M611 Final Exam, Selected Solutions

4a. First, recall from the homework the inequalities

$$\|u\|_{L^1(\mathbb{R}^n)} \leq \|g\|_{L^1(\mathbb{R}^n)}$$
$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-n/2}\|g\|_{L^1(\mathbb{R}^n)}.$$

These are the cases $p = 1$ and $p = \infty$ respectively. (In principle, these should be established as part of the solution.) Now, for $1 < p < \infty$, compute,

$$\begin{aligned}\|u\|^p_{L^p(\mathbb{R})} &= \int_{\mathbb{R}^n} |u|^p d\vec{x} = \int_{\mathbb{R}^n} |u|^{p-1}|u| d\vec{x} \leq \|u\|^{p-1}_{L^\infty(\mathbb{R}^n)}\|u\|_{L^1(\mathbb{R}^n)} \\ &\leq \Big(Ct^{-n/2}\|g\|_{L^1(\mathbb{R}^n)}\Big)^{p-1}\|g\|_{L^1(\mathbb{R}^n)} = C^{p-1}t^{-\frac{n}{2}(p-1)}\|g\|^p_{L^1(\mathbb{R}^n)}.\end{aligned}$$

The claim is now immediate upon taking a p-root on each side, with constant $C^{1-\frac{1}{p}}$.

Note. Though we didn't develop this machinery, I'll mention that this problem can also be solved using the following useful lemma: For $1 \leq p \leq \infty$, $f \in L^p(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$, we have

$$\|f * g\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)}.$$

(See, e.g., Papa Rudin, Chapter 7, Problem 4, for the case $n = 1$.) For our purposes, this gives an expected triangle-type inequality

$$\Big\| \int_{\mathbb{R}^n} \Phi(\vec{x} - \vec{y}, t) g(\vec{y}) d\vec{y} \Big\|_{L^p_x(\mathbb{R}^n)} \leq \int_{\mathbb{R}^n} \|\Phi(\cdot - \vec{y}, t)\|_{L^p_x(\mathbb{R}^n)} |g(\vec{y})| d\vec{y}.$$

(Here, I have used the fact that the L^p norm eliminates dependence on $\vec{y}$ in Φ.)

5b. We use Poisson's formula

$$u(\vec{x}, t) = \frac{1}{2} \fint_{B(\vec{x},t)} \frac{tg(\vec{y}) + tDg(\vec{y}) \cdot (\vec{y} - \vec{x}) + t^2 h(\vec{y})}{\sqrt{t^2 - |\vec{x} - \vec{y}|^2}} d\vec{y},$$

and though it's not necessary, I think it clarifies things to work in polar coordinates,

$$u(\vec{x}, t) = \frac{1}{2\pi t^2} \int_0^t \int_{\partial B(\vec{x},r)} \frac{tg(\vec{y}) + tDg(\vec{y}) \cdot (\vec{y} - \vec{x}) + t^2 h(\vec{y})}{\sqrt{t^2 - r^2}} dS_y dr.$$

Now for $|\vec{x}| \leq \theta t$, $r = |\vec{x} - \vec{y}| \leq |\vec{x}| + |\vec{y}| \leq \theta t + |\vec{y}|$, and so $|\vec{y}| \geq r - \theta t$. Fix $\epsilon > 0$ so that $\theta + \epsilon < 1$, and notice that if $r \geq (\theta + \epsilon)t$ then $|\vec{y}| \geq \epsilon t$, and for t sufficiently large $\vec{y}$ will be off the support of g and h. We conclude that for t sufficiently large

$$\begin{aligned}u(\vec{x}, t) &= \frac{1}{2\pi t^2} \int_0^{(\theta+\epsilon)t} \int_{\partial B(\vec{x},r)} \frac{tg(\vec{y}) + tDg(\vec{y}) \cdot (\vec{y} - \vec{x}) + t^2 h(\vec{y})}{\sqrt{t^2 - r^2}} dS_y dr \\ &\leq \frac{1}{2\pi t^2} \int_0^{(\theta+\epsilon)t} \int_{\partial B(\vec{x},r)} \frac{tg(\vec{y}) + tDg(\vec{y}) \cdot (\vec{y} - \vec{x}) + t^2 h(\vec{y})}{\sqrt{t^2 - (\theta+\epsilon)^2 t^2}} dS_y dr \\ &\leq \frac{1}{2\pi\sqrt{1 - (\theta+\epsilon)^2} t^3} \int_{B(\vec{x},(\theta+\epsilon)t)} tg(\vec{y}) + tDg(\vec{y}) \cdot (\vec{y} - \vec{x}) + t^2 h(\vec{y}) d\vec{y},\end{aligned}$$

and the claim now follows from $g, h \in L^1(\mathbb{R}^2)$. The estimate is trivial for small t since it only requires boundedness and

$$u(\vec{x}, 0) = g(\vec{x}).$$