbb{Z}^n} (\tau_{\vec{k}\epsilon}\varphi)\psi(\vec{k}\epsilon)\epsilon^n \overset{\epsilon \to 0}{\to} \phi * \psi \quad \text{in } \mathcal{D}(\mathbb{R}^n).$$

Note. First, show that there is a compact set K so that $\zeta_\epsilon(\vec{x}) := \sum_{\vec{k} \in \mathbb{Z}^n} (\tau_{\vec{k}\epsilon}\varphi)\psi(\vec{k}\epsilon)\epsilon^n$ has support in K for each ϵ. Then set

$$R_{\vec{k}\epsilon} := \{\vec{x} : k_k \epsilon \leq x_j \leq (k_j + 1)\epsilon\},$$

and explain why:

(i)

$$\int_{R_{\vec{k}\epsilon}} (\tau_{\vec{y}}\varphi(\vec{x}))\psi(\vec{y})d\vec{y} = (\tau_{\vec{\xi}(\vec{k})}\varphi(\vec{x}))\psi(\vec{\xi}(\vec{k}))\epsilon^n$$

for some $\vec{\xi}(\vec{k}) \in R_{\vec{k}\epsilon}$, and

(ii)

$$\sum_{\vec{k}\in\mathbb{Z}^n} \int_{R_{\vec{k}\epsilon}} (\tau_{\vec{y}}\varphi(\vec{x}))\psi(\vec{y})d\vec{y} = \varphi * \psi(\vec{x}).$$

Now use these.

10. [10 pts] Evans 2.5.10.