Integral of the Poisson kernel over $\partial\mathbb{R}^n_+$.

We show that for $\vec{x} \in \mathbb{R}^n_+$

$$\int_{\partial\mathbb{R}^n_+} K(\vec{x},\vec{y})dy_1\cdots dy_{n-1} = 1,$$

where $K(\vec{x},\vec{y})$ denotes the Poisson kernel for $\mathbb{R}^n_+$. Fix $\vec{x} \in \mathbb{R}^n_+$ and write $\vec{x} = (\bar{x}, x_n)$. We have

$$\begin{aligned}
\frac{2x_n}{n\alpha(n)}\int_{\partial\mathbb{R}^n_+}\frac{1}{|\vec{x}-\vec{y}|^n}dy_1\cdots dy_{n-1} &= \frac{2x_n}{n\alpha(n)}\int_0^\infty\int_{\partial B(\bar{x},r)}\frac{1}{(r^2+x_n^2)^{n/2}}dS_y dr \\
&= \frac{2x_n}{n\alpha(n)}(n-1)\alpha(n-1)\int_0^\infty\frac{r^{n-2}}{(r^2+x_n^2)^{n/2}}dr,
\end{aligned}$$

where in this last equality we have observed that $B(\bar{x},r) \subset \mathbb{R}^{n-1}$ and so $|\partial B(\bar{x},r)| = (n-1)\alpha(n-1)r^{n-2}$. In order to analyze the integral, we set

$$r = x_n \tan\theta,$$

so that

$$\int_0^\infty\frac{r^{n-2}}{(r^2+x_n^2)^{n/2}}dr = \frac{1}{x_n}\int_0^{\frac{\pi}{2}}\sin^{n-2}\theta d\theta = \frac{1}{x_n}\frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})},$$

where I've cribbed this last equality from an integral table. The result is now clear from the relations

$$\begin{aligned}
n\alpha(n) &= \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \\
(n-1)\alpha(n-1) &= \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})}.
\end{aligned}$$